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Intuitionistic Fuzzy Multigroups

Shinoj T K[1] **and *Sunil Jacob John***[2]

[1]Department of Mathematics, National Institute of Technology, Calicut
Calicut-673 601, Kerala, India. E-mail: shinojthrissur@gmail.com

[2]Department of Mathematics, National Institute of Technology, Calicut
Calicut-673 601, Kerala, India. E-mail: sunil@nitc.ac.in

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Abstract. To study the various algebraic structures of intuitionistic fuzzy multisets, in this paper the concept of intuitionistic fuzzy multi groups are introduced and its various properties are discussed

Keywords : Multiset, fuzzy multiset, intuitionistic fuzzy multiset, multi groups, fuzzy multi groups, intuitionistic fuzzy multi group

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1. Introduction

The theory of sets introduced by George Cantor based on the notion of element membership to sets has proved itself to be one of the most powerful tools of modern mathematics. In classical set theory, a set is a well-defined collection of distinct objects. If repeated occurrences of any object are allowed in a set, then the mathematical structure is called as multiset. Thus, a multiset differs from a set in the sense that each element has a multiplicity. A complete account of the development of multiset theory can be seen in [1, 2, 3, 4].

Most of the real life situations are complex and for modeling them we need a simplification of the complex system. The simplification must be in such a way that the information lost should be minimum. One way to do this is to allow some degree of uncertainty into it. To handle situations like this, many tools were suggested. They include Fuzzy sets, Rough sets, Soft sets etc.

Considering the uncertainty factor, Lofti Zadeh [5] introduced Fuzzy sets in 1965, in which a membership function assigns to each element of the universe of discourse, a number from the unit interval [0,1] to indicate the degree of belongingness to the set under consideration. Fuzzy sets were introduced with a view to reconcile mathematical modeling and human knowledge in the engineering sciences. Since then, a considerable body of literature has blossomed around the concept of fuzzy sets in an incredibly wide range of areas, from mathematics and logics to traditional and advanced engineering methodologies. Owing to the fact that set theory is the corner stone of modern mathematics, a new and more general framework of mathematics was established.

In 1983, Atanassov [6,7] introduced the concept of Intuitionistic Fuzzy sets. The same time a theory called `Intuitionistic Fuzzy set theory' was independently introduced by Takeuti and Titani [8] as a theory developed in (a kind of) Intuitionistic logic. An Intuitionistic Fuzzy set is characterized by two functions expressing the degree of membership and the degree of nonmembership of elements of the universe to the Intuitionistic Fuzzy set. Among the various notions of higher-order Fuzzy sets, Intuitionistic Fuzzy sets proposed by Atanassov provide a flexible framework to explain uncertainty and vagueness. It is well-known that in the beginning of the last century L.Brouwer introduced the concept of Intuitionism. The name Intuitionistic Fuzzy set is due to George Gargove, with the motivation that their fuzzification denies the law of excluded middle-one of the main ideas of Intuitionism.

As a generalization of multiset, Yager [9] introduced fuzzy multisets and suggested possible applications to relational databases. An element of a Fuzzy Multiset can occur more than once with possibly the same or different membership values.

The concept of Intuitionistic Fuzzy Multiset is introduced in [10] which have applications in medical diagnosis and robotics.

In mathematics, Abstract algebra is the study of algebraic structures and more specifically the term algebraic structure generally refers to a set (called carrier set or underlying set) with one or more finitely operations defined on it. Examples of algebraic structures include groups, rings, fields, and lattices. We introduced algebraic structures on Fuzzy multisets in [11]. In this work we are extending these algebraic structures on Intuitionistic Fuzzy Multisets by introducing a new concept named **Intuitionistic Fuzzy Multigroups**.

2. Preliminaries

Definition 2.1. Let X be a set. A *multiset* (mset) M drawn from X is represented by a function Count M or C_M defined as $C_M : X \to \{0,1, 2, 3,\ldots\}$.
For each $x \in X$, $C_M(x)$ is the characteristic value of x in M. Here $C_M(x)$ denotes the number of occurrences of x in M.

Definition 2.2. ([12]) Let X be a group. A multi set G over X is a *multi group* over X if the count of G satisfies the following two conditions.

1. $C_G(xy) \geq C_G(x) \wedge C_G(y) \;\; \forall\, x, y \in X$;
2. $C_G(x^{-1}) \geq C_G(x) \;\; \forall\, x \in X$

Definition 2.3. If X is a collection of objects, then a *fuzzy set A* in X is a set of ordered pairs: $A = \{(x, \mu_A(x)) : x \in X, \mu_A : X \to [0,1]\}$ where μ_A is called the membership function of A, and is defined from X into [0, 1].

Definition 2.4. ([13]) Let G be a group and $\mu \in FP(G)$ (fuzzy power set of G), then μ is called *fuzzy subgroup* of G if

1. $\mu(xy) \geq \mu(x) \wedge \mu(y) \;\; \forall\, x, y \in G$ and
2. $\mu(x^{-1}) \geq \mu(x) \;\; \forall\, x \in G$.

