Annals of Pure and Applied Mathematics
Vol. 8, No. 2, 2014, 131-141
ISSN: 2279-087X (P), 2279-0888(online)
Published on 17 December 2014
www.researchmathsci.org

Annals of
Pure and Applied Mathematics

On Certain Subclass of Analytic Functions with Respect to 2*k*-Symmetric Conjugate Points

B.Srutha Keerthi[1] and P. Lokesh[2]

[1]Mathematics Division, School of Advanced Sciences
VIT University Chennai Campus, Vandallur Kellambakkam Road
Chennai – 600 127, India. E-mail: sruthilaya06@yahoo.co.in

[2]Department of Mathematics, Bharathiar University, Coimbatore, Tamilnadu, India.
E-mail: lokeshpandurangan@gmail.com

Received 15 October 2014; accepted 21 November 2014

Abstract. In the present paper, we introduce new subclass $\mathcal{P}_{SC}^{(k)}(\rho,\lambda,\alpha)$ of analytic function with respect to 2k-symmetric conjugate points. Such results as integral representations, convolution conditions and coefficient inequalities for this class are provided.

Keywords: Analytic functions, Hadamard product, 2k-symmetric conjugate points

AMS Mathematics Subject Classification (2010): 30C45

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk.

$$\mathbb{U} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

Also let $S^*(\alpha)$ and $\mathcal{C}(\alpha)$ denote the familiar subclasses of $\mathcal{A}$ consisting of functions which are starlike and convex of order α $(0 \le \alpha < 1)$ in U, respectively.

Let $S_{SC}^{(k)}(\alpha)$ denote the class of functions in $\mathcal{A}$ satisfying the following inequality:

$$\mathbb{R}\left(\frac{zf'(z)}{f_{2k}(z)}\right) > \alpha. \quad (z \in \mathbb{U}), \tag{1.2}$$

where $0 \le \alpha < 1$, $k \ge 2$ is a fixed positive integer and $f_{2k}(z)$ is defined by the following equality:

$$f_{2k}(z) = \frac{1}{2k}\sum_{\upsilon=0}^{k-1}\left(\varepsilon^{-\upsilon} f(\varepsilon^{\upsilon} z) + \varepsilon^{\upsilon}\overline{f(\varepsilon^{\upsilon}\overline{z})}\right), \quad \left(\varepsilon = \exp\left(\frac{2\pi i}{k}\right);\ z \in \mathbb{U}\right) \tag{1.3}$$

And a function $f(z) \in \mathcal{A}$ is in the class $\mathcal{C}_{SC}^{(k)}(\alpha)$ if and only if $zf'(z) \in S_{SC}^{(k)}(\alpha)$. The class $S_{SC}^{(k)}(0)$ of functions starlike with respect to 2k-symmetric conjugate points was introduced and investigated by Al-Amiri et al. [1].

Let $\mathcal{T}$ be the subclass of $\mathcal{A}$ consisting of all functions which are of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0).$$

We denote by S^*, $\mathcal{K}$, $\mathcal{C}$ and $\mathcal{C}^*$ the familiar subclass of $\mathcal{A}$ consisting of functions which are, respectively, starlike, convex, close-to-convex and quasi-convex in $\mathbb{U}$. Thus, by definition, we have (see, for details [4, 6, 7, 8]).

$$\mathcal{S}^* = \left\{ f : f \in \mathcal{A} \text{ and } \mathbb{R}\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad (z \in \mathbb{U}) \right\},$$

$$\mathcal{K} = \left\{ f : f \in \mathcal{A} \text{ and } \mathbb{R}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, \quad (z \in \mathbb{U}) \right\}, \text{ and}$$

$$\mathcal{C} = \left\{ f : f \in \mathcal{A}, \exists g \in \mathcal{S}^* : \text{ such that } \mathbb{R}\left\{\frac{zf'(z)}{g(z)}\right\} > 0, \quad (z \in \mathbb{U}) \right\}.$$

Definition 1.1. Let $\mathcal{T}(\rho, \lambda, \alpha)$ be the subclass of $\mathcal{T}$ consisting of functions f(z) which satisfy the inequality:

$$\mathbb{R}\left(\frac{\dfrac{zf'(z) + \rho z^2 f''(z)}{(1-\rho)f(z) + \rho z f'(z)}}{\lambda\left(\dfrac{zf'(z) + \rho z^2 f''(z)}{(1-\rho)f(z) + \rho z f'(z)}\right) + (1-\lambda)} \right) > \alpha \quad (z \in \mathbb{U}) \tag{1.4}$$

for some α $(0 \leq \alpha < 1)$, λ $(0 \leq \lambda < 1)$ and ρ $(0 \leq \rho \leq 1)$. If $\rho = 0$, a function $f(z) \in \mathcal{A}$ is in the class $\mathcal{C}(\lambda, \alpha)$. This class was first introduced and investigated by Altintas and Owa [2], then was studied by Aouf et al. [3].

