Matrix Representation of Double Layered Fuzzy Graph and its Properties

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Abstract. Uncertainties in a problem are represented as fuzzy matrices using fuzzy principles. Recent days fuzzy matrices have become very famous. In this paper unlike the usual matrix representation of a fuzzy graph with respect to vertices, a new matrix representation with edge membership values as rows and columns is introduced. The relationship between the double layered fuzzy graph and the given fuzzy graph whose crisp graph is a cycle are analyzed.

Keywords: fuzzy graph, strong fuzzy graph, double layered fuzzy graph, matrix representation of double layered fuzzy graph, edge matrix representation of fuzzy graph

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1. Introduction
The concept of fuzzy set was introduced by Zadeh in 1965. Fuzzy graph theory was introduced by Rosenfeld in 1975 [5]. It is well known that matrices play a major role in various areas such as mathematics, physics, statistics, engineering etc. Matrices with entries from \([0, 1]\) and matrix operation defined by fuzzy logical operations are fuzzy matrices. Fuzzy matrices play a fundamental role in fuzzy set theory. They provide us with a logical framework within which many problems of practical applications can be formulated.

Fuzzy matrices can be successfully used when fuzzy uncertainty occurs in a problem. Fuzzy matrix has been proposed to represent fuzzy relation in a system based on fuzzy set theory [1]. Fuzzy matrices were introduced first time by Thomson [2], who discussed the convergence of powers of fuzzy matrices. Two new operations in fuzzy graphs were introduced by Shayamal and Pal [4]. The determinant and adjoint of a square fuzzy matrix are introduced by Ragab and Emam [13]. Pathinathan and Jesintha Rosline had defined the double layered fuzzy graph [12]. In this Paper the edge matrix representation is defined, using it the matrix representation of the double layered fuzzy graph is given. The relationship between the DLFG and the given fuzzy graph are given as propositions and simple examples are presented with verification.
Definition 2.1. [5] A fuzzy graph $G$ is a pair of functions $G:(\sigma, \mu)$ where $\sigma$ is a fuzzy subset of a non-empty set $S$ and $\mu$ is a symmetric fuzzy relation on $\sigma$. The underlying crisp graph of $G:(\sigma, \mu)$ is denoted by $G^*: (\sigma^*, \mu^*)$

Definition 2.2. [8] Let $G(\sigma, \mu)$ be a fuzzy graph, the order of $G$ is defined as $O(G) = \sum_{u \in V} \sigma(u)$

Definition 2.3. [8] Let $G(\sigma, \mu)$ be a fuzzy graph, the size of $G$ is defined as $S(G) = \sum_{u, v \in V} \mu(u, v)$

Definition 2.4. [10] Let $G$ be a fuzzy graph, the degree of a vertex $u$ in $G$ is defined as $d(u) = \sum_{v \in V} \mu(u, v)$ and is denoted as $d_G(u)$.

Definition 2.5. [12] Let $G:(\sigma, \mu)$ be a fuzzy graph with the underlying crisp graph $G^*:(\sigma^*, \mu^*)$. The pair $DL(G):(\sigma_{DL}, \mu_{DL})$ is defined as follows. The node set of $DL(G)$ be $\sigma^* \cup \mu^*$. The fuzzy subset $\sigma_{DL}$ is defined as $\sigma_{DL} = \begin{cases} \sigma(u) & \text{if } u \in \sigma^* \\ \mu(uv) & \text{if } uv \in \mu^* \end{cases}$

The fuzzy relation $\mu_{DL}$ on VUE is defined as $\mu_{DL}(u, v) = \begin{cases} \mu(uv) & \text{if } u, v \in \sigma^* \\ \mu(e_i) \land \mu(e_j) & \text{if the edge } e_i \text{ and } e_j \text{ have a node in common between them} \\ \mu(u_i) \land \mu(e_i) & \text{if } u_i \in \sigma^* \land e_i \in \mu^* \text{ and each } e_i \text{ incident with single } u_i \text{ only either clockwise or anticlockwise.} \\ 0 & \text{otherwise.} \end{cases}$

By definition, $\mu_{DL}(u, v) \leq \sigma_{DL}(u) \land \sigma_{DL}(v) \forall u, v \in \sigma^* \cup \mu^*$. Here $\mu_{DL}$ is a fuzzy relation on the fuzzy subset $\sigma_{DL}$. Hence the pair $DL(G):(\sigma_{DL}, \mu_{DL})$ is a fuzzy graph and is termed as Double Layered Fuzzy Graph.