Definition 2.5. ([10]) Let X be a nonempty set. An *Intuitionistic Fuzzy Multiset* A denoted by IFMS drawn from X is characterized by two functions : ‘count membership’ of A (CM_A) and 'count non membership' of A (CN_A) given respectively by CM_A: $X \to Q$ and CN_A: $X \to Q$ where Q is the set of all crisp multisets drawn from the unit interval [0, 1] such that for each $x \in X$, the membership sequence is defined as a decreasingly ordered sequence of elements in $CMA(x)$ which is denoted by $(\mu^1{}_A(x), \mu^2{}_A(x),...,\mu^P{}_A(x))$ where $(\mu^1{}_A(x) \geq \mu^2{}_A(x) \geq ,... \geq ,\mu^P{}_A(x)$ and the corresponding non membership sequence will be denoted by $(v^1{}_A(x), v^2{}_A(x),...,v^P{}_A(x))$ such that $0 \leq \mu^i{}_A(x) + v^i{}_A(x) \leq 1$ for every $x \in X$ and $i = 1,2,...,p$.

An IFMS A is denoted by
$A = \{< x : (\mu^1{}_A(x), \mu^2{}_A(x),...,\mu^P{}_A(x)), (v^1{}_A(x), v^2{}_A(x), ... ,v^P{}_A(x)) > : x \in X\}$

Remark: We arrange the membership sequence in decreasing order but the corresponding non membership sequence may not be in decreasing or increasing order.

Definition 2.6. ([10]) Length of an element x in an IFMS A is defined as the Cardinality of CM_A (x) or $CN_A(x)$ for which $0 \leq \mu^j{}_A(x) + v^j{}_A(x) \leq 1$ and it is denoted by L(x: A). That is

$$L(x : A) = |CM_A(x)| = |CN_A(x)|$$

Definition 2.7. ([10]) If A and B are IFMSs drawn from X then $L(x : A,B) = Max\{L(x : A), L(x : B)\}$. Alternatively we use L(x) for L(x :A, B).

Definition 2.8. ([10]) For any two IFMSs A and B drawn from a set X, the following operations and relations will hold. Let $A = \{< x : (\mu^1{}_A(x), \mu^2{}_A(x),...,\mu^P{}_A(x)), (v^1{}_A(x), v^2{}_A(x),...,v^P{}_A(x)) > : x \in X\}$ and $B = \{< x : (\mu^1{}_B(x), \mu^2{}_B(x),...,\mu^P{}_B(x)), (v^1{}_B(x), v^2{}_B(x),...,v^P{}_B(x)) > : x \in X\}$ then

1. Inclusion
   $A \subset B \Leftrightarrow \mu^j{}_A(x) \leq \mu^j{}_A(x)$ and $v^j{}_A(x) \geq v^j{}_B(x)$;
   $j = 1,2,..., L(x), x \in X$
   $A = B \Leftrightarrow A \subset B$ and $B \subset A$
2. Complement
   $\neg A = \{< x : (v^1{}_A(x),...,v^P{}_A(x)), (\mu^1{}_A(x),\ldots\ldots ...,\mu^P{}_A(x)) > : x \in X\}$
3. Union ($A \cup B$)
   In $A \cup B$ the membership and non membership values are obtained as follows.
   $\mu^j{}_{A\cup B}(x) = \mu^j{}_A(x) \vee \mu^j{}_B(x)$
   $v^j{}_{A\cup B}(x) = v^j{}_A(x) \wedge v^j{}_B(x)$
   $j = 1, 2,...,L(x), x \in X$.
4. Intersection ($A \cap B$)
   In $A \cap B$ the membership and non membership values are obtained as follows.
   $\mu^j{}_{A\cap B}(x) = \mu^j{}_A(x) \wedge \mu^j{}_B(x)$
   $v^j{}_{A\cap B}(x) = v^j{}_A(x) \vee v^j{}_B(x)$
   $j = 1, 2,...,L(x), x \in X$.

Definition 2.9. ([11]) Let X be a group. A fuzzy multiset G over X is a *fuzzy multi group* (FMG) over X if the count (count membership) of G satisfies the following two conditions.

1. $CM_G(xy) \geq CM_G(x) \wedge CM_G(y) \ \forall x, y \in X$.
2. $CM_G(x^{-1}) = CM_G(x) \ \forall x \in X$.

Theorem 2.10. ([11]) Let $A \in FM(X)$. Then $A \in FMG(X)$ iff $CM_A(xy^{-1}) \geq CM_A(x) \wedge CM_A(y) \ \forall x, y \in X$.

Definition 2.11. ([11]) Let $A \in FM(X)$. Then
$A[\alpha,n] = \{ x \in X: \mu^j_A(x) \geq \alpha ; L(x) \geq j \geq n \text{ and } j, n \in \mathbb{N}\}$. This is called *n-α level* set of A.

Definition 2.12. ([11]) Let $A \in FM(X)$. Then define $A^* = \{ x \in X: CM_A(x) = CM_A(e)\}$.

3. Intuitionistic fuzzy multigroups

Throughout this section, let *X* be a group with a binary operation and the identity element is *e*. Also we assume that the intuitionistic fuzzy multisets are taken from IFMS(X) and IFMG(*X*) denotes the set of all intuitionistic fuzzy multi groups (IFMG) over the group *X*.

Definition 3.1. Let $A \in IFMS(X)$. Then A^{-1} is defined as $CM_{A^{-1}}(x) = CM_A(x^{-1})$ and $CN_{A^{-1}}(x) = CN_A(x^{-1})$.