We now introduce the following subclass of $\mathcal{A}$ with respect to 2k-symmetric conjugate points and obtain some interesting results.

A function $f(z) \in \mathcal{A}$ is in the class $\mathcal{P}_{SC}^{(k)}(\rho, \lambda, \alpha)$ if it satisfies the following inequality:

$$\mathbb{R}\left(\frac{\dfrac{zf'(z) + \rho z^2 f''(z)}{(1-\rho)f_{2k}(z) + \rho z f'_{2k}(z)}}{\lambda\left(\dfrac{zf'(z) + \rho z^2 f''(z)}{(1-\rho)f_{2k}(z) + \rho z f'_{2k}(z)}\right) + (1-\lambda)} \right) > \alpha \quad (z \in \mathbb{U}), \tag{1.5}$$

where $0 \leq \alpha < 1$, $0 \leq \lambda < 1$, $0 \leq \rho \leq 1$ and $f_{2k}(z)$ is defined the equality (1.3). If $\rho = 0$, a function $f(z) \in \mathcal{A}$ is in the class $\mathcal{P}_{SC}^{(k)}(\lambda, \alpha)$ which was studied by Luo and Wang [5].

For $\lambda = 0$ in $\mathcal{P}_{SC}^{(k)}(\rho, \lambda, \alpha)$ we get $S_{SC}^{(k)}(\rho, \alpha)$ [10].

Lemma 1.1. Let $\gamma \geq 0$ and $f \in \mathcal{C}$, then

$$F(z) = \frac{1+\gamma}{z^{\gamma}} \int_0^z f(t)t^{\gamma-1}dt \in \mathcal{C}.$$

This lemma is a special case of Theorem 4 in [9].

Lemma 1.2. [6] Let $0 < \rho \leq 1$ and $f \in \mathcal{C}^*$, then

$$F(z) = \frac{1}{\rho} z^{1-\frac{1}{\rho}} \int_0^z f(t)t^{\frac{1}{\rho}-2}dt \in \mathcal{C}^* \subset \mathcal{C}.$$

Lemma 1.3. Let $0 \leq \rho \leq 1$ and $0 \leq \alpha < 1$, then we have $\mathcal{P}_{sc}^{(k)}(\rho, \alpha) \subset \mathcal{C} \subset \mathcal{S}$.

Proof: Let $F(z) = (1-\rho)f(z) + \rho z f'(z)$, $F_{2k}(z) = (1-\rho)f_{2k}(z) + \rho z f'_{2k}(z)$ with $f(z) \in \mathcal{P}_{sc}^{(k)}(\rho, \alpha)$, substituting z by $\varepsilon^{\mu} z$ ($\mu = 0, 1, 2, ..., k-1$) in (1.5), we get

$$\mathbb{R}\left(\frac{\dfrac{\varepsilon^{\mu} z f'(\varepsilon^{\mu} z) + \rho(\varepsilon^{\mu} z)^2 f''(\varepsilon^{\mu} z)}{(1-\rho)f_{2k}(\varepsilon^{\mu} z) + \rho(\varepsilon^{\mu} z)f'_{2k}(\varepsilon^{\mu} z)}}{\lambda\left(\dfrac{\varepsilon^{\mu} z f'(\varepsilon^{\mu} z) + \rho(\varepsilon^{\mu} z)^2 f''(\varepsilon^{\mu} z)}{(1-\rho)f_{2k}(\varepsilon^{\mu} z) + \rho(\varepsilon^{\mu} z)f'_{2k}(\varepsilon^{\mu} z)}\right) + (1-\lambda)} \right) > \alpha, \tag{1.6}$$

From inequality (1.6) we have

$$\mathbb{R}\left(\frac{\dfrac{\overline{\varepsilon^{\mu}\bar{z}}\,\overline{f'(\varepsilon^{\mu}\bar{z})} + \rho(\overline{\varepsilon^{\mu}\bar{z}})^2\, \overline{f''(\varepsilon^{\mu}\bar{z})}}{(1-\rho)\overline{f_{2k}(\varepsilon^{\mu}\bar{z})} + \rho(\overline{\varepsilon^{\mu}\bar{z}})\overline{f'_{2k}(\varepsilon^{\mu}\bar{z})}}}{\lambda\left(\dfrac{\overline{\varepsilon^{\mu}\bar{z}}\,\overline{f'(\varepsilon^{\mu}\bar{z})} + \rho(\overline{\varepsilon^{\mu}\bar{z}})^2\, \overline{f''(\varepsilon^{\mu}\bar{z})}}{(1-\rho)\overline{f_{2k}(\varepsilon^{\mu}\bar{z})} + \rho(\overline{\varepsilon^{\mu}\bar{z}})\overline{f'_{2k}(\varepsilon^{\mu}\bar{z})}}\right) + (1-\lambda)} \right) > \alpha, \tag{1.7}$$