Definition 2.6. A fuzzy graph $G:(\sigma, \mu)$ with the fuzzy relation $\mu$ to be reflexive and symmetric is completely determined by the fuzzy matrix $M_G$, where $(M_G)_{ij} = \begin{cases} \mu(v_i, v_j) & \text{if } i \neq j \\ \sigma(v_i) & \text{if } i = j \end{cases}$

If $\sigma^*$ has $n$ elements then $M_G$ has $n \times n$ elements.
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**Remark 2.1.** In this paper, we are named the above matrix as Matrix representation of a fuzzy graph with respect to vertices and is denoted as $M_{G_z}$. 

3. **Matrix representation of DLFG**
Consider a fuzzy graph $G$ with $n = 3$ vertices.

![Figure 1: $G:(\sigma, \mu)$](image1)

The matrix representation with respect to vertices for the fuzzy graph $G$ is given by

$$M_{G_z} = \begin{bmatrix} v_1 & v_2 & v_3 \\ v_1 & 0.6 & 0.5 & 0.4 \\ v_2 & 0.5 & 0.8 & 0.3 \\ v_3 & 0.4 & 0.3 & 0.5 \end{bmatrix}$$

The double layered fuzzy graph for $G$ is given by

![Figure 2: Double layered fuzzy graph $DL(G):(\sigma_{DL}, \mu_{DL})$](image2)

The matrix representation of DLFG is

$$M_{DL(G_z)} = \begin{bmatrix} \sigma_1 & \sigma_2 & \sigma_3 & e_1 & e_2 & e_3 \\ \sigma_1 & 0.6 & 0.5 & 0.4 & 0.5 & 0 \\ \sigma_2 & 0.5 & 0.8 & 0.3 & 0 & 0.3 \\ \sigma_3 & 0.4 & 0.3 & 0.5 & 0 & 0 & 0.4 \\ e_1 & 0.5 & 0 & 0 & 0.5 & 0.3 & 0.4 \\ e_2 & 0 & 0.3 & 0 & 0.3 & 0.3 & 0.3 \\ e_3 & 0 & 0 & 0.4 & 0.4 & 0.3 & 0.4 \end{bmatrix}$$
3.1. Edge matrix representation of fuzzy graphs
For a fuzzy graph $G : (\sigma, \mu)$ with the fuzzy relation $\mu$ to be reflexive and symmetric, the edge matrix $M_{G_{\mu}}$ is defined as follows,

$$
\begin{cases}
\min \{\mu(e_i), \mu(e_j)\} & \text{if } v_i \text{ is the common vertex between } e_i \text{ and } e_j \\
\mu(e_i) & \text{if } i = j \\
0 & \text{otherwise}
\end{cases}
$$

If $\mu'$ contains 'n' elements then $M_{G_{\mu'}}$ is a square matrix of order n.

Example 3.1 For Figure 1, the edge matrix representation is given by

$$
M_{G_{\mu}} = \begin{bmatrix}
e_1 & e_2 & e_3 \\
e_2 & 0.5 & 0.3 & 0.4 \\
e_3 & 0.3 & 0.3 & 0.3 \\
e_4 & 0.4 & 0.3 & 0.4
\end{bmatrix}
$$

Thus for Figure 2, the matrix representation becomes

$$
M_{DL(G_{\sigma})} = \begin{bmatrix}
M_{G_{\sigma}} & D_{G_{\sigma}} \\
D_{G_{\sigma}} & M_{G_{\sigma}}
\end{bmatrix}
$$

4. Theoretical concepts

Theorem 4.1. $M_{DL(G_{\sigma})}$ is a symmetric matrix.

