Annals of Pure and Applied Mathematics
Vol. 9, No. 1, 2015, 53-65
ISSN: 2279-087X (P), 2279-0888(online)
Published on 13 January 2015
www.researchmathsci.org

Annals of
Pure and Applied
Mathematics

Rees Nilpotent Radical and Semisimple Semigroups

***Hai-zhen Liu*[1] and *Pei Yang*[2]**

[1]College of Science, University of Shanghai for Science and Technology
Shanghai-200093, China. Email: 592663972@qq.com
[2]College of Science, Univesity of Shanghai for Science and Technology
Shanghai-200093, China. Email: 742155149@qq.com

Received 14 December 2014; accepted 5 January 2015

Abstract. In this paper, the structure of completely semisimple semigroups and the structure of the Rees (completely) nilpotent radical extension semigroups were fully described by introducing the concepts of the Rees nilpotent radical of a semigroup and (completely) semisimple semigroup, which differs from the method of the nilpotent extension in the article [1,2] and the concept of the radical of a semigroup in the article [3]-[7]. The properties of the Rees nilpotent radical were discussed. These results illustrate that the radical theory is an effective means of studying the structure of semigroup.

Keywords: Nilpotent left (right) ideal; nilpotent congruence; (rees) nilpotent radical; completely semisimply semigroup; expansion P-congruence

AMS Mathematics Subject Classification (20M10): 20M10

1. Introduction and preliminaries

It is well known that rings and algebras all have nilpotent radical concepts. The famous Wedderburn-Artin theory formed by proving a right Artin ring having nilpotent radical. But so far , there is no nilpotent radical concept on semigroups. The main reason is the quotient semigroups are determined by congruences.

In [1,2], the nilpotent extension of semigroups is associated with the radical concept of semigroups which are determined by ideals. Recently, the [3]-[6] have overcomed the difficulties mentioned above by introducing several radicals of semigroups and investigated the structure of semigroups which have radicals.

This paper tries to use different method from nilpotent extension method in [1, 2] and different concept from radicals concept in [3]-[7]. We from another point of view, introduce the concept of (Rees) nilpotent radical of semigroups, nilpotent extension semigroups and (completely) semisimple semigroup to study properties of the radical.

Finally, the structure of the completely semisimple semigroup and Rees nilpotent radical (ideal) extension semigroup are given.

From this section, we always assume that semigroup mentioned below contains zero element, unless otherwise stated. Some concepts and resurlts without being marked are invoked from [1]-[7]. The following Lemma is clear.

Lemma 1.1. Let I be an ideal of semigroup S, then

(1) If S is nilpotent, then the subsemigroup of S and Rees quotient semigroup S/ρ_I are all nilpotent;

(2) If Rees quotient semigroup S/ρ_I is nilpotent, then for every $x \in S$, there exist $n \in N^+$, such that $x^n \in I$.

(3) If I and quotient semigroup is nilpotent, then S/ρ_I is nilpotent, then S is also nilpotent.

(4)

Lemma 1.2. Let S be a right (left) completely semigroup with non-zero idempotents, then

(1) S has primitive idempotent elements.

(2) If $0 \neq e \in E(S)$ is a primitive idempotent element of S, then $\forall 0 \neq f \in R_e$ is a primitive idempotent element.

Proof: We need only to prove the conclusion holds for right completely semigroup.

(1) Since S is a right completely semigroup and satisfies the $\min_R$ condition, then its non-empty right ideal $W = \{eS \mid \forall 0 \neq e \in E(S)\}$ has minimal element, denoted by $eS (e \in E(S))$. If there exist $0 \neq f \in E(S)$, such that $f \leq_n e$, where "$\leq_n$" is the natural order on $E(S)$, i.e. $fe = ef = f$ .Then $fS \subseteq eS$. Since eS is minimal, so we obtain that $fS = eS$, that is, fRe. According to $e = fe$,we have $e=f$, which implies that e is a primitive idempotent element of S.

(2) Suppose that $0 \neq f \in R_e$, then $fS = eS$ .If there exist a $0 \neq h \in E(S)$, such that $h \leq_n e$, which implies $fh = hf = h$, so $hS \subseteq fS = eS$. Since eS is minimal, we have $hS = eS = fS$, i.e. f is a primitive idempotent element of S. □

Lemma 1.3. Suppose that S is a right completely semigroup, then every right ideal K of S and every Rees quotient semigroup S/ρ_I are right complete.

Proof: We need only to prove the conclusion holds for right ideal K of S.