Note that $f_{2k}(\varepsilon^{\mu} z) = \varepsilon^{\mu} f_{2k}(z)$, $f'_{2k}(\varepsilon^{\mu} z) = f'_{2k}(z)$, $\overline{f_{2k}(\varepsilon^{\mu}\bar{z})} = \varepsilon^{-\mu} f_{2k}(z)$ and $\overline{f'_{2k}(\varepsilon^{\mu}\bar{z})} = f'_{2k}(z)$, thus, inequalities (1.6) and (1.7) can be written as

$$\mathbb{R}\left(\frac{\dfrac{z f'(\varepsilon^{\mu} z) + \rho z^2 \varepsilon^{\mu} f''(\varepsilon^{\mu} z)}{(1-\rho)f_{2k}(z) + \rho z f'_{2k}(z)}}{\lambda\left(\dfrac{z f'(\varepsilon^{\mu} z) + \rho z^2 \varepsilon^{\mu} f''(\varepsilon^{\mu} z)}{(1-\rho)f_{2k}(z) + \rho z f'_{2k}(z)}\right) + (1-\lambda)} \right) > \alpha, \tag{1.8}$$

and

$$\Re\left(\frac{\dfrac{z\overline{f'(\varepsilon^\mu \overline{z})}+\rho z^2\varepsilon^{-\mu}\overline{f''(\varepsilon^\mu \overline{z})}}{(1-\rho)f_{2k}(z)+\rho z f'_{2k}(z)}}{\lambda\left(\dfrac{z\overline{f'(\varepsilon^\mu \overline{z})}+\rho z^2\varepsilon^{-\mu}\overline{f''(\varepsilon^\mu \overline{z})}}{(1-\rho)f_{2k}(z)+\rho z f'_{2k}(z)}\right)+(1-\lambda)}\right)>\alpha \qquad (1.9)$$

Summing inequalities (1.8) and (1.9), we can obtain

$$\Re\left(\frac{\dfrac{z\left[f'(\varepsilon^\mu z)+\overline{f'(\varepsilon^\mu \overline{z})}\right]+\rho z^2\left[\varepsilon^\mu f''(\varepsilon^\mu z)+\varepsilon^{-\mu}\overline{f''(\varepsilon^\mu \overline{z})}\right]}{(1-\rho)f_{2k}(z)+\rho z f'_{2k}(z)}}{\lambda\left(\dfrac{z\left[f'(\varepsilon^\mu z)+\overline{f'(\varepsilon^\mu \overline{z})}\right]+\rho z^2\left[\varepsilon^\mu f''(\varepsilon^\mu z)+\varepsilon^{-\mu}\overline{f''(\varepsilon^\mu \overline{z})}\right]}{(1-\rho)f_{2k}(z)+\rho z f'_{2k}(z)}\right)+(1-\lambda)}\right)>2\alpha. \qquad (1.10)$$

Let $\mu = 0, 1, 2, ..., k-1$ in (1.10), respectively, and summing them we can get

$$\Re\left(\frac{\dfrac{z\dfrac{1}{2k}\sum_{\mu=0}^{k-1}\left[f'(\varepsilon^\mu z)+\overline{f'(\varepsilon^\mu \overline{z})}\right]+\rho z^2\dfrac{1}{2k}\sum_{\mu=0}^{k-1}\left[\varepsilon^\mu f''(\varepsilon^\mu z)+\varepsilon^{-\mu}\overline{f''(\varepsilon^\mu \overline{z})}\right]}{(1-\rho)f_{2k}(z)+\rho z f'_{2k}(z)}}{\lambda\left(\dfrac{z\dfrac{1}{2k}\sum_{\mu=0}^{k-1}\left[f'(\varepsilon^\mu z)+\overline{f'(\varepsilon^\mu \overline{z})}\right]+\rho z^2\dfrac{1}{2k}\sum_{\mu=0}^{k-1}\left[\varepsilon^\mu f''(\varepsilon^\mu z)+\varepsilon^{-\mu}\overline{f''(\varepsilon^\mu \overline{z})}\right]}{(1-\rho)f_{2k}(z)+\rho z f'_{2k}(z)}\right)+(1-\lambda)}\right)>\alpha,$$

or equivalently,

$$\Re\left(\frac{\dfrac{zf'_{2k}(z)+\rho z^2 f''_{2k}(z)}{(1-\rho)f_{2k}(z)+\rho z f'_{2k}(z)}}{\lambda\left(\dfrac{zf'_{2k}(z)+\rho z^2 f''_{2k}(z)}{(1-\rho)f_{2k}(z)+\rho z f'_{2k}(z)}\right)+(1-\lambda)}\right)=\Re\left(\frac{\dfrac{zF'_{2k}(z)}{F_{2k}(z)}}{\lambda\left(\dfrac{zF'_{2k}(z)}{F_{2k}(z)}\right)+(1-\lambda)}\right)>\alpha,$$

that is for $\lambda=0$ $F_{2k}(z) \in S^*(\alpha)$, which is the class of starlike functions of order α in $\mathbb{U}$. Note that $S^*(0) = S^*$, this implies that $F(z) = (1-\rho) f(z) + \rho z f'(z) \in \mathcal{C}$. We now split it into two cases to prove

Case (i) When $\rho = 0$, $\lambda=0$, it is obvious that $f(z) = F(z) \in \mathcal{C}$.