Proof: By the definition of $M_{DL}$, the relation $\mu$ is a symmetric relation. Hence,

$$
\begin{align*}
\left(M_{DL(G_{\sigma})}\right)_{i,j} &= \mu(v_i, v_j) \\
&= \mu(v_j, v_i) \quad \because \mu \text{ is symmetric} \\
&= \left(M_{DL(G_{\sigma})}\right)_{j,i}
\end{align*}
$$

$\therefore M_{DL(G_{\sigma})} = M_{DL(G_{\sigma})}^T$

Thus $M_{DL(G_{\sigma})}$ is a symmetric matrix

Theorem 4.2. Trace ($M_{DL(G_{\sigma})}$) = Order(G) + Size(G)

Proof: Trace ($M_{DL(G_{\sigma})}$) = Sum of the diagonal entries in $M_{DL(G_{\sigma})}$.

$$
= \sum_{i=1}^{n} \mu_{DL(G)}(v_i, v_i) = \sum_{v_i \in \sigma_{DL}} \sigma_{DL(G)}(v_i)
$$
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\[
\begin{align*}
&= \sum_{v_i \in \sigma^*} \sigma_G(v_i) + \mu_G(e_i) \quad \text{by the definition of node set in DLFG.} \\
&= \sum_{v_i \in \sigma^*} \sigma_G(v_i) + \sum_{e_i \in \mu^*} \mu_G(e_i) = \text{Order (G) + Size (G).}
\end{align*}
\]

**Theorem 4.3.** The sum of all the entries in \( M_{DL(G)} \) except the diagonal element in the row or column is degree of DLFG, i.e.,

(i) If \( v_i \notin \sigma^* \), then \( d_{DL(G)}(v_i) = \sum_{j=1 \atop \sigma^*} (M_{\sigma^*})_{ij} + (M_{\mu^*})_{ik} \)

(ii) if \( v_i \notin \sigma^* \), then \( d_{DL(G)}(v_i) = \sum_{j=1 \atop \mu^*} (M_{\sigma^*})_{ij} + (M_{\mu^*})_{ik} \), where

\[
k = \begin{cases} 
  i+1 & \text{if } i+1 \leq n \\
  \Re m \left( \frac{i+1}{n} \right) & \text{if } i+1 > n
\end{cases}
\]

**Proof:** The sum of all the entries in \( M_{DL(G)} \) except the diagonal element in the row or column is

\[
\begin{align*}
&= \sum_{j=1 \atop \sigma^*} (M_{DL(G)})_{ij} = \sum_{j=1 \atop \mu^*} (M_{DL(G)})_{ij} = d_{DL(G)}(v_i)
\end{align*}
\]

**Case i:** If \( v_i \notin \sigma^* \) in \( G \), then \( d_{DL(G)}(v_i) = \sum_{j=1 \atop \sigma^*} (M_{\sigma^*})_{ij} + (M_{\mu^*})_{ik} \)

\[
= \sum_{j=1 \atop \sigma^*} \mu_G(v_i, v_j) + \mu_G(v_i, v_k) = \sum_{j=1 \atop \sigma^*} (M_{\sigma^*})_{ij} + (M_{\mu^*})_{ik} ,
\]

where \( k = \begin{cases} 
  i+1 & \text{if } i+1 \leq n \\
  \Re m \left( \frac{i+1}{n} \right) & \text{if } i+1 > n
\end{cases} \)

**Case ii:**

If \( v_i \notin \mu^* \) in \( G \), then \( d_{DL(G)}(v_i) = \sum_{j=1 \atop \sigma^*} (M_{\sigma^*})_{ij} + (M_{\mu^*})_{ik} \)

\[
= \sum_{j=1 \atop \mu^*} (M_{\sigma^*})_{ij} + (M_{\mu^*})_{ik} ,
\]

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where \( k = \begin{cases} 
  i+1 & \text{if } i+1 \leq n \\
  \text{Re}(\frac{i+1}{n}) & \text{if } i+1 > n 
\end{cases} \)

**Theorem 4.4.** The sum of all entries in \( M_{DL(G)} \) except the diagonal element is

\[
4 \text{size}(G) + 2 \sum_{i=1}^{n} (\mu(e_i) \land \mu(e_j)) .
\]

