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Transitive Compatible Pair Congruence

Pei Yang[1], ***Zhenlin Gao***[2] ***Hai-zhen Liu***[3] **and** ***Jing Zhang***[4]

[1]College of Science, University of Shanghai for Science and Technology
Shanghai-200093, China. Email: 742155149@qq.com

[2]College of Science, University of Shanghai for Science and Technology
Shanghai-200093, China. Email: zlgao@sina.com

[3]College of Science, University of Shanghai for Science and Technology
Shanghai-200093, China. Email: 592663972@qq.com

[4]College of Science, Dalian University of Technology
Dalian-116024, China. Email: 731689657@qq.com
Corresponding author: Pei Yang

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Abstract. In this paper, we introduce the concepts of transitive compatible pair congruence $\rho_{(H,K)}$. Prove that any congruence on a semigroup S can be expressed as a transitive compatible pair congruence.

Keywords: Semigroup ; Transitive compatible pair; congruence

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1. Introduction and preliminaries

The Green's star relations $\mathscr{L}^*$, $\mathscr{R}^*$, $\mathscr{D}^*$, $\mathscr{H}^*$, $\mathscr{J}^*$ on semigroup S are defined as follows.

$$(a,b)\in \mathscr{L}^* \Leftrightarrow (\forall x, y\in S^1)\ ax = ay \Leftrightarrow bx = by;$$

$$(a,b)\in \mathscr{R}^* \Leftrightarrow (\forall x, y\in S^1)\ xa = ya \Leftrightarrow xb = yb;$$

$$\mathscr{D}^* = \mathscr{L}^* \vee \mathscr{R}^*,\quad \mathscr{H}^* = \mathscr{L}^* \wedge \mathscr{R}^*,\quad a\ \mathscr{J}^*\ b \Leftrightarrow J^*(a) = J^*(b).$$

where $J^*(x)(x\in S)$ represents the minimal $\mathscr{J}^*$-ideal generated by x.

Definition 1.1. Assume that P represents a kind property about semigroups, if the semigroup S has P property, then S is called a P semigroup. A congruence ρ on the semigroup S is called P congruence, if S/ρ is a P semigroup.

We shall call a semigroup S a rpp semigroup, if every $\mathscr{L}^*$-class of S contains idempotents of S and for any $a\in S$, $e\in E(L_a^*)$, then $a = ae$. A rpp semigroup S is called adequate, if for any $a\in S$, there exists a unique $a^+ \in E(L_a^*)$ such that $a = a^+a$.

The following basic conclusions are needed throughout this paper.

Lemma 1.1. Let S be a semigroup, for any $a \in S$, $e \in E(S)$ the following statements are equivalent:

(1) $a \mathscr{L}^* e$ ($a \mathscr{R}^* e$) ;

(2) $ae = a$ $(ea = a)$, and for any $s, t \in S^1$, $as = at(sa = ta)$ implies $es = et(se = te)$.

Lemma 1.2. (1) L and L^* both are right congruences on semigroup S;

(2) $R \subseteq R^*$, $L \subseteq L^*$, $D \subseteq D^*, H \subseteq H^*, J \subseteq J^*$;

(3) $R^* |_{\mathrm{Re}g(S)} = R, L^* |_{\mathrm{Re}g(S)} = L$.

Lemma 1.3. (1) Band B has a semilattice decomposition $B = \bigcup_{\alpha \in Y} J_\alpha$ which rest with the semilattice Y, where J_α $(\forall \alpha \in Y)$ are J -classes of B.

(2) Left regular band I is a semilattice of left zero bands L_α, where L_α are L -classes of I.

2. Transitive compatible pair congruences

From this section, we always assume that semigroup S mentioned below contains zero element, unless otherwise stated.

Definition 2.1. Let H and K be two subsemigroups of semigroup S ,such that $S = H \cup K$, ρ^H and ρ^K be two relations on H and K respectively. We define a relation on S by ρ^H and ρ^K as follows:

$$\rho_{(H,K)} = \rho^H \cup \rho^K = \{ (a,b) \in S \times S \mid (a,b) \in \rho^H \text{ or } (a,b) \in \rho^K \}. \tag{2.1}$$

If ρ^H and ρ^K satisfy the following conditions:

$$(a,b) \in \rho^H,\ (b,c) \in \rho^K \Rightarrow (a,c) \in \rho_{(H,K)}\ ; \tag{2.2}$$

$$(\forall x \in S)(a,b) \in \rho^H\ (resp, (a,b) \in \rho^K)$$

$$\Rightarrow \begin{cases} (xa, xb) \in \rho^H \text{ or } (xa, xb) \in \rho^K, \\ or\ (\exists\ c \in H \cap K)(xa, c) \in \rho^H, (c, xb) \in \rho^K; \end{cases} \tag{2.3}$$

$$\begin{cases} (ax, bx) \in \rho^H \text{ or } (ax, bx) \in \rho^K, \\ or\ (\exists\ c \in H \cap K)(ax, c) \in \rho^H, (c, bx) \in \rho^K, \end{cases} \tag{2.4}$$

then, (H,K) is called a *transitive compatible pair of* $\rho_{(H,K)}$.