Case (ii) When $\lambda=0$ and $0 < \rho \le 1$. From $F(z) = (1-\rho) f(z) + \rho z f'(z)$ and $0 < \rho \le 1$, we have

$$f(z)=\frac{1}{\rho}z^{1-\frac{1}{\rho}}\int_0^z F(t)t^{\frac{1}{\rho}-2}dt.$$

Since $\gamma = \frac{1}{\rho} - 1 \geq 0$, by Lemma 1.1, we obtain that f(z) ∈ $\mathcal{C} \subset$ S. Hence $\mathcal{P}_{sc}^{(k)}(\rho, \alpha) \subset \mathcal{C} \subset$ S, and the proof is complete. □

2. Integral representations

We first give some integral representations of functions in the class $\mathcal{P}_{SC}^{(k)}(\rho, \lambda, \alpha)$.

Theorem 2.1. Let f(z) ∈ $\mathcal{P}_{SC}^{(k)}(\rho, \lambda, \alpha)$ with 0 < ρ ≤ 1. Then

$$f_{2k}(z) = \frac{1}{\rho} z^{1-\frac{1}{\rho}} \int_0^z \exp\left(\frac{1}{2k} \sum_{\mu=0}^{k-1} \int_0^u \frac{2(1-\alpha)}{\zeta} \left(\frac{\omega(\varepsilon^\mu \zeta)}{1-\lambda-(1+\lambda-2\alpha\lambda)\omega(\varepsilon^\mu \zeta)} + \frac{\overline{\omega(\varepsilon^\mu \overline{\zeta})}}{1-\lambda-(1+\lambda-2\alpha\lambda)\overline{\omega(\varepsilon^\mu \overline{\zeta})}} \right) d\zeta \right) u^{\frac{1}{\rho}-1} du, \tag{2.1}$$

where $f_{2k}(z)$ is defined by equality (1.3), ω(z) is analytic in U and ω(0) = 0, |w(z)| < 1.

Proof: Suppose that f(z) ∈ $\mathcal{P}_{SC}^{(k)}(\rho, \lambda, \alpha)$, we know that the condition (1.5) can be written as follows:

$$\frac{\dfrac{zf'(z)+\rho z^2 f''(z)}{(1-\rho)f_{2k}(z)+\rho z f'_{2k}(z)}}{\lambda\left(\dfrac{zf'(z)+\rho z^2 f''(z)}{(1-\rho)f_{2k}(z)+\rho z f'_{2k}(z)}\right)+(1-\lambda)} \prec \frac{1+(1-2\alpha)z}{1-z},$$

where ≺ stands for the subordination. It follows that

$$\frac{\dfrac{zf'(z)+\rho z^2 f''(z)}{(1-\rho)f_{2k}(z)+\rho z f'_{2k}(z)}}{\lambda\left(\dfrac{zf'(z)+\rho z^2 f''(z)}{(1-\rho)f_{2k}(z)+\rho z f'_{2k}(z)}\right)+(1-\lambda)} \prec \frac{1+(1-2\alpha)\omega(z)}{1-\omega(z)},$$

where ω(z) is analytic in U and ω(0) = 0, |ω(z)| < 1. This yields

$$\frac{zf'(z)+\rho z^2 f''(z)}{(1-\rho)f_{2k}(z)+\rho z f'_{2k}(z)} = \frac{(1-\lambda)[1+(1-2\alpha)\omega(z)]}{1-\lambda-(1+\lambda-2\alpha\lambda)\omega(z)}, \tag{2.2}$$

Substituting z by ε^μ z (μ = 0, 1, 2, ..., k–1) in (2.2), respectively, we get

$$\frac{\varepsilon^\mu z f'(\varepsilon^\mu z)+\rho(\varepsilon^\mu z)^2 f''(\varepsilon^\mu z)}{(1-\rho)f_{2k}(\varepsilon^\mu z)+\rho\varepsilon^\mu z f'_{2k}(\varepsilon^\mu z)} = \frac{(1-\lambda)[1+(1-2\alpha)\omega(\varepsilon^\mu z)]}{1-\lambda-(1+\lambda-2\alpha\lambda)\omega(\varepsilon^\mu z)}. \tag{2.3}$$