If K has a right ideal descending chain:

$K_1 \supseteq K_2 \supseteq \cdots \supseteq K_n \supseteq \cdots$,

then there exist two descending chains of S

$K_1K^1 \supseteq K_2K^1 \supseteq \cdots \supseteq K_nK^1 \supseteq \cdots; K_1S^1 \supseteq K_2S^1 \supseteq \cdots \supseteq K_nS^1 \supseteq \cdots$.

Since S is right complete, then there exist n, $m \in N^+$, such that

$K_nK^1 = K_{n+1}K^1 = \cdots, K_mS^1 = K_{m+1}S^1 = \cdots$.

Take $t = max\{n, m\}$. Notice that $K_tK^1 \subseteq K_t \subseteq K_tS^1$, thus $K_t = K_{t+1} = \cdots$, i.e. K is right complete.

Lemma 1.4. Let S be a right completely semigroup, then its nilpotent element right ideal is nilpotent.

Proof: Suppose that I is a nilpotent element right ideal of S, then

$I \supseteq I^2 \supseteq \cdots \supseteq I^n \supseteq \cdots$ is a descending chain of S.

By condition, we have that there exists $n \in N^+$, such that $B = I^n = I^{n+1} = \cdots$. Hence $B^2 = B$, we obtain that $b \in B, and\ bB \neq \{0\}$. Let $W = \{bB \neq \{0\} | b \in B\}$, then W is a non-zero right ideal set of S. So we know that W has the minimal element, denoted by $b_0B \neq \{0\}$. Hence we get that there exist $c \in B$, such that $\{0\} \neq b_0cB \subseteq b_0B$ from $b_0B^2 = b_0B \neq \{0\}$. By the condition b_0B is minimal, we know that $b_0B = b_0cB$. That is, there exist $x \in B$, such that $b_0c = b_0cx$. Therefore for $\forall k \in N^+$, we have

$0 \neq b_0c = b_0cx = \cdots = b_0cx^k$.

But x is a nilpotent element, so there exist $k \in N^+$, such that $x^k = 0$, then

$0 \neq b_0c = b_0cx = \cdots = b_0cx^k = 0$.

This contradiction implies $B = \{0\}$, i.e. I is a right nilpotent ideal.

2. Definitions and basic results

For the sake of convenience, we first cite some necessary and basic results which will be needed in the sequel.

Definition 2.1. Let ρ_I be a Rees congruence on semigroup S with zero elements, if I is a nilpotent subsemigroup of S, then we call ρ_I a nilpotent (Rees) congruence of S.

Clearly, $\{0\}$ is an ordinary nilpotent ideal of semigroup S which has zero elements. Therefore $\rho_{\{0\}} = I_S$ is a minimum nilpotent Rees congruence on S.

Lemma 2.1. For semigroup with S zero elements, the following conclusions hold:

(1) Denote the least upper bound of the whole nilpotent Rees congruence $\rho_\alpha (\alpha \in w)$ by ρ_{rni}, i.e.

$$\rho_{rni} = \vee_{\alpha\in\omega}\{\rho_\alpha | (\forall \alpha \in w) \rho_\alpha \textit{take over all thenilpotent Rees congruence on } S\}, \quad (2.1)$$

If ρ_{rni} be a nilpotent Rees congruence of S, then ρ_{rni} – set

$$N(S) = \cup_{\alpha\in\omega}\{N_\alpha | N_\alpha \textit{take over all the nilpotent ideals of } S\} \quad (2.2)$$

is existent, and $N(S)$ is a nilpotent ideal of S.

(2) Nilpotent Rees congruence $\rho_N = \rho_{rni}$ if and only if N is the maximum nilpotent ideal of S.

Proof: (1) If ρ_{rni} is a nilpotent Rees congruence on S, clearly, $\rho_{rni}-$set can be denoted by (2.1). By Definition 2.1, we know that $N(S)$ is a nilpotent ideal of S.

(2) $\Rightarrow$ We need only prove that $N(S)$ is the maximum nilpotent ideal of S. If it is not, we can suppose that there is a nilpotent ideal $N \supset N(S)$ of S, then the Rees congruence

ρ_N which regards N as $\rho-$set of S, i.e.

$$\rho_N \in \vee_{\alpha\in\omega}\{\rho_\alpha \mid (\forall \alpha \in w)\, \rho_\alpha \textit{take over all the nilpotent Rees congruence on } S\}.$$

This is contrary to the least upper bound of this set. Therefore $N(S)$ is the maximum nilpotent ideal of S.