Theorem 2.1. Let (H,K) be a transitive compatible pair of $\rho_{(H,K)}$ defined as Definition 2.1. Then the following conclusions hold :

(1) $\rho_{(H,K)}$ is an equivalent relation on S if and only if ρ^H and ρ^K are equivalent relations on H and K respectively. And we also have

$$\rho^H = \rho_{(H,K)}|_H \text{ or } \rho^K = \rho_{(H,K)}|_K ; \tag{2.5}$$

(2) $\rho_{(H,K)}$ is a congruence on S if and only if ρ^H and ρ^K are congruences on H and K respectively.

Proof: (1)$\Rightarrow$. Assume that the relation $\rho_{(H,K)}$ is an equivalent relation on S, for any $a \in H$, $(a,a) \in \rho_{(H,K)}$ implies $(a,a) \in \rho^H$, thus the relation ρ^H is reflexive. Let $(a,b) \in \rho^H$, by formula (2.1) we have $(a,b) \in \rho_{(H,K)}$, thus $(b,a) \in \rho_{(H,K)}$, namely $(b,a) \in \rho^H$ by $(b,a) \in H \times H$ and formula (2.1), this shows that the relation ρ^H is symmetric. Similarly, we have

$$(a,b),(b,c) \in \rho^H \Rightarrow (a,b),(b,c) \in \rho_{(H,K)} \Rightarrow (a,c) \in \rho_{(H,K)} \Rightarrow (a,c) \in \rho^H .$$

So we conclude that ρ^H is an equivalent relation on H. Also, using the same argument as above, we obtain that ρ^K is an equivalent relation on K.

$\Leftarrow$. Let ρ^H, ρ^K be equivalent relations on H and K respectively, for any $a \in S$, then $a \in H$ or $a \in K$. Hence $(a,a) \in \rho^H$ or $(a,a) \in \rho^K$, this implies $(a,a) \in \rho_{(H,K)}$ by formula (2.1), thus the relation $\rho_{(H,K)}$ is reflexive. Let $(a,b) \in \rho_{(H,K)}$, then $(a,b) \in \rho^H$ or $(a,b) \in \rho^K$ by formula (2.1). Hence, we can infer that $(b,a) \in \rho^H$ or $(b,a) \in \rho^K$, this fact shows $(b,a) \in \rho_{(H,K)}$.Namely, $\rho_{(H,K)}$ is symmetric. Next, we shall illustrate that $\rho_{(H,K)}$ is transitive. Let $(a,b),(b,c) \in \rho_{(H,K)}$, according to formula (2.1), there exist four cases as follows:

(i) $(a,b),(b,c) \in \rho^H$; (ii) $(a,b),(b,c) \in \rho^K$;

(iii) $(a,b) \in \rho^H, (b,c) \in \rho^K$; (iv) $(a,b) \in \rho^K, (b,c) \in \rho^H$.

For case (i) (resp, case(ii)),we know $(a,c) \in \rho^H$ (resp, $(a,c) \in \rho^K$), it allows that $(a,c) \in \rho_{(H,K)}$. Now we considers the last two cases. Since (H,K) is a transitive compatible pair of $\rho_{(H,K)}$, by formula (2.2) we have $(a,c) \in \rho_{(H,K)}$ by formula(2.1). Summing up the above cases, we know that the relation $\rho_{(H,K)}$ is transitive. Consequently, the relation $\rho_{(H,K)}$ is an equivalent relation on S. Finally, we shall prove formula (2.5). Clearly,

$$\rho^H \subseteq \rho_{(H,K)}|_H , \rho^K \subseteq \rho_{(H,K)}|_K .$$

We assume that $\rho^K \subset \rho_{(H,K)}|_K$, then we will illustrate $\rho^H = \rho_{(H,K)}|_H$. Since $\rho_{(H,K)}|_K = \rho^K \cup (\rho_{(H,K)}|_{(H \cap K)})$, thus according to Definition 2.1 we have that

$$\rho_{(H,K)} = (\rho_{(H,K)}|_H) \cup (\rho_{(H,K)}|_K) \supseteq (\rho_{(H,K)}|_H) \cup \rho^K \supseteq \rho^H \cup \rho^K = \rho_{(H,K)} .$$

So, we can include that $\rho^H = \rho_{(H,K)}|_H$, and $\rho^H \supseteq \rho_{(H,K)}|_{(H \cap K)}$.