From (2.3), we have

$$\frac{\overline{(\varepsilon^\mu \overline{z})}\,\overline{f'(\varepsilon^\mu \overline{z})}+\rho\overline{(\varepsilon^\mu \overline{z})}^2\,\overline{f''(\varepsilon^\mu \overline{z})}}{(1-\rho)\overline{f_{2k}(\varepsilon^\mu \overline{z})}+\rho\overline{\varepsilon^\mu \overline{z}}\,\overline{f'_{2k}(\varepsilon^\mu \overline{z})}} = \frac{(1-\lambda)[1+(1-2\alpha)\overline{\omega(\varepsilon^\mu \overline{z})}]}{1-\lambda-(1+\lambda-2\alpha\lambda)\overline{\omega(\varepsilon^\mu \overline{z})}}. \tag{2.4}$$

Note that $f_{2k}(\varepsilon^{\mu} z) = \varepsilon^{\mu} f_{2k}(z)$ and $\overline{f_{2k}(\varepsilon^{\mu}\overline{z})} = \varepsilon^{-\mu} f_{2k}(z)$, summing equalities (2.3) and (2.4), we can obtain

$$\frac{z(f'(\varepsilon^{\mu}z)+\overline{f'(\varepsilon^{\mu}\overline{z})})+\rho z^2(\varepsilon^{\mu}f''(\varepsilon^{\mu}z)+\varepsilon^{-\mu}\overline{f''(\varepsilon^{\mu}\overline{z})})}{(1-\rho)f_{2k}(z)+\rho z f'_{2k}(z)} = \frac{(1-\lambda)[1+(1-2\alpha)\omega(\varepsilon^{\mu}z)]}{1-\lambda-(1+\lambda-2\alpha\lambda)\omega(\varepsilon^{\mu}z)} + \frac{(1-\lambda)[1+(1-2\alpha)\overline{\omega(\varepsilon^{\mu}\overline{z})}]}{1-\lambda-(1+\lambda-2\alpha\lambda)\overline{\omega(\varepsilon^{\mu}\overline{z})}}. \tag{2.5}$$

Let $\mu = 0, 1, 2, \ldots, k-1$ in (2.5), respectively, and summing them we can get

$$\frac{zf'_{2k}(z)+\rho z^2 f''_{2k}(z)}{(1-\rho)f_{2k}(z)+\rho z f'_{2k}(z)} = \frac{1}{2k}\sum_{\mu=0}^{k-1}\frac{(1-\lambda)[1+(1-2\alpha)\omega(\varepsilon^{\mu}z)]}{1-\lambda-(1+\lambda-2\alpha\lambda)\omega(\varepsilon^{\mu}z)} + \frac{(1-\lambda)[1+(1-2\alpha)\overline{\omega(\varepsilon^{\mu}\overline{z})}]}{1-\lambda-(1+\lambda-2\alpha\lambda)\overline{\omega(\varepsilon^{\mu}\overline{z})}}. \tag{2.6}$$

From (2.6), we can get

$$\frac{f'_{2k}(z)+\rho z f''_{2k}(z)}{(1-\rho)f_{2k}(z)+\rho z f'_{2k}(z)} - \frac{1}{z} = \frac{1}{2k}\sum_{\mu=0}^{k-1}\frac{1}{z}\left(\frac{(1-\lambda)[1+(1-2\alpha)\omega(\varepsilon^{\mu}z)]}{1-\lambda-(1+\lambda-2\alpha\lambda)\omega(\varepsilon^{\mu}z)} + \frac{(1-\lambda)[1+(1-2\alpha)\overline{\omega(\varepsilon^{\mu}\overline{z})}]}{1-\lambda-(1+\lambda-2\alpha\lambda)\overline{\omega(\varepsilon^{\mu}\overline{z})}} - 2\right).$$

Integrating (2.7), we have

$$\log\left(\frac{(1-\rho)f_{2k}(z)+\rho z f'_{2k}(z)}{z}\right) = \frac{1}{2k}\sum_{\mu=0}^{k-1}\int_0^z \frac{2(1-\alpha)}{\zeta} \left(\frac{\omega(\varepsilon^{\mu}\zeta)}{1-\lambda-(1+\lambda-2\alpha\lambda)\omega(\varepsilon^{\mu}\zeta)} + \frac{\overline{\omega(\varepsilon^{\mu}\bar{\zeta})}}{1-\lambda-(1+\lambda-2\alpha\lambda)\overline{\omega(\varepsilon^{\mu}\bar{\zeta})}}\right)d\zeta.$$

That is

$$(1-\rho)f_{2k}(z)+\rho z f'_{2k}(z) = z\exp\frac{1}{2k}\sum_{\mu=0}^{k-1}\int_0^z \frac{2(1-\alpha)}{\zeta} \left(\frac{\omega(\varepsilon^{\mu}\zeta)]}{1-\lambda-(1+\lambda-2\alpha\lambda)\omega(\varepsilon^{\mu}\zeta)} + \frac{\overline{\omega(\varepsilon^{\mu}\bar{\zeta})}}{1-\lambda-(1+\lambda-2\alpha\lambda)\overline{\omega(\varepsilon^{\mu}\bar{\zeta})}}\right)d\zeta. \tag{2.8}$$