$\Leftarrow$ If N is the maximum nilpotent ideal of S, then $N(S)$=N from

$N \in \{\mathrm{N}_\alpha \mid \mathrm{N}_\alpha(\alpha \in \omega)$ *is the nilpotent* Re*es congruence* $\rho_\alpha - set\}$,

thus $\rho_N = \rho_{rni}$.

Lemma 2.2. Let $N = \cup_{\alpha\in\omega}\{\mathrm{N}_\alpha \mid \mathrm{N}_\alpha \textit{take over all the nilpotent ideals of } S\}$, S be a completely semigroup, then N is the minimum nilpotent ideal. Thus $\rho_N = \vee_{\alpha\in\omega}\rho_{N_\alpha} = \rho_{rni}$ is the nilpotent Rees congruence on S.

Proof: According to [2], N is an ideal of S, let $x \in N$, then there exist $\alpha \in \omega$ and $x \in N_\alpha$. Therefore x is the nilpotent element of S. So we obtain N is the nilpotent elements ideal of S. Then N is the nilpotent ideal of S. Clearly, N is the minimum nilpotent ideal. Thus its corresponding nilpotent Rees congruence $\rho_N = \vee_{\alpha\in\omega}\rho_{N_\alpha} = \rho_{rni}$ is nilpotent Rees congruence on S.

By the lemma above, the concept of Rees nilpotent radical different from [3] to [6] is introduced.

Definition 2.2. For semigroup S with zero elements, if the least upper bound of the whole nilpotent Rees congruence $\rho_\alpha(\alpha \in w)$ by ρ_{rni} (2.1) on S are the nilpotent Rees congruence, then we call ρ_{rni} the Rees nilpotent radical (congruence) and call the nilpotent radical ideal $N(S)$ (2.2) nilpotent radical ideal. Sometimes we call Rees nilpotent radical congruence ρ_{rni} and nilpotent radical ideal $N(S)$ are both Rees nilpotent radical of S. If Rees nilpotent radical congruence $\rho_{rni} = \rho_{\{0\}} = 1_S$, then we call S a semisimple semigroup. If semisimple semigroup S is a completely semigroup, then we call S a completely semisimple semigroup.

Clearly, the Rees nilpotent radical of nilpotent semigroup is itself. A (completely) 0- simple semigroup is a (completely) semisimple semigroup.

Proposition 2.1. Let S be a semigroup with zero elements, then the following conclusions hold:

(1) If Rees nilpotent radical congruence ρ_{rni} is existent, then

(a) ρ_{rni} is the minimum Rees nilpotent congruence;

(b) $N(S)$ is the minimum nilpotent ideal ;

(c) S/ρ_{rni} is a semisimple semigroup.

(2) If S is a completely semigroup, then the Rees nilpotent radical ρ_{rni} and $N(S)$ of S is existent, and $N(S)$ contains all the nilpotent elements right (left) ideal.

(3) If $N(S)$ is a completely semigroup, S/ρ_{rni} is a completely regular semigroup, then S is a completely π- regular semigroup.

Proof: (1) (a) By Definition 2.2, we can easily know.

(b) By Lemma 2.2 and Definition 2.2 , we can get $N(S)$ is the minimum nilpotent ideal .

(c) If the Rees nilpotent radical $N(S/\rho_{rni}) \neq \{\bar{0}\}$ of S/ρ_{rni}, let its nilpotent index be t, then by [2], we have there exist non-zero ideal N of S, such that $N(S/\rho_K)= N \cup N(S)/N(S)$, where $N \cap N(S) = \{0\}$.

Notice that for $\forall n_k \in N (k = 1, \cdots t)$, we have

$$n_k\rho_{rni} \in N(S/\rho_{rni}),\ (n_1 n_2 \cdots n_t)\rho_{rni} = n_1\rho_{rni} \cdot n_2\rho_{rni} \cdots n_t\rho_{rni} = \bar{0},$$

i.e. $n_1 n_2 \cdots n_t \in N \cap N(S) = \{0\}$. Therefore N is the non-zero ideal of S which is contradict to the condition $N(S)$ is the minimum nilpotent ideal . Thus $N(S/\rho_{rni})=\{\bar{0}\}$, $N(S/\rho_{rni})=\{\bar{0}\}$, that is, S/ρ_{rni} is a semisimple semigroup .

(2) According to Lemma 2.1 ,Lemma 2.2 and Definition 2.2 ,we can know easily. Then Rees nilpotent radical ρ_{rni} and $N(S)$ of S are existent . By Lemma 1.6 and Lemma 1.7, we can obtain that $N(S)$ contains all the nilpotent elements right (left) ideal of S.