Definition 3.2. Let $A, B \in IFMS(X)$. Then define A o B as
$CM_{AoB}(x) = \vee \{ CM_A(y) \wedge CM_B(z) \ ; y, z \in X \text{ and } yz = x\}$.
$CN_{AoB}(x) = \wedge \{ CN(y) \vee CN(z) \ ; y, z \in X \text{ and } yz = x\}$

Proposition 3.3. Let $A, B, A_i \in IFMS(X)$, then the following results hold

a) $[A^{-1}]^{-1} = A$.

b) $A \subseteq B \Rightarrow A^{-1} \subseteq B^{-1}$.

c) $[\bigcup_{i=i}^{n} A_i]^{-1} = \bigcup_{i=i}^{n} [A_i^{-1}]$.

d) $[\bigcap_{i=i}^{n} A_i]^{-1} = \bigcap_{i=i}^{n} [A_i^{-1}]$.

e) $(A \text{ o } B)^{-1} = B^{-1} \text{o } A^{-1}$

f) $CM_{AoB}(x) = \vee_{y\in X} \{ CM_A(y) \wedge CM_B(y^{-1}x) \} \ \forall x \in X$
$\quad = \vee_{y\in X} \{ CM_A(xy^{-1}) \wedge CM_B(y) \} \ \forall x \in X$
$CN_{AoB}(x) = \wedge_{y\in X} \{ CN_A(y) \vee CN_B(y^{-1}x) \} \ \forall x \in X$
$\quad = \wedge_{y\in X} \{ CN_A(xy^{-1}) \vee CN_B(y) \} \ \forall x \in X$.

Proof:

a) $CM_{(A^{-1})^{-1}}(x) = CM_{(A^{-1})}(x^{-1})$
$= CM_A((x^{-1})^{-1})$
$= CM_A(x) \ \forall x \in X$. Since X is a group $((x^{-1})^{-1}) = x$
$CN_{(A^{-1})^{-1}}(x) = CN_{(A^{-1})}(x^{-1})$
$= CN_A((x^{-1})^{-1})$

$= \quad CN_A(x)\ \forall\ x \in X$. Since X is a group $((x^{-1})^{-1}) = x$

$\Rightarrow A \quad = \quad (A^{-1})^{-1}$.

b) Given $A \subseteq B$

$\Rightarrow CM_A(x^{-1}) \leq CM_B(x^{-1}) \quad \forall x \in X$

$CM_{(A^{-1})}(x) \leq CM_{(B^{-1})}(x)$

And $CN_A(x^{-1}) \geq CN_B(x^{-1}) \quad \forall x \in X$

$CN_{(A^{-1})}(x) \geq CN_{(B^{-1})}(x)$

$\Rightarrow A^{-1} \subseteq B^{-1}$

c) $CM_{(\cup_{i=1}^{n} A_i)^{-1}}(x) = CM_{(\cup_{i=1}^{n} A_i)}(x^{-1})$

$= \vee\{CM_{A_i}(x^{-1})\ ;\ i = 1, \dots, n\}$ by definition of union

$= \vee\{CM_{A_i^{-1}}(x)\ ;\ i = 1, \dots, n\}$

$= CM_{\cup_{i=1}^{n} A_i^{-1}}(x)$. by definition of union

$CN_{(\cup_{i=1}^{n} A_i)^{-1}}(x) = CN_{(\cup_{i=1}^{n} A_i)}(x^{-1})$

$= \wedge\{CN_{A_i}(x^{-1})\ ;\ i = 1, \dots, n\}$ by definition of union

$= \wedge\{CN_{A_i^{-1}}(x)\ ;\ i = 1, \dots, n\}$

$= CN_{\cup_{i=1}^{n} A_i^{-1}}(x)$. by definition of union.

$$\Rightarrow [\bigcup_{i=1}^{n} A_i]^{-1} = \bigcup_{i=1}^{n} (A_i^{-1}).$$

d) $CM_{(\cap_{i=1}^{n} A_i)^{-1}}(x) = CM_{(\cap_{i=1}^{n} A_i)}(x^{-1})$

$= \wedge\{CM_{A_i}(x^{-1})\ ;\ i = 1, \dots, n\}$

$= \wedge\{CM_{A_i^{-1}}(x)\ ;\ i = 1, \dots, n\}$

$= CM_{\cap_{i=1}^{n} A_i^{-1}}(x)$. by definition of intersection.

$$\Rightarrow [\bigcap_{i=1}^{n} A_i]^{-1} = \bigcap_{i=1}^{n} (A_i^{-1})$$

$CN_{(\cap_{i=1}^{n} A_i)^{-1}}(x) = CN_{(\cap_{i=1}^{n} A_i)}(x^{-1})$

$= \vee\{CN_{A_i}(x^{-1})\ ;\ i = 1, \dots, n\}$

$= \vee\{CN_{A_i^{-1}}(x)\ ;\ i = 1, \dots, n\}$

$= CN_{\cap_{i=1}^{n} A_i^{-1}}(x)$. by definition of intersection.

$$\Rightarrow [\bigcap_{i=1}^{n} A_i]^{-1} = \bigcap_{i=1}^{n} (A_i^{-1})$$

e) $CM_{(A\ o\ B)}{}^{-1}(x) = CM_{(A\ o\ B)}(x^{-1}) = \vee\{CM_A(y)\ \ CM_B(z)\ ;\ y, z \in X \text{ and } yz = x^{-1}\}$

$= \vee\{CM_B(z) \wedge CM_A(y)\ ;\ y, z \in X \text{ and } (yz)^{-1} = x\}$

$= \vee\{CM_B(z^{-1})^{-1} \wedge CM_A(y^{-1})^{-1}\ ;\ y, z \in X \text{ and } (yz)^{-1} = x\}$

$= \vee \{CM_{B^{-1}}(z^{-1}) \wedge CM_{A^{-1}}(y^{-1})\ ;\ y^{-1},\ z^{-1} \in X \text{ and } z^{-1}y^{-1} = x\}$