The assertion (2.1) in Theorem 2.1 can now easily be derived from (2.8). □

Theorem 2.2. Let $f(z) \in \mathcal{P}_{SC}^{(k)}(\rho,\lambda,\alpha)$ with $k \geq 2$. Then

$$f(z) = \frac{1}{\rho} z^{1-\frac{1}{\rho}} \int_0^z \int_0^u \exp\left(\frac{1}{2k} \sum_{\mu=0}^{k-1} \int_0^{\xi} \frac{2(1-\alpha)}{\zeta} \left(\frac{\omega(\varepsilon^{\mu}\zeta)}{1-\lambda-(1+\lambda-2\alpha\lambda)\omega(\varepsilon^{\mu}\zeta)} + \frac{\overline{\omega(\varepsilon^{\mu}\overline{\zeta})}}{1-\lambda-(1+\lambda-2\alpha\lambda)\overline{\omega(\varepsilon^{\mu}\overline{\zeta})}} \right) d\zeta \right) \frac{(1-\lambda)[1+(1-2\alpha)\omega(\xi)]}{1-\lambda-(1+\lambda-2\alpha\lambda)\omega(\xi)} d\xi \, u^{\frac{1}{\rho}-2} du. \qquad (2.9)$$

where $\omega(z)$ is analytic in $\mathbb{U}$ and $\omega(0) = 0$, $|\omega(z)| < 1$.

Proof: Suppose that $f(z) \in \mathcal{P}_{SC}^{(k)}(\rho, \lambda, \alpha)$, from equalities (2.1) and (2.2), we can get

$$zf'(z) + \rho z^2 f''(z) = ((1-\rho)f_{2k}(z) + \rho z f'_{2k}(z)) \frac{(1-\lambda)[1+(1-2\alpha)\omega(z)]}{1-\lambda-(1+\lambda-2\alpha\lambda)\omega(z)}$$

$$= \exp \frac{1}{2k} \sum_{\mu=0}^{k-1} \int_0^z \frac{2(1-\alpha)}{\zeta} \left(\frac{\omega(\varepsilon^{\mu}\zeta)}{1-\lambda-(1+\lambda-2\alpha\lambda)\omega(\varepsilon^{\mu}\zeta)} + \frac{\overline{\omega(\varepsilon^{\mu}\overline{\zeta})}}{1-\lambda-(1+\lambda-2\alpha\lambda)\overline{\omega(\varepsilon^{\mu}\overline{\zeta})}} \right) d\zeta \, \frac{(1-\lambda)[1+(1-2\alpha)\omega(z)]}{1-\lambda-(1+\lambda-2\alpha\lambda)\omega(z)}.$$

Integrating the equality, we can easily get (2.9). □

3. Convolution conditions

In this section, we provide some convolution conditions for the class $\mathcal{P}_{SC}^{(k)}(\rho, \lambda, \alpha)$. Let f, g ∈ $\mathcal{A}$, where f(z) is given by (1.1) and g(z) is defined by

$$g(z) = z + \sum_{n=2}^{\infty} c_n z^n.$$

Then the Hadamard product (or convolution) f * g is defined (as usual) by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n c_n z^n = (g * f)(z).$$

Theorem 3.1. A function $f(z) \in \mathcal{P}_{SC}^{(k)}(\rho, \lambda, \alpha)$ if and only if

$$\frac{1}{z}\left\{ f * \left[\left(\frac{z}{(1-z)^2} \left((1-e^{i\theta}) - \lambda(1+(1-2\alpha)e^{i\theta}) \right) - \frac{(1-\rho)(1-\lambda)(1+(1-2\alpha)e^{i\theta})}{2} h \right) \right. \right.$$

$$+\rho z\left(\frac{z}{(1-z)^2}\left((1-e^{i\theta})-\lambda(1+(1-2\alpha)e^{i\theta})\right)-\frac{(1-\lambda)(1+(1-2\alpha)e^{i\theta})}{2}h\right)'\Bigg](z)$$

$$-(1-\lambda)(1+(1-2\alpha)e^{i\theta}\overline{f*\left(\frac{1-\rho}{2}h+\frac{\rho}{2}zh'\right)(\overline{z})}\Bigg\}\neq 0 \tag{3.1}$$

for all $z\in \mathbb{U}$ and $0\leq\theta<2\pi$, where

$$h(z)=\frac{1}{k}\sum_{\nu=0}^{k-1}\frac{z}{1-\varepsilon^{\nu}z}. \tag{3.2}$$

Proof: Suppose that $f(z)\in \mathcal{P}_{SC}^{(k)}(\rho,\lambda,\alpha)$, since the condition (1.5) is equivalent to