(2)$\Leftarrow$. $\rho_{(H,K)}$ is an equivalent relation on S. Let $(a,b) \in \rho_{(H,K)}$, then $(a,b) \in \rho^H$ or $(a,b) \in \rho^K$. Now we consider the situation $(a,b) \in \rho^H$. For any $x \in S$ .If $x \in H$, since

$(a,b)\in\rho^H$ and ρ^H is a congruence on H, thus we have $(xa,xb)\in\rho^H$, $(ax,bx)\in\rho^H$. If $x\in K$, by formula (2.3) given in Definition 2.1, we have

$$(xa,xb)\in\rho^H \text{ or } (xa,xb)\in\rho^K, \text{or } (\exists\, c\in H\cap K)(xa,c)\in\rho^H,(c,xb)\in\rho^K.$$

Hence, we obtain $(xa,xb)\in\rho_{(H,K)}$. Similarly, we have $(ax,bx)\in\rho_{(H,K)}$.

For the situation $(a,b)\in\rho^K$, similarly, we also have $(xa,xb),(ax,bx)\in\rho_{(H,K)}$. Thus $\rho_{(H,K)}$ is a congruence on S.

$\Rightarrow$. Let $(a,b)\in\rho^H$, then $(a,b)\in\rho_{(H,K)}$. For $h\in H$, since H is a subsemigroup of S, thus $(ha,hb),(ah,bh)\in H\times H$ .By the given condition, $\rho_{(H,K)}$ is a congruence on S, we immediately have $(ha,hb),(ah,bh)\in\rho_{(H,K)}$, therefore $(ha,hb),(ah,bh)\in\rho^H$ by formula(2.1) and $(ha,hb),(ah,bh)\in H\times H$. This shows that ρ^H is a congruence on H. In a similar way, we also can obtain that ρ^K is a congruence on K.

Definition 2.2. If the relation $\rho_{(H,K)}=\rho^H\cup\rho^K$ defined by formula (2.1) is a congruence on semigroup S ,then we call $\rho_{(H,K)}$ a *transitive compatible pair congruence on* S .For the *transitive compatible pair congruence* $\rho_{(H,K)}=\rho^H\cup\rho^K$ *on* S, if at least one of H and K is a proper subsemigroup of S and $\rho^H\neq 1_H$ *or* $\rho^K\neq 1_K$,then we call $\rho_{(H,K)}$ a *proper transitive compatible pair congruence on* S ,and (H,K) a *proper transitive compatible pair of* $\rho_{(H,K)}$ .Otherwise, we call $\rho_{(H,K)}$ trivial.

Obviously, by Definition 2.2, the Rees congruence $\rho=(H\times H)\cup 1_S$ determined by a proper ideal is the proper transitive compatible pair congruence $\rho_{(H,S)}$ on S and (H,S) is a proper transitive compatible pair of ρ .In order to study a nontrivial congruence ρ, we need to obtain some proper transitive compatible pair congruence representation of ρ. So, we need to prove the following theorem.

Theorem 2.2. For any nontrivial congruence ρ on semigroup S can be expressed as a proper transitive compatible pair congruence $\rho_{(H,K)}$ on S.

Proof: Let $1_S\neq\rho\subset S\times S$ .Now, we shall prove that ρ can be expressed as a proper transitive compatible pair congruence $\rho_{(H,K)}$ on S.

Denote the zero element of S/ρ by $\overline{0}$, let

$$0_\rho=\{\,a\in S\mid a\in\overline{0}\,\},\quad K=\{\,x\in S\mid \overline{0}\neq\overline{x}\in S/\rho\,\}\cup\{\,0\,\},$$

$$K_0=\{\,x\in K\mid \overline{x} \text{ is not a zero divisor of } S/\rho\,\},\quad H=0_\rho\cup(K\setminus K_0).$$

If $K_0\neq\varnothing$, then for any $x,y\in K_0$, $\overline{x}\cdot\overline{y}=\overline{xy}\neq\overline{0}$, thus $0\neq xy\in K_0$. This fact shows that

K_0 is a proper subsemigroup of S without zero divisors. Since for any $a \in 0_\rho$, $x \in S$, $\overline{0} = \overline{a} \cdot \overline{x} = \overline{ax}$, therefore we have $ax \in 0_\rho$. Similarly, we obtain $xa \in 0_\rho$. Hence we can deduce that 0_ρ is a proper ideal of S. Next, we shall prove that H is a subsemigroup of S .For any $a, b \in H$, we distinguish two cases as follows.

Case 1. At least one of elements a and b belong to 0_ρ;

Case 2. Elements a and b both are in $K \setminus K_0$, i.e. $a, b \in K \setminus K_0$.