$= CM_{B^{-1} o A^{-1}}(x)\ \forall x \in X$ (1)

And $CN_{(A o B)^{-1}}(x) = CN_{(A o B)}(x^{-1})$

$= \wedge \{CN_A(y) \vee CN_B(z)\ ;\ y, z \in X \text{ and } yz = x^{-1}\}$

$= \wedge \{CN_B(z) \vee CN_A(y)\ ;\ y, z \in X \text{ and } (yz)^{-1} = x\}$

$= \wedge \{CN_B(z^{-1})^{-1} \vee CN_A(y^{-1})^{-1}\ ;\ y, z \in X \text{ and } (yz)^{-1} = x\}$

$= \wedge \{CN_{B^{-1}}(z^{-1}) \vee CN_{A^{-1}}(y^{-1})\ ;$
$y^{-1},\ z^{-1} \in X \text{ and } z^{-1}y^{-1} = x\}$

$= CN_{B^{-1} o A^{-1}}(x)\ \forall x \in X$ (2)

From (1) and (2) $(A o B)^{-1} = B^{-1} o A^{-1}$

f) Since X is a group, it follows that for each x, y ∈ X, ∃ a unique $z(=y^{-1}x) \in X$, such that yz = x. Then $CM_{AoB}(x) = \vee_{y\in X}\{CM_A(y) \wedge CM_B(y^{-1}x)\}\ \forall x \in X$ and

$CN_{AoB}(x) = \wedge_{y\in X}\{CN_A(y) \vee CN_B(y^{-1}x)\}\ \forall x \in X$

Also $CM_{AoB}(x) = \vee\{CM_B(z) \wedge CM_A(y)\ ;\ y, z \in X \text{ and } yz = x\}$ and

$CN_{AoB}(x) = \wedge\{CN_B(z) \vee CN_A(y)\ ;\ y, z \in X \text{ and } yz = x\}$.

Since X is a group, it follows that for each x, y ∈ X, ∃ a unique $z(=xy^{-1}) \in X$, such that zy= x. Then $CM_{AoB}(x) = \vee_{y\in X}\{CM_B(xy^{-1}) \wedge CM_A(y)\}\ \forall x \in X$ and

$CN_{AoB}(x) = \wedge_{y\in X}\{CN_B(xy^{-1}) \vee CN_A(y)\}\ \forall x \in X$.

Definition 3.4. Let *X* be a group. An Intuitionistic fuzzy multiset *G* over *X* is an *Intuitionistic fuzzy multi group* (IFMG) over *X* if the counts (count membership and non membership) of *G* satisfies the following two conditions.

1. $CM_G(xy) \geq CM_G(x) \wedge CM_G(y)\ \forall x, y \in X$.
2. $CM_G(x^{-1}) \geq CM_G(x)\ \forall x \in X$.
3. $CN_G(xy) \leq CN_G(x) \wedge CN_G(y)\ \forall x, y \in X$.
4. $CN_G(x^{-1}) \leq CN_G(x)\ \forall x \in X$.

Example 3.5. $(Z_4, +_4)$is a group. Then
A = {<0: (0.9, 0.8, 0.7, 0.5, 0.1, 0.1), (0.1, 0.2, 0.3, 0.5, 0.9, 0.9)>, <1: (0.6, 0.4 0.3, 0.1), (0.4, 0.6, 0.7, 0.9)>, <2:(0.8, 0.7, 0.7, 0.5, 0.1, 0.1), (0.2, 0.3, 0.3, 0.5, 0.9, 0.9)>, <3: (0.6, 0.4, 0.3, 0.1), (0.4, 0.6, 0.7, 0.9)>} is an Intuitionistic fuzzy multi group.
But B = {<0: (0.9, 0.8, 0.7, 0.5, 0.1, 0.1), (0.1, 0.2, 0.3, 0.5, 0.9, 0.9)>, <1: (0.9, 0.7, 0.7, 0.5, 0.1, 0.1), (0.1, 0.3, 0.3, 0.5, 0.9, 0.9)>, <2:(0.6, 0.4, 0.3, 0.1), (0.4, 0.6, 0.7, 0.9)>, <3: (0.8, 0.7, 0.7, 0.5, 0.1, 0.1), (0.2, 0.3, 0.3, 0.5, 0.9, 0.9)>} is not an IFMG. Because $CM_B(1^{-1})$ is not greater than or equal to $CM_B(1)$.

From the definition and above example it is clear that IFMG is a generalized case of FMG.

Proposition 3.6. Let A ∈ IFMS(X) and $CM_A(x^{-1}) \geq CM_A(x)$ and $CN_A(x^{-1}) \leq CN_A(x)$.
Then $L(x : A) = L(x^{-1} : A)$

Proof:

$CM_A(x^{-1}) \geq CM_A(x)$ (given)

Now $CM_A(x) = CM_A((x^{-1})^{-1}) \geq CM_A(x^{-1})$ Then

$CM_A(x) \quad = \quad CM_A(x^{-1})$.
$CN_A(x^{-1}) \quad \leq \quad CN_A(x)$ (given)
Now $CN_A(x) = CN_A((x^{-1})^{-1}) \leq CN_A(x^{-1})$ Then
$CN_A(x) \quad = \quad CN_A(x^{-1})$.
Now $L(x;A) = |CM_A(x)| = |CN_A(x)|$ (By definition)
Hence $L(x;A) = |CM_A(x^{-1})| = |CN_A(x^{-1})|$
$= L(x^{-1}: A)$