$$\frac{\dfrac{zf'(z)+\rho z^2f''(z)}{(1-\rho)f_{2k}(z)+\rho zf'_{2k}(z)}}{\lambda\left(\dfrac{zf'(z)+\rho z^2f''(z)}{(1-\rho)f_{2k}(z)+\rho zf'_{2k}(z)}\right)+(1-\lambda)}\neq\frac{1+(1-2\alpha)e^{i\theta}}{1-e^{i\theta}},$$

for all $z\in \mathbb{U}$ and $0\leq\theta<2\pi$. And the condition (3.2) can be written as follows:

$$\frac{1}{z}\Big\{(zf'(z)+\rho z^2f''(z))((1-e^{i\theta})-\lambda(1+(1-2\alpha)e^{i\theta})) \\ -(1-\lambda)[(1-\rho)f_{2k}(z)+\rho zf'_{2k}(z)](1+(1-2\alpha)e^{i\theta})\Big\}\neq 0. \tag{3.4}$$

On the other hand, it is well known that

$$zf'(z)=f(z)*\frac{z}{(1-z)^2}. \tag{3.5}$$

And from the definition of $f_{2k}(z)$, we know

$$f_{2k}(z)=z+\sum_{n=2}^{\infty}\frac{a_n+\overline{a_n}}{2}c_nz^n=\frac{1}{2}((f*h)(z)+\overline{(f*h)(z)}), \tag{3.6}$$

where h(z) is given by (3.6). Substituting (3.5) and (3.6) into (3.4), we can easily get (3.1). This completes the proof of Theorem 3.1. □

4. Coefficient inequalities

In this section, we provide the sufficient conditions for functions belonging to the class $\mathcal{P}_{SC}^{(k)}(\rho,\lambda,\alpha)$.

Theorem 4.1. Let $0\leq\alpha<1$, $0\leq\lambda<1$ and $0\leq\rho<1$. If

$$\sum_{n=1}^{\infty}[(1-\rho)+\rho(nk+1)][(1-\lambda)\,|\,(nk+1)a_{nk+1}-R(a_{nk+1})\,| \\ +(1-\alpha)(\lambda(nk+1)a_{nk+1}+(1-\lambda)\,|\,R(a_{nk+1})\,|] \\ +\sum_{\substack{n=2\\ n\neq lk+1}}^{\infty}(1-\lambda)n[(1-\rho)+\rho n]\,|\,a_n\,|\leq 1-\alpha \tag{4.1}$$

Then $f(z) \in \mathcal{P}_{SC}^{(k)}(\rho, \lambda, \alpha)$.

Proof: It suffices to show that

$$\left| \frac{\dfrac{zf'(z) + \rho z^2 f''(z)}{(1-\rho)f_{2k}(z) + \rho z f_{2k}'(z)}}{\lambda\left(\dfrac{zf'(z) + \rho z^2 f''(z)}{(1-\rho)f_{2k}(z) + \rho z f_{2k}'(z)}\right) + (1-\lambda)} - 1 \right| < 1 - \alpha,$$

Note that for $|z| = r < 1$, we have

$$\left| \frac{\dfrac{zf'(z) + \rho z^2 f''(z)}{(1-\rho)f_{2k}(z) + \rho z f_{2k}'(z)}}{\lambda\left(\dfrac{zf'(z) + \rho z^2 f''(z)}{(1-\rho)f_{2k}(z) + \rho z f_{2k}'(z)}\right) + (1-\lambda)} - 1 \right|$$

$$= \left| \frac{\sum_{n=2}^{\infty}(1-\lambda)[(1-\rho) + \rho n](na_n - \mathbb{R}(a_n)b_n)z^{n-1}}{1 + \sum_{n=2}^{\infty}[(1-\rho) + \rho n](\lambda n a_n + (1-\lambda)\mathbb{R}(a_n)b_n)z^{n-1}} \right|$$

$$\le \frac{\sum_{n=2}^{\infty}(1-\lambda)[(1-\rho) + \rho n]\left|na_n - \mathbb{R}(a_n)b_n\right||z|^{n-1}}{1 - \sum_{n=2}^{\infty}[(1-\rho) + \rho n]\left[\lambda n a_n + (1-\lambda)b_n\left|\mathbb{R}(a_n)\right|\right]|z|^{n-1}}$$

$$\le \frac{\sum_{n=2}^{\infty}(1-\lambda)[(1-\rho) + \rho n]\left|na_n - \mathbb{R}(a_n)b_n\right|}{1 - \sum_{n=2}^{\infty}[(1-\rho) + \rho n]\left[\lambda n a_n + (1-\lambda)b_n\left|\mathbb{R}(a_n)\right|\right]}$$

where

$$b_n = \frac{1}{k}\sum_{\nu=0}^{k-1}\varepsilon^{(n-1)\nu} = \begin{cases} 1, & n = lk+1, \\ 0, & n \ne lk+1. \end{cases} \tag{4.2}$$