(3) By [2], the Rees congruence ρ_{rni} satisfies the condition

$S = N(S) \cup M, M \cong S/\rho_{rni}$, and it is a completely regular semigroup. If H is the right ideal of S, then we have

$HS = H(N(S) \cup M) = HN(S) \cup HM \subseteq H$ from Lemma1.1. Hence $HN(S) \subseteq H, HM \subseteq H$, thus $H^N \triangleq H \cap N(S), H^M \triangleq H \cap M$ are the ideal of $N(S)$ and M ,with $H = H^N \cup H^M$. If S has the descending chain:

$$H_1 \supseteq H_2 \supseteq \cdots \supseteq H_n \supseteq \cdots \qquad (2.3)$$

According to Lemma 1.1 and the above analysis, we get the right ideal descending chain of $N(S)$ and S:

$$H_1^N \supseteq H_2^N \supseteq \cdots \supseteq H_n^N \supseteq \cdots, H_1^M \supseteq H_2^M \supseteq \cdots \supseteq H_n^M \supseteq \cdots,$$

but $N(S)$ and M are completely semigroups, so there exist natural number $k, t \in N^+$, such that $H_k^N = H_{k+1}^N = \cdots, H_t^M = H_{t+1}^M = \cdots$. Take $n = max\{k, t\}$, then $H_n = H_n^N \cup H_n^M = H_{n+1}^N \cup H_{n+1}^M = H_{n+1} = \cdots$.

That is, the right ideal descending chain (2.3) stops in finite steps. Therefore S is right complete. Similarly, S is left complete. Since $N(S)$ is a nilpotent semigroup, then for $\forall x \in N(S)$, there exsist $n \in N^+$, such that $x^n = 0 \in \mathrm{Re}\,g(S)$, i.e. $N(S)$ is a π-regular semigroup. By the condition M is a regular semigroup, with $S = N(S) \cup M$, so S is a π-regular semigroup. To sum up, we have that S is a completely π-regular semigroup.

3. The structure characteristics of completely semisimple semigroup

Lemma 3.1. If the minimal 0- ideal K of semigroup S satisfies $K^2 \neq \{0\}$, then K is a 0- simple semigroup.

Proof: Let I be an ideal of K. Since $K^2 \neq \{0\}$, then we have $K^1 I K^1 \subseteq I$ .Thus $I = \cup_{0 \neq a \in I} K^1 a K^1$, $\forall 0 \neq a \in I, \{0\} \neq K^1 a K^1 \subseteq I$ .By $SK^1 a K^1 S \subseteq K^1 a K^1 \subseteq K$,we can get that $K^1 a K^1$ is a non-zero ideal contained in the K of S. We obtain $K^1 a K^1 = K$ from the minimality of K. Hence

$$I = \cup_{0 \neq a \in I} K^1 a K^1 = \cup_{0 \neq a \in I} K = K,$$

then K is a 0- simple semigroup.

Lemma 3.2. Let S be a completely semigroup, then the following conclusions hold:

(1) If the minimal 0- ideal K of semigroup S satisfies $K^2 \neq \{0\}$, then K is a completely 0-simple semigroup.

(2) If the minimal 0- ideal K of semigroup S satisfies $K^2 \neq \{0\}$, then K has primitive idempotent generators.

(3) If S is a semisimple semigroup, then its non zero 0-ideal K satisfies $K^2 \neq \{0\}$, and K contains the the minimal 0-ideal of S .

Proof: (1) If $K^2 \neq \{0\}$, then there exist $a \in K$, such that $\{0\} \neq aS^1 \subseteq K$. By the minimality of K we have $aS^1 = K$ .We easily know $I = S^1 a S^1 \neq \{0\}$ is a non zero ideal of S. In case H is a non-zero ideal of S contained in the I. Take $0 \neq h = tas \in H$. By $0 \neq asS^1 \subseteq aS^1$ and the minimality of $aS^1 = K$,we have $asS^1 = aS^1$. Hence

S^1asS^1=S^1aS^1=$I \subseteq H$, thus $H = I$.

That is, I is a minimal 0-ideal of S. By lemma 3.2, I is a regular semigroup. Therefore, there exist $e \in E(R_a)$, such that $eS^1 = eS$=aS^1=K , i.e. K is generated by e . If there exist $0 \neq f \in E(S) \cap K$, such that $fe = ef = f$, then $\{0\} \neq fS$=$efS \subseteq eS = K$. We have fS=K=eS from the minimality of K . Thus there exist $u \in S$ such that $e = fu$,then $f = fe = fu = e$. That is , e is a primitive idempotent generator of K. Similarly, we can prove it holds for the minimal 0-left ideal of S.