Proposition 3.7. Let $A \in$ IFMG (X). Then

a) $CM_A(e) \geq CM_A(x) \quad \forall x \in X$
b) $CN_A(e) \leq CN_A(x) \quad \forall x \in X$
c) $CM_A(x^n) \geq CM_A(x) \quad \forall x \in X$
d) $CN_A(x^n) \leq CN_A(x) \quad \forall x \in X$
e) $A^{-1} \supseteq A$

Proof: Let $x, y \in X$.

a) $CM_A(e) = CM_A(xx^{-1})$
$\geq CM_A(x) \wedge CM_A(x^{-1})$
$\geq CM_A(x) \wedge CM_A(x)$
$= CM_A(x)$.

b) $CN_A(e) = CN_A(xx^{-1})$
$\leq CN_A(x) \vee CN_A(x^{-1})$
$\leq CN_A(x) \vee CN_A(x)$
$= CN_A(x)$.

c) $CM_A(x^n) \geq CM_A(x^{n-1}) \wedge CM_A(x)$
$\geq CM_A(x) \wedge CM_A(x) \wedge \ldots \wedge CM_A(x)$ by recursion
$= CM_A(x)$.

d) $CN_A(x^n) \leq CN_A(x^{n-1}) \vee CN_A(x)$
$\leq CN_A(x) \vee CN_A(x) \vee \ldots \vee CN_A(x)$ by recursion
$= CN_A(x)$.

e) $CM_{A^{-1}}(x) = CM_A(x^{-1})$
$\geq CM_A(x)$
$CN_{A^{-1}}(x) = CN_A(x^{-1})$
$\leq CN_A(x)$
$\Rightarrow A^{-1} \supseteq A$

Theorem 3.8. Let $A \in$ IFMS(X). Then $A \in$ IFMG(X) iff $CM_A(xy^{-1}) \geq CM_A(x) \wedge CM_A(y)$ and $CN_A(xy^{-1}) \leq CN_A(x) \vee CN_A(y) \ \forall x, y \in X$.

Proof: Let $A \in$ IFMG(X)

Then $CM_A(xy^{-1}) \geq CM_A(x) \wedge CM_A(y^{-1})$
$\geq CM_A(x) \wedge CM_A(y) \ \forall x, y \in X$
Then $CN_A(xy^{-1}) \leq CN_A(x) \vee CN_A(y^{-1})$
$\leq CN_A(x) \vee CN_A(y) \ \forall x, y \in X$

Conversely, let the given condition be satisfied.
Also $CM_A(x^{-1}) = CM_A(ex^{-1})$

$\geq$ $CM_A(e) \wedge CM_A(x)$
$=$ $CM_A(x)$

Now $CM_A(xy) \geq CM_A(x) \wedge CM_A(y^{-1})$
$= CM_A(x) \wedge CM_A(y)$

And $CN_A(x^{-1}) = CN_A(ex^{-1})$
$\leq CN_A(e) \vee CN_A(x)$
$= CN_A(x)$

Now $CN_A(xy) \leq CN_A(x) \vee CN_A(y^{-1})$
$= CN_A(x) \vee CN_A(y)$

Hence the proof.

Definition 3.9. Let $A \in IFMS(X)$. Then $A[\alpha, \beta, n] = \{ x \in X: \mu^j_A(x) \geq \alpha, v^j_A(x) \leq \beta ; L(x) \geq j \geq n$ and $j, n \in \mathbb{N}\}$. This is called *(n,α,β) level set* of A.

Proposition 3.10. Let $A \in IFMG(X)$. Then $A[\alpha,\beta, n]$ are subgroups of X.

Proof:

Let $x, y \in A[\alpha, \beta, n]$.

It implies that for $j \geq n$ $\mu^j_A(x) \geq \alpha$ and $\mu^j_A(y) \geq \alpha$; $v^j_A(x) \leq \beta$ and $v^j_A(y) \leq \beta$

Then $\mu^j_A(xy^{-1}) \geq \alpha$ and $v^j_A(xy^{-1}) \leq \beta$

This $\Rightarrow$ if $x, y \in A[\alpha,\beta, n]$ then $xy^{-1} \in A[\alpha,\beta, n]$.

Hence $A[\alpha,\beta, n]$ is a subgroup of *X*.

Definition 3.11. Let $A \in IFMS(X)$. Then define

$A^* = \{ x \in X: CM_A(x) = CM_A(e)$ and $CN_A(x) = CN_A(e) \}$.

Proposition 3.12. Let $A \in IFMG(X)$. Then A^* is a subgroup of X.

Proof:

Let $x, y \in A^*$

Then $CM_A(x) = CM_A(y) = CM_A(e)$ (1)

and $CN_A(x) = CN_A(y) = CN_A(e)$ (2)

Then $CM_A(xy^{-1}) \geq CM_A(x) \wedge CM_A(y)$
$= CM_A(e) \wedge CM_A(e)$ by (1)
$= CM_A(e)$

But $CM_A(xy^{-1}) \leq CM_A(e)$

i.e. $CM_A(xy^{-1}) = CM_A(e)$

Then $CN_A(xy^{-1}) \leq CN_A(x) \vee CN_A(y)$
$= CN_A(e) \vee CN_A(e)$ by (2)
$= CN_A(e)$

But $CN_A(xy^{-1}) \leq CN_A(e)$

i.e. $CN_A(xy^{-1}) = CN_A(e)$

$\Rightarrow xy^{-1} \in A^*$. Hence A^* is a subgroup of X.