**Proof:** The sum of all entries in \( M_{DL(G)} \) except the diagonal element is

\[
= \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \mu(v_i, v_j) = \sum_{i=1}^{n} d_{DL(G)}(v_i)
\]

**Case i:**

If \( v_i \in \sigma^* \), then

\[
= \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \mu(v_i, v_j) = \sum_{i=1}^{n} d_{DL(G)}(v_i) = \sum_{i=1}^{n} \left( \sum_{j=1, j \neq i}^{n} (M_{G_{\sigma}})_{ij} + (M_{G_{\sigma}})_{ik} \right)
\]

\[
= \sum_{i=1}^{n} d_G(v_i) + (M_{G_{\sigma}})_{ik} = \sum_{i=1}^{n} d_G(v_i) + \sum_{i=1}^{n} (M_{G_{\sigma}})_{ik}
\]

\[
= 2 \text{size}(G) + \text{Size}(G) = 3 \text{size}(G).
\]

**Case ii:**

If \( v_i \in \mu^* \), then

\[
= \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \mu(v_i, v_j) = \sum_{i=1}^{n} d_{DL(G)}(v_i) = \sum_{i=1}^{n} \left( \sum_{j=1, j \neq i}^{n} (M_{G_{\mu}})_{ij} + (M_{G_{\mu}})_{ik} \right)
\]

\[
= \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \mu(e_i) \land \mu(e_j) + (M_{G_{\mu}})_{ik} = 2 \sum_{i=1}^{n} \mu(e_i) \land \mu(e_j) + \sum_{i=1}^{n} (M_{G_{\mu}})_{ik}
\]

\[
= 2 \sum_{i=1}^{n} (\mu(e_i) \land \mu(e_j)) + \text{size}(G).
\]

Thus, if \( v_i \in \sigma_{DL^*} \), then

\[
= \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \mu(v_i, v_j) = 2 \sum_{i=1}^{n} (\mu(e_i) \land \mu(e_j)) + 4 \text{size}(G)
\]

**Example 4.1.** Consider the fuzzy graph with \( n = 4 \) vertices.

![Figure 3: G: (\sigma, \mu)](image_url)
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Here, Size (G) = 1.5 and \( \sum_{i,j=1\atop i \neq j}^{n} (\mu(e_i) \land \mu(e_j)) = 1.1. \)

\[
\begin{pmatrix}
0.7 & 0.4 & 0 & 0.6 & 0.4 & 0 & 0 & 0 \\
0.4 & 0.8 & 0.3 & 0 & 0 & 0.3 & 0 & 0 \\
0.3 & 0.3 & 0.5 & 0.2 & 0 & 0 & 0.2 & 0 \\
0.6 & 0.2 & 0.9 & 0 & 0 & 0 & 0.6 & 0 \\
0.4 & 0 & 0 & 0.4 & 0.3 & 0 & 0.4 & 0 \\
0.3 & 0 & 0 & 0.3 & 0.3 & 0.2 & 0 & 0 \\
0 & 0 & 0.2 & 0 & 0 & 0.2 & 0.2 & 0 \\
0 & 0 & 0.6 & 0.4 & 0 & 0.2 & 0.6 & 0 \\
\end{pmatrix}
\]

\( \text{Sum of all entries except the diagonal elements} = 8.2 \)

\[
4 \text{ size}(G) + 2 \sum_{i=1}^{n} (\mu(e_i) - \mu(e_j)) = 2(1.1) + 4(1.5) = 2.2 + 6 = 8.2.
\]

Thus, Sum of all entries except the diagonal elements
\[
= 4 \text{ size}(G) + 2 \sum_{i=1}^{n} (\mu(e_i) - \mu(e_j))
\]

5. Conclusion
In this paper, we have defined a new matrix representation using edge membership values. The relationship between the matrix representation of double layered fuzzy graph using vertices and given fuzzy graph whose crisp graph is found to be a cycle is examined. Numerical example is given to verify the results. Further analysis will lead to application of DLFG in different networks.

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