(2) We prove the conclusion by reduction to absurdity. By $K \neq \{0\}$, K^2=$\{0\}$ we have that K is a non-zero nilpotent 0-right (left) ideal of S. By lemma 1.7, S has non-zero nilpotent 0-ideals which is opposite to the condition S is a completely semisimple semigroup.

(3) Combine the conclusion (1)and (2),we can easily know.

Lemma 3.2. Let S is a completely semigroup, then the the following conclusions hold:
(1) If the minimal 0-right (left) ideal K of semigroup S satisfies $K^2 \neq \{0\}$, then K has primitive idempotent generators.
(2) If S is a semisimple semigroup, then its non zero 0-right (left) ideal K satisfies $K^2 \neq \{0\}$.
(3) If S is a semisimple semigroup, then its non zero minimal 0-right (left) ideal K has primitive idempotent generators.

Proof: (1). If $K^2 \neq \{0\}$, then there exist a $a \in K$, such that $\{0\} \neq aS^1 \subseteq K$ .By the minimality of K we have aS^1=K .We easily know $I = S^1aS^1 \neq \{0\}$ is a non zero ideal of S.In case H is a non zero ideal of S contained in the I .Take $0 \neq h$=$tas \in H$ .By $0 \neq asS^1 \subseteq aS^1$ and the minimality of aS^1=K , we have
asS^1=$aS^1 \Rightarrow S^1asS^1$=$S^1aS^1$=$I \subseteq H \Rightarrow H = I$.
That is, I is a minimal 0-ideal of S.By lemma 3.2, I is a regular semigroup. Therefore there exist $e \in E(R_a)$, such that $eS^1 = eS$=aS^1=K , i.e. K generated by e. If there exist $0 \neq f \in E(S) \cap K$, such that $fe = ef = f$, then $\{0\} \neq fS$=$efS \subseteq eS = K$. We have fS=K=eS from the minimality of K. Thus there exist $u \in S$, such that $e = fu$, then $f = fe = fu = e$. That is, e is a primitive idempotent generator of K. Similarly, we can prove it holds for the minimal 0-left ideal of S.

(2) We prove the conclusion by reduction to absurdity. By $K \neq \{0\}$, K^2=$\{0\}$, we have that K is a non-zero nilpotent 0-right (left) ideal of S. By [2], S has non-zero nilpotent 0-ideals which is opposite to the condition that S is a completely semisimple semigroup.

(3) Combine the conclusion (1)and (2),we can easily know.

Note: The reverse side of Lemma 3.2(2) and lemma 3.3(1) is not necessarily holding on completely semigroup. That is, in the completely semigroups, the ideal generated by primitive idempotent elements is not necessary a minimal 0-right ideal. We here give the following example.

Example 1.1. Let S be a completely semigroup, e be a primitive idempotent generator, but $eS = S$ is not the minimal 0-right ideal of S

S	0	e	a
0	0	0	0
e	0	e	a
a	0	0	0

Therefore, based on Lemma 3.2 and 3.3 we know the following concept is necessary.

Definition 3.1. In the completely semigroup S,we call the idempotent generator of its minimal 0-right (left) ideal the right (left) completely primitive idempotent generator of S. If e is not only the right completely primitive idempotent generator but also the left completely primitive idempotent generator, then we call e the completely primitive idempotent generator. And denote the Set by $CPE(S)$. If $CPE(S) = E(S)$, then S is called a completely primitive semigroup.

By Definition 3.1, we know that $CPE(S) \subseteq PE(S) \subseteq E(S)$, then a completely primitive semigroup is a primitive semigroup. But a primitive regular semigroup is not necessary to be a completely semigroup.

Lemma 3.1. Let S be a completely semisimple semigroup with zero elements, then the following conclusions hold:

(1) $CPE(S) \subseteq PE(S)$;

(2) Let $\{A_\alpha\}_{\alpha\in\omega}$ be the different non-zero minimal 0- ideal class of S, then $A = \cup_{\alpha\in\omega} A_\alpha$ is the 0- direct union ideal of S;

(3) S is the 0-direct union of its finite number of minimal 0-ideal $SeS(e \in CPE(S))$,

$$S = \bigcup_{e\in CPE(S)} SeS;$$

(4) S is the 0-direct union of its finite number of completely 0-simple semigroup, then S is a completely primitive regular semigroup.