Definition 3.13. Let $A \in IFMS(X)$. Let $j \in \mathbb{N}$. Then define

$A^j = \{x \in X: \mu_A^j(x) > 0,\ \mu_A^{j+1}(x) = 0 \text{ and } \nu_A^j(x) = 0\}$.

Proposition 3.14. Let $A \in IFMG(X)$. Then A^j is a subgroup of X iff $\mu_A^{j+1}(xy^{-1}) = 0$ and $\nu_A^{j+1}(xy^{-1}) = 0\ \forall\, x, y \in A^j$

Proof:

Let $x, y \in A^j$. It implies that

$\mu_A^j(x) > 0$, $\mu_A^j(y) > 0$ and $\mu_A^{j+1}(x) = 0$, $\mu_A^{j+1}(y) = 0$

$\nu_A^j(x) = 0$, $\nu_A^j(y) = 0$ and $\nu_A^{j+1}(x) = 0$, $\nu_A^{j+1}(y) = 0$

Assume $\mu_A^{j+1}(xy^{-1}) = 0$ and $\nu_A^{j+1}(xy^{-1}) = 0\ \forall\, x, y \in A^j$

Then by the above theorem,

$\mu_A^j(xy^{-1}) > 0$ and $\nu_A^j(xy^{-1}) = 0$

$\Rightarrow xy^{-1} \in A^j$. Then A^j is a subgroup of X. Hence the proof.

Conversely,

A^j is a subgroup of X. Then $x, y \in A^j \Longrightarrow xy^{-1} \in A^j$.

$\Rightarrow \mu_A^{j+1}(xy^{-1}) = 0$ and $\nu_A^{j+1}(xy^{-1}) = 0$ Hence the proof.

Theorem 3.15. Let $A \in IFMS(X)$. Then $A \in IFMG(X)$ iff $A \circ A \subseteq A$ and $A \subseteq A^{-1}$.

Proof: Let $A \in IFMG(X)$ and $x, y, z \in X$.

$\Rightarrow CM_A(xy) \geq CM_A(x) \wedge CM_A(y)$

$\Rightarrow CM_A(z) \geq \vee\{CM_A(x) \wedge CM_A(y)\ ;\ xy = z\}$

$= CM_{A \circ A}(z)$

And $CN_A(xy) \leq CN_A(x) \vee CN_A(y)$

$\Rightarrow CN_A(z) \leq \wedge\{CN_A(x) \vee CN_A(y)\ ;\ xy = z\}$

$= CN_{A \circ A}(z)$

$\Rightarrow A \circ A \subseteq A$. Now by (3.7) (e) we get the 2nd condition.

Conversely,

Assume $A \circ A \subseteq A$ (1)

and $A \subseteq A^{-1}$

$\Rightarrow CM_A^{-1}(x) \geq CM_A(x)$

But $CM_A^{-1}(x) = CM_A(x^{-1})$

$\Rightarrow CM_A(x^{-1}) \geq CM_A(x)$ (2)

Also $CN_A^{-1}(x) \leq CN_A(x)$

But $CN_A^{-1}(x) = CN_A(x^{-1})$

$\Rightarrow CN_A(x^{-1}) \leq CN_A(x)$ (3)

Since $A \in IFMS(X)$, then to prove $A \in IFMG(X)$ it is enough to prove that $CM_A(xy^{-1}) \geq CM_A(x) \wedge CM_A(y)$ and $CN_A(xy^{-1}) \leq CN_A(x) \vee CN_A(y)\ \forall\, x, y \in X$

Now $CM_A(xy^{-1}) \geq CM_{A \circ A}(xy^{-1})$ by (1)

$= \vee_{z \in X}\{CM_A(z) \,\square\, CM_A(z^{-1}xy^{-1})\}$ by (3.3(f))

$\geq \{CM_A(x) \wedge CM_A(y^{-1})\}$; $z = x$

$\geq CM_A(x) \wedge CM_A(y)$ by (2)

And $CN_A(xy^{-1}) \leq CN_{A \circ A}(xy^{-1})$ by (1)

$= \wedge_{z\in X}\{CN_A(z) \vee CN_A(z^{-1}xy^{-1})\}$ by (3.3(f))

$\leq \{CN_A(x) \vee CN_A(y^{-1})\}$; $z = x$

$\leq CN_A(x) \vee CN_A(y)$ by (3)

Hence the proof.

Corollary 3.16. Let A ∈ IFMS(X). Then A ∈ IFMG(X) iff A o A = A and A ⍰ A^{-1}.

Proof: Let $A \in$ IFMG(X). Then

$$CM_{AoA}(x) = \vee\{ CM_A(y) \wedge CM_A(z) ;\ y,z \in X \text{ and } yz = x \}$$
$$\geq \{ CM_A(e) \wedge CM_A(e^{-1}x) \}$$
$$= CM_A(x)$$
$$CN_{AoA}(x) = \wedge\{ CN_A(y) \vee CN_A(z) ;\ y,z \in X \text{ and } yz = x \}$$
$$\leq \{ CN_A(e) \wedge CN_A(e^{-1}x) \}$$
$$= CN_A(x)$$

So $A \subseteq A$ o A.

Hence by the above theorem the proof is complete.

Proposition 3.17. Let A∈ IFMS(X). Then A ∈ IFMG(X) if A o $A^{-1} \subseteq$ A.