This last expression is bounded above by $1 - \alpha$ if

$$\sum_{n=2}^{\infty}[(1-\rho+\rho n][(1-\lambda)\left|na_n - \mathbb{R}(a_n)b_n\right| + (1-\alpha)(\lambda n a_n + (1-\lambda)b_n\left|\mathbb{R}(a_n)\right|] \le 1-\alpha, \tag{4.3}$$

Since inequality (4.3) can be written as inequality (4.1), hence f(z) satisfies the condition (1.5). This completes the proof of Theorem 4.1. □

Theorem 4.2. Let $0 \le \alpha < 1$, $0 \le \lambda < 1$ and $0 \le \rho \le 1$ and $f(z) \in \mathcal{T}$. Then $f(z) \in \mathcal{TP}_{SC}^{(k)}(\rho, \lambda, \alpha)$ if and only if

$$\sum_{n=1}^{\infty}[(1-\rho)+\rho(nk+1)][(nk+1)-\alpha([\lambda(nk+1)+(1-\lambda)]a_{nk+1})]$$
$$+\sum_{\substack{n=2\\ n\neq lk+1}}^{\infty} n[(1-\rho)+\rho n]a_n \le 1-\alpha. \tag{4.4}$$

Proof: In view of Theorem 4.1, we need only to prove the necessity. Suppose that $f(z) \in \mathcal{TP}_{SC}^{(k)}(\rho,\lambda,\alpha)$, then from (1.5), we can get

$$\mathbb{R}\left(\frac{1-\sum_{n=2}^{\infty} n[(1-\rho)+\rho n]a_n z^{n-1}}{1-\sum_{n=2}^{\infty}\{\lambda[n((1-\rho)+\rho n)a_n]+(1-\lambda)[((1-\rho)+\rho n)a_n b_n]\}z^{n-1}}\right) > \alpha, \tag{4.5}$$

where b_n is given by 4.2. By letting $z \to 1^-$ through real values in (4.5), we can get

$$\frac{1-\sum_{n=2}^{\infty} n[(1-\rho)+\rho n]a_n}{1-\sum_{n=2}^{\infty}\{\lambda[n((1-\rho)+\rho n)a_n]+(1-\lambda)[((1-\rho)+\rho n)a_n b_n]\}} \ge \alpha,$$

or equivalently,

$$\sum_{n=2}^{\infty}[(1-\rho)+\rho n]+(n-\alpha(\lambda n+(1-\lambda))a_n b_n) \le 1-\alpha, \tag{4.6}$$

substituting (4.2) into inequality (4.6), we can get inequality (4.4) easily. This completes the proof of Theorem 4.2. □

Acknowledgement. The first author thanks the support provided by Science and Engineering Research Board (DST), New Delhi. Project No: SR/S4/MS:716/10 with titled “On Certain Subclass of Analytic Functions with respect to 2k-Symmetric Conjugate Points”.

REFERENCES

1. H.Al-Amiri, D.Coman and P.T.Mocanu, Some properties of starlike functions with respect to symmetric conjugate points, *Internat. J. Math. Math. Sci.*, 18 (1995) 469–474.
2. O.Altintas and S.Owa, On subclasses of univalent functions with negative coefficients, *Pusan Kyongnam Math. J.*, 4 (1988) 41–56.
3. M.K.Aouf, H.M.Hossen and A.Y.Lashin, Convex subclass of starlike functions, *Kyungpook Math. J.*, 40 (2000) 287–297.
4. P.L.Duren, *Univalent Functions*, Springer-Verlag, New York, 1983.
5. Hong.-C.Luo and Z.-G.Wang, Some subclasses of analytic functions with respect to 2k-symmetric conjugate points, *Research Group in Mathematical Inequalities and Applications*, 11(3) (2008) 1–7.
6. K.I.Noor, On quasi-convex functions and related topics, *Internat. J. Math. Math. Sci.*, 10 (1987) 241–258.
7. S.Owa, M.Nunokawa, H.Saitoh and H.M.Srivastava, Close-to-convexity, starlikeness and convexity of certain analytic functions, *Appl. Math. Lett.*, 15 (2002) 63–69.

8. H.M.Srivastava and S. Owa (Eds.), *Current Topics in Analytic Function Theory*, World Scientific, Singapore, 1992.
9. Z.-R.Wu, The integral operator of starlikeness and the family of Bazilevic functions, *Acta Math. Sinica*, 27 (1984) 394–409.
10. Z.-G.Wang and D.-Z.Chen, On subclasses of close-to-convex and quasi-convex functions with respect to 2k-symmetric conjugate points, *Lobachevskii Journal of Mathematics*, 26 (2007) 127–135.