For the former, since 0_ρ is an ideal of S, then $ab \in 0_\rho \subseteq H$. For the latter, let $a, b \in K \setminus K_0$, if $\overline{ba} = \overline{0}$ (or $\overline{ab} = \overline{0}$), then ba (or ab) $\in K \setminus K_0$. If $\overline{ba} \neq \overline{0}, \overline{ab} \neq \overline{0}$, since $\overline{a} \neq \overline{0}$ is a zero divisor in S/ρ, then there exists $x \in K \setminus K_0$ such that $\overline{a}\,\overline{x} = \overline{ax} = \overline{0}$ (or $\overline{x}\,\overline{a} = \overline{0}$). Therefore $\overline{ba}\,\overline{x} = \overline{b}\,\overline{ax} = \overline{0}$ (or $\overline{x}\,\overline{ab} = \overline{xa}\,\overline{b} = \overline{0}$), and consequently $\overline{ba} \neq \overline{0}$ (or $\overline{ab} \neq \overline{0}$) is a zero divisor in S/ρ. We immediately have ba (or ab) $\in K \setminus K_0$. Hence, we obtain that H is a subsemigroup of S. In particular, let $K_0 = \varnothing$, then $K = K \setminus K_0$ is a proper subsemigroup of S.

Now we distinguish the following two situations.

(i) If $K_0 = \varnothing$, then $H = 0_\rho \cup K = S$. Since 0_ρ is a proper ideal of S, it is obvious that $\rho_{(0_\rho, K)} = (\rho|_{0_\rho}) \cup (\rho|_K)$ is a proper transitive compatible pair congruence on S and $\rho = \rho_{(0_\rho, K)}$.

(ii) If $K_0 \neq \varnothing$, we shall prove that $\rho_{(H, K_0)} = (\rho|_H) \cup (\rho|_{K_0})$ is a proper transitive compatible pair congruence on S. Clearly, $\rho|_H$ and $\rho|_{K_0}$ are congruences on H and K_0 respectively. Since $H \cap K_0 = \varnothing$, thus formula (2.2) naturally holds.

Let $(a,b) \in \rho|_H$, for any $x \in S$, then $x \in H$ or $x \in K_0$. If $x \in H$, we have $(xa, xb), (ax, bx) \in \rho|_H$. If $x \in K_0$, since $(a,b) \in \rho|_H$, then $(a,b) \in \rho$, and therefore $(xa, xb) \in \rho$, $(ax, bx) \in \rho$. For the case $(xa, xb) \in \rho$, we will illustrate that either $(xa, xb) \in H \times H$ or $(xa, xb) \in K_0 \times K_0$. If $xa \in K_0$, let $\vartheta : S \to S/\rho$, $x\vartheta = x\rho$ be the natural homomorphism. We have $xb \in ((xa)\rho)\vartheta^{-1} \subseteq K_0$ by $(xa)\rho = (xb)\rho$, hence $xb \in K_0$. If $xa \in H$, we also can get $xb \in H$ using the same means as above. Thus, we deduce that either $(xa, xb) \in H \times H$ or $(xa, xb) \in K_0 \times K_0$. Since $(xa, xb) \in \rho$, then either $(xa, xb) \in \rho|_H$ or $(xa, xb) \in \rho|_{K_0}$ by $(xa, xb) \in H \times H$ or $(xa, xb) \in K_0 \times K_0$. Consider the other case $(ax, bx) \in \rho$, $x \in S$,we also obtain that either $(ax, bx) \in \rho|_H$ or $(ax, bx) \in \rho|_{K_0}$ using the similar arguments.

Similarly, for the situation $(a,b) \in \rho|_{K_0}$, $x \in S$, we also conclude that formulas (2.3),(2.4) hold. Summing up the discussion above, we immediately know that formulas (2.3) and (2.4) hold. Thus, (H, K_0) is a proper transitive compatible pair of ρ.

Finally, we will prove $\rho = \rho_{(H,K_0)}$. Suppose that $(a,b) \in \rho$, if $(a,b) \notin \rho|_{K_0}$, since $H \cap K_0 = \varnothing$, then $(a,b) \in H \times H$ and consequently $(a,b) \in \rho|_H$. If $(a,b) \notin \rho|_H$, similarly we can get $(a,b) \in \rho|_{K_0}$. This fact implies $\rho \subseteq \rho_{(H,K_0)}$. On the contrary, let $(a,b) \in \rho_{(H,K_0)}$, then either $(a,b) \in \rho|_H$ or $(a,b) \in \rho|_{K_0}$, namely $(a,b) \in \rho$. To sum up, we know $\rho = \rho_{(H,K_0)}$.

By Theorem 2.2, we know that transitive compatible pair congruence on semigroup S is a kind of universal representation method fitting to any congruence on S. Concretely speaking, let H and K be subsemigroups of semigroup S, for a congruence ρ on S, so long as (H,K) is a transitive compatible pair of ρ, then ρ can be represented as $\rho = \rho_{(H,K)} = (\rho|_H) \cup (\rho|_K)$.

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