Proof:

Assume A o $A^{-1} \subseteq$ A (1)

Since A ∈ IFMS (X), then to prove A ∈ IFMG (X) it is enough to prove that

$CM_A(xy^{-1}) \geq CM_A(x) \wedge CM_A(y)\ \forall x, y \in X$

$CN_A(xy^{-1}) \leq CN_A(x) \vee CN_A(y)\ \forall x, y \in X$

$$CM_A(xy^{-1}) \geq CM_{AoA^{-1}}(xy^{-1}) \quad \text{by (1)}$$
$$= \vee_{z\in X}\{CM_A(z) \wedge CM_{A^{-1}}(z^{-1}xy^{-1})\}$$
$$\geq \{CM_A(x) \wedge CM_{A^{-1}}(y^{-1})\};\ z = x$$
$$= CM_A(x) \wedge CM_A(y).$$
$$CN_A(xy^{-1}) \leq CN_{AoA^{-1}}(xy^{-1}) \quad \text{by (1)}$$
$$= \wedge_{z\in X}\{CN_A(z) \vee CN_{A^{-1}}(z^{-1}xy^{-1})\}$$
$$\leq \{CN_A(x) \vee CN_{A^{-1}}(y^{-1})\};\ z = x$$
$$= CN_A(x) \vee CN_A(y).$$

Hence the proof.

Theorem 3.18. Let A, B ∈ IFMG(X). Then A∩B ∈ IFMG(X).

Proof: Let x, y ∈ A∩B ∈ IFMS(X)

⇒ x, y ∈ A and x, y ∈ B

⇒ $CM_A(xy^{-1}) \geq CM_A(x) \wedge CM_A(y^{-1})$, $CM_B(xy^{-1}) \geq CM_B(x) \wedge CM_B(y^{-1})$ and

$CN_A(xy^{-1}) \leq CN_A(x) \vee CN_A(y^{-1})$, $CN_B(xy^{-1}) \leq CN_B(x) \vee CN_B(y^{-1})$

Now $CM_{A\cap B}(xy^{-1}) = CM_A(xy^{-1}) \wedge CM_B(xy^{-1})$ by definition of intersection

$\geq [CM_A(x) \wedge CM_A(y^{-1})] \wedge [CM_B(x) \wedge CM_B(y^{-1})]$

$= [CM_A(x) \wedge CM_B(x)] \wedge [CM_A(y^{-1}) \wedge CM_B(y^{-1})]$

(by commutative property of minimum)

$\geq [CM_A(x) \wedge CM_B(x)] \wedge [CM_A(y) \wedge CM_B(y)]$ Since A, B∈ IFMG(X)

$= CM_{A\cap B}(x) \wedge CM_{A\cap B}(y)$ by definition of intersection

⇒ $CM_{A\cap B}(xy^{-1}) \geq CM_{A\cap B}(x) \wedge CM_{A\cap B}(y)$ (1)

And $CN_{A\cap B}(xy^{-1}) = CN_A(xy^{-1}) \vee CN_B(xy^{-1})$ by definition of intersection

$\leq$ $[CN_A(x) \vee CN_A(y^{-1})] \vee [CN_B(x) \vee CM_B(y^{-1})]$

$=$ $[CN_A(x) \vee CN_B(x)] \vee [CN_A(y^{-1}) \vee CN_B(y^{-1})]$

(by commutative property of maximum)

$\leq$ $[CN_A(x) \vee CN_B(x)] \vee [CN_A(y) \vee CN_B(y)]$ Since A, B$\in$ IFMG(X)

$=$ $CN_{A\cap B}(x) \vee CN_{A\cap B}(y)$ by definition of intersection

$\Rightarrow$ $CN_{A\cap B}(xy^{-1}) \leq CN_{A\cap B}(x) \wedge CN_{A\cap B}(y)$ (2)

From (1) and (2) A$\cap$B $\in$ IFMG(X). Hence the proof.

Remark 3.19. If $\{A_i\ ;\ i \in I\}$ is a family of IFMG over *X*, then their intersection $\bigcap_{i\in I} A_i$ is also a IFMG over *X*.

Proposition 3.20. Let A, B $\in$ IFMG(X). Then $CM_{A\cup B}(x) \leq CM_{A\cup B}(x^{-1})$, $CN_{A\cup B}(x) \geq CN_{A\cup B}(x^{-1})$.

Proof:

$CM_{A\cup B}(x^{-1}) = \vee\ \{CM_A(x^{-1}),\ CM_B(x^{-1})\}$

$\geq \vee \{CM_A(x),\ CM_B(x)\}$ Since A, B$\in$ IFMG(X)

$= CM_{A\cup B}(x)$.

$CN_{A\cup B}(x^{-1}) = \wedge\ \{CN_A(x^{-1}),\ CN_B(x^{-1})\}$

$\leq \wedge \{CN_A(x),\ CN_B(x)\}$ Since A, B$\in$ IFMG(X)

$= CN_{A\cup B}(x)$.

Hence the proof.

From this it is clear that, if A, B $\in$ IFMG(X) then A $\cup$ B $\in$ IFMG(X) iff $CM_{A\cup B}(xy) \geq CM_{A\cup B}(x) \wedge CM_{A\cup B}(y)$ and $CN_{A\cup B}(xy) \leq CN_{A\cup B}(x) \vee CN_{A\cup B}(y)$

Corollary 3.21. Let A, B $\in$ IFMG(X). Then A$\cup$B need not be an element of IFMG(X).