Proof: (1). We need only prove for $\forall e \in PE(S)$, it is clearly eS is a minimal 0-right ideal of S. If eS is not a minimal 0-right ideal of S, since S is a completely semisimple semigroup, then

$W = \{H \mid H \textit{ is the non zero right ideal of } S, H \subset eS\}$

is non-empty and W has the minimal element denoted by H. Clearly, H is a minimal 0-right ideal contained in $eS \neq \{0\}$. ByLemma3.2, there exist $h \in CPE(S)$ such that

$H = hS \subseteq eS$. which lead to h=eh=he, $h \leq_n e$. Since e is a primitive idempotent element, so $h = e$, i.e. eS is a minimal 0-right ideal of S.

(2). Clearly, A is an ideal of S. Since S is a completely semisimple semigroup, By Lemma 3.2, the non-zero minimal 0-ideal A_α($\alpha \in \omega$)of S satisfies $A_\alpha^2 \neq \{0\}$. Assume there exist α, $\beta \in \omega$, $\alpha \neq \beta$,such that $A_\alpha \cap A_\beta \neq \{0\}$, then $A_\alpha \cap A_\beta$ is a non-zero 0-ideal of S contained in A_α and A_β. By the minimality of A_α, A_β we have A_α=$A_\alpha \cap A_\beta$=A_β. It is opposite to the assumption .Therefore $A_\alpha \cap A_\beta$={0}. We can get $A = \cup_{\alpha\in\omega} A_\alpha$ is a 0-direct union ideal of S from $A_\alpha \cdot A_\beta \subseteq A_\alpha \cap A_\beta$={0};

(3). By Lemma 3.2, Definition 3.1 and conclusion (2), we have that $K = \bigcup_{e\in CPE(S)} SeS$ is the non-zero minimal 0-ideal of S. By [2], quotient semigroup $S/K \overset{\phi}{\cong} S \setminus K \cup \{0\} \triangleq M$, then S=$M \cup K$ $M \cap K$={0}. If $M \neq \{0\}$, we can prove M is a completely semisimple semigroup.By lemma 1.5, M is a completely semigroup. Let

$\varphi : S \to S/K = S/\rho_K, \varphi(x) = x\rho_K$,

If Rees nilpotent radical $N(S/\rho_K) \neq \{\bar{0}\}$ of S/ρ_K, suppose its nilpotent index is t, by [2] we have, there exist a non-zero ideal N of S, such that $N \cup K = ((N(S/\rho_K))\varphi^{-1} \neq \{0\}$ is a non-zero ideal of S, where $N \cap K = \{0\}$. For $\forall n_1, n_2, \cdots, n_t \in N$,

$(n_1 n_2 \cdots n_t)\,\varphi$= $(n_1 n_2 \cdots n_t)\,\rho_K = n_1\rho_K \cdot n_2\rho_K \cdots n_t\rho_K = \{\bar{0}\}$,

i.e. $n_1 n_2 \cdots n_t \in N \cap K = \{0\}$. Thus N is a non-zero nilpotent ideal of S, which is opposite to $N(S) = \{0\}$.So $N(S/\rho_K)$={$\bar{0}$}, that is, S/ρ_K is a completely semisimple semigroup. From $S/K \overset{\phi}{\cong} M$ we can get M is a completely semisimple semigroup. If M has non-zero minimal 0-ideal A, then by Lemma 1.1 and 3.2, we have that there exist $f \in E(S) \cap CPE(M)$, such that $A = MfM \neq \{0\}$ .If f is not the primitive idempotent element of S, then according to the conclusion (1), we have $e \in PE(S)$=C$PE(S)$, such that $ef = fe = e$.Thus

$\{0\} \neq MeM = MefM \subseteq MfM \Rightarrow MeM = MfM \subseteq SeS \subseteq K \Rightarrow M \cap K \neq \{0\}$,

it is impossible, so $f \in PE(S)$=C$PE(S)$. But it lead to $MfM \subseteq SfS \subseteq K$, so $M \cap K \neq \{0\}$,it is also impossible. These contradiction imply M={0}, i.e. S=K. Let I_J denote a J-class representative element set of C$PE(S)$, $I_J \subseteq CPE(S)$, then $S = \bigcup_{e\in I_J} SeS$. If I_J contains infinite number of idempotent elements. We might suppose I_J={$e_1, e_2, \cdots, e_k, \cdots$}, make $W = \{S_k = \bigcup_{e\in I_J \setminus \{e_i\}_1^k} SeS (k \in N^+)\}$, then W is the ideal set of S.

Clearly, W has no minimal element, thus S is not a completely semigroup. Therefore I_J should be a finite set. So the conclusion (3) holds.

(4). By the conclusion (3), paper [2] and Definition 3.1 we can get it easily.

By the conclusion above, we can prove the following structure of the completely semisimple semigroup.

Lemma 3.2. For a semigroup S with zero elements, the following conclusions are equivalent.

(1) S is a semisimple semigroup;

(2) S is a 0-direct unions of a completely 0-simple semigroup ;

(3) S is a primitive regular semigroup .

Proof: According to Lemma 3.1 and paper [2], we need only to prove (2)=> (1) .

Let $S = \underset{e\in PE(S)}{\cup} SeS$ is a 0-direct unions of a completely 0-simple semigroup , we need only to prove S has no non-zero nilpotent ideal . If N is a non-zero nilpotent ideal of S, by the expression of 0- direct union of S, we know easily , there exsist $\Phi \neq \omega \subseteq PE(S)$, such that $N = \underset{e\in\omega}{\cup} SeS$. Let its nilptent index be t, then

$$N^t = (\underset{e\in\omega}{\cup} SeS)^t = \underset{e\in\omega}{\cup}(SeS)^t = \{0\} .$$

Since($\forall e \in \omega$) SeS is a 0-simple semigroup, so
for $\forall t \in N^+, 0 \neq e \in (SeS)^t \Rightarrow (SeS)^t \neq \{0\}$.
That is, the formular above doesn't hold. So S has no non-zero nilpotent ideal.

4. The structure of the nilpotent ideal extention semigroup

In this section, the structure of the nilpotent ideal extention semigroup is given using the‘nilpotent extention’ concept in basic theory of semigroup’s ideal extention (see [7], but not [2]) .

Lemma 4.1. [2] Let C be a semigroup with zero elements and N be a nilpotent semigroup with its nilpotent index be n, $C \cap N = \{0\}$. If θ is a homomorphic mapping of C to N, then

(1) $C^n \subseteq (0)\theta^{-1}$;

(2) If C is a regular semigroup, then $(0)\theta^{-1} = C$, i.e. θ is a zero homomorphic mapping of C to N .

Lemma 4.2. Let C be a semigroup with zero elements and N be a nilpotent semigroup with its nilpotent index be n, $C \cap N = \{0\}$. If θ is a homomorphic mapping of C to N, make $S = N \cup C$, define the operation '$\circ$' determined by (a) to (d)on S:

(a) $\forall x, y \in C^*, x \circ y = xy$ is in C;

(b) $\forall x \in C, n \in N, x \circ n = x\theta \cdot n$ is in N;

(c) $\forall x \in C^*, n \in N, n \circ x = n \cdot x\theta$ is in N;

(d) $\forall n, m \in M, n \circ m = nm$ is in N.

then S form a semigroup. By [2], S is called an extension semigroup from semigroup C to the ideal N.

Lemma 4.3. If C and N are all right (left) completely semigroup, then

(1) The extension semigroup S is a right (left) completely semigroup;

(2) If θ is a zero homomorphic mapping of C to N in lemma 4.2, then the extension semigroup S is a $0-$direct union semigroup, i.e.

$S = C \cup N, C \cap N = \{0\}, CN = NC = \{0\}$.

Proof: (1) The method to prove is same to property 2.1(3).We can get S is a right (left) completely semigroup.

(3) Since θ is a zero homomorphic mapping of C to N ,then we have the conclusion by the operation definite in lemma 4.2.

Theorem 4.1. Let S be an extension semigroup of the ideal N, with S=C$\cup N$, $S/\rho_N \cong C$, then the following statements are equivalent:

(1) S is an extension semigroup of completely regular semigroup C about the completely Rees nilpotent radical N.

(2) S is an extension semigroup of completely semisimple semigroup C about the completely Rees nilpotent radical N.

(3) S is a completely π- regular semigroup, with C=Reg(S)=$\cup_{e\in E} CeC, E$=$E(S)$;

(4) S is a 0-direct union of completely 0-simple semigroup $C_\alpha(\alpha \in \omega)$ and completely nilpotent semigroup. With $C = \cup_{\alpha\in\omega} C_\alpha$, S=$C \cup N, C \cap N = \{0\}$.

Proof: (1)<=>(2). By Proposition 2.1 and Lemma 3.2, we can get directly.