Proof:

X = {a, b, c, e} is Klein's 4 group. Then A = {<a: (0.6, 0.4, 0.3, 0.1),(0.3, 0.4, 0.7, 0.5)>, <e: (0.9, 0.8, 0.7, 0.5, 0.1, 0.1), (0.0, 0.1, 0.2, 0.4, 0.5, 0.7)} and B = {<b: (0.8, 0.8, 0.5, 0.5, 0.1, 0.1), (0.1, 0.0, 0.4, 0.3, 0.7, 0.3), <e: (0.9, 0.8, 0.7, 0.5, 0.1, 0.1),(0.1, 0.1, 0.2, 0.5, 0.5, 0.6)>} are intuitionistic fuzzy multi groups. $A\cup B$ = {<a: (0.6, 0.4, 0.3, 0.1), (0.0, 0.0, 0.0, 0.0)>, <b: (0.8, 0.8, 0.5, 0.5, 0.1, 0.1), (0.0, 0.0, 0.0, 0.0, 0.0, 0.0) >, <e: (0.9, 0.8, 0.7, 0.5, 0.1, 0.1), (0.0, 0.1, 0.2, 0.4, 0.5, 0.6)>}. But $CM_{A\cup B}(c) < CM_{A\cup B}(a) \wedge CM_{A\cup B}(b)$ as ab = c in Klein's 4 group. Then A$\cup$B $\notin$ IFMG(X).

Proposition 3.22. Let A $\in$ IFMS(X) and A o $A^{-1} \subseteq$ A. Then A $\in$ IFMG(X). Also $CM_A(xy^{-1}) \geq CM_{AoA}(xy)$ and $CN_A(xy^{-1}) \leq CN_{AoA}(xy)$.

Proof: First part follows by 3.17.

Now $CM_A(xy^{-1}) \geq CM_{AoA^{-1}}(xy^{-1})$ (given)

$= \bigvee_{z\in X}\{CM_A(z) \wedge CM_{A^{-1}}(z^{-1}xy^{-1})\}$ by (3.3(f))

$\geq \{CM_A(x) \wedge CM_{A^{-1}}(y^{-1})\}$ (when z = x)

$= \{CM_A(x) \wedge CM_A(y)\}$

$= CM_{AoA}(xy)$.

And $CN_A(xy^{-1}) \leq CN_{AoA^{-1}}(xy^{-1})$ (given)

$= \wedge_{z\in X}\{CN_A(z) \vee CN_{A^{-1}}(z^{-1}xy^{-1})\}$ by 3.3(f)

$\leq \{CN_A(x) \vee CN_{A^{-1}}(y^{-1})\}$ (when z = x)

$= \{ CN_A(x) \vee CN_A(y) \}$
$= CN_{AoA}(xy).$

Definition 3.23. Let A, B ∈ IFMG(X). Then A is said to be a *sub-intuitionistic fuzzy multi group* of *B* if $A \subseteq B$.

Example 3.24. (Z_4, $+_4$) is a group. Then A = {<2: (0.6, 0.4, 0.3, 0.1), (0.4, 0.6, 0.7, 0.9)>, <1: (0.8, 0.7, 0.7, 0.5, 0.1, 0.1),(0.2, 0.3, 0.3, 0.5, 0.9, 0.9)>, <3: (0.8, 0.7, 0.7, 0.5, 0.1, 0.1), (0.2, 0.3, 0.3, 0.5, 0.9, 0.9)>, <0: (0.9, 0.8, 0.7, 0.5, 0.1, 0.1), (0.1, 0.8, 0.3, 0.5, 0.9, 0.9)>} is a fuzzy multi group. And B = {<2: (0.6, 0.4, 0.3, 0.1), (0.4, 0.6, 0.7, 0.9)>, <1: (0.7, 0.6, 0.5, 0.5, 0.1, 0.1), (0.3, 0.4, 0.5, 0.5, 0.9,0.9)>, <3: (0.7, 0.6, 0.5, 0.5, 0.1, 0.1), (0.3, 0.4, 0.5, 0.5, 0.9, 0.9)>, <0: (0.9, 0.8, 0.7, 0.5, 0.1, 0.1), (0.1, 0.2, 0.3, 0.5,0.9, 0.9)>} is a sub-fuzzy multi group of *A*.

Definition 3.25. Let A ∈ IFMS(X). Then <A> = {□ A_i : $A \subseteq A_i \in$ IFMG(X) }is called the *intuitionistic fuzzy muli subgroup* of X generated by A.

Remark 3.26. <A> is the smallest intuitionistic fuzzy multi subgroup of X that contains A.

Proposition 3.27. *If A* ∈ IFMG(X), and H is a subgroup of X, then $A|_H$ (i.e. A restricted to H) ∈ IFMG(H) and is an intuitionistic fuzzy multi subgroup of A.

Proof:

Let $x, y \in H$. Then $xy^{-1} \in H$. Now

$CM_{A|_H}(xy^{-1}) = CM_A(xy^{-1}) \geq CM_A(x) \wedge CM_A(y) = CM_{A|_H}(x) \wedge CM_{A|_H}(y)$

$CN_{A|_H}(xy^{-1}) = CN_A(xy^{-1}) \leq CN(x) \wedge CN_A(y) = CN_{A|_H}(x) \wedge CN_{A|_H}(y)$

The second part is trivial.

4. **Conclusion**

In this paper, the algebraic structure of intuitionistic fuzzy multiset is introduced as intuitionistic fuzzy multigroup. Intuitionistic fuzzy multigroup is a generalized case of fuzzy multigroup. The various basic operations, definitions and theorems related to intuitionistic fuzzy multigroup have been discussed. The foundations which we made through this paper can be used to get an insight into the higher order structures of group theory.

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