(1)=>(3). Acoording to C is a regular semigroup, N is nilpotent ideal of S, by Proposition 2.1(3) we have that S is a completely π- regular semigroup.

Since $E \not\subset N \Rightarrow \forall e \in E, CeC \subseteq C$,then $\bigcup_{e\in E} CeC \subseteq C$. Conversely, for $\forall a \in C$, there exist $e \in E$, such that $a = ea = eea \in CeC$. So $C \subseteq \bigcup_{e\in E} CeC$, C=$\bigcup_{e\in E} CeC, E$=$E(S)$.

Furthermore, by S=$C \cup N, C \cap N = \{0\}$, N is a nilpotent ideal of S, we have

C=Reg(S). i.e. the conclusion (3) holds**.**

(3)=>(1). Since S is a completely π- regular semigroup, $C = \text{Re}\,g(S) = \bigcup_{e\in E} CeC$, $N = S/C \cup \{0\}$, $S = C \cup N$, so N is the set of all non regular elements in S. Since S is a π- regular semigroup, thus N is the set of π-regular elements of all non-regular elements in S. By $NS \subseteq N$, $SN \subseteq N$, we have that for $\forall x \in N, \exists n \in N^+$, such that $x^n \in C \cap N = \{0\} \Rightarrow x^n = 0$. i.e. N is a nilpotent elements ideal.

According to the completeness of S and Lemma 1.5, we have that N is the completely nilpotent ideal of S, so the conclusion (1) holds.

(1)=>(4). By lemma 3.2, completely regular semigroup C is a 0-direct union semigroup of completely 0-simple semigroup C_α($\alpha \in \omega$), i.e.

$C = \bigcup_{\alpha\in\omega} C_\alpha$,(α , $\beta \in \omega$, $\alpha \neq \beta$) $C_\alpha C_\beta = \{0\}$.

Notice that C is a regular semigroup and N is a nilpotent semigroup and Lemma 4.1, the homomorphic mapping θ of C to N can only be a zero homomorphic. By lemma 4.3, S is a 0-direct union semigroup of C and N ,
i.e. $S = C \cup N$, $C \cap N = \{0\}$, $CN = NC = \{0\}$. So the conclusion (4) holds.

(4)=>(1) We need only to prove N is the Rees nilpotent ideal of the extension semigroup S. By Lemma 2.2, we need only to prove is the maximum nilpotent ideal of S. From the 0- direct union composition

$S = C \cup N$, $C \cap N = \{0\}$, $CN = NC = \{0\}$ of S , by(α , $\beta \in \omega$, $\alpha \neq \beta$) $C_\alpha C_\beta = \{0\}$ we know that C is a completely regular subsemigroup of S. N is a completely nilpotent ideal of S. If M is a completely ideal of S, such that $N \subset M$, then there exist $0 \neq x \in M \setminus N$, then $x \in C$. But x is a regular element of S, thus there exist $e \in E(R_x)$, such that $0 \neq x = ex = e \cdots ex$, i.e. M can not be the nilpotent ideal of S. Therefore N is the maximum nilpotent ideal of S. So the conclusion (1) holds.

REFERENCES

1. S.Bogdanovic and M.Ciric, Primitive π-regular semigroups, *Proc. Japan Acad.*, 68, Ser. A .1992.

2. A.H.Clifford and G.B.Preston, Algebraic Theory of Semigroups, American Mathematical Society, 1961.
3. Zhenlin Gao and Chengsu Xu, The expantion structure of cwrpp rees radical of semigroup, *Journal of University of Shanghai for Science and Technology*, 34(6) (2012) 604-608.
4. Zhenlin Gao and Tufang Song, The cwrpp radical of semigroup and semi -cwrpp semigroup, *Journal of University of Shanghai for Science and Technology*, 34(5) (2012) 499- 504.
5. Z.Gao and H.Li, Primary Semigroup with strong $C-$wrpp rees radical, *Journal of University of Shanghai for Science and Technology*, 34(6) (2012) 613 -618.
6. Huan Li and Zhenlin Gao, CRCE- semigroup, *Journal of University of Shanghai for Science and Technology*, 35(1) (2013) 97-102.
7. J.M.Howie, An Introduction to Semigroup Theory, Academic Press, 1976.
8. T.Vasanthi and M.Amala, Some special classes of semirings and ordered semirings, *Annals of Pure and Applied Mathematics*, 4(2) (2013) 145-159.
9. T.Vasanthi and N.Sulochana, On the additive and multiplicative structure of semirings, *Annals of Pure and Applied Mathematics*, 3(1) (2013) 78-84.