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Skew Chromatic Index of Circular Ladder Graphs

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Abstract. A skew edge coloring of a graph *G* is defined to be a set of two edge colorings such that no two edges are assigned the same unordered pair of colors. The skew chromatic index *s*(*G*) is the minimum number of colors required for a skew edge coloring of *G*. In this paper, an algorithm is determined for skew edge coloring of circular ladder graphs. Alsothe skew chromatic index of circular ladder graphs is solved in polynomial time.

Keywords: Skew edge coloring; skew chromatic index; circular ladder graph

AMS Mathematics Subject Classification (2010):05C15,05C85

1. Introduction

Let *G* = (*V*, *E*) be a finite, simple connected undirected graph. Graph coloring (vertex coloring or edge coloring) problems are important to model various real time applications [6] such as traffic planning, VLSI design, circuit routing, psychology, scheduling, transportation, etc. An edge coloring of a graph *G* is an assignment of colors to the edges of *G* so that no two adjacent edges are assigned the same color. The minimum number of colors required for an edge coloring of *G* is the edge chromatic number or the chromatic index and is denoted by $\chi'(G)$. Let $\Delta(G)$ denote the maximum degree of vertices of a graph Vizing [10] has shown that for any graph G, $\chi'(G)$ is either $\Delta(G)$ or $\Delta(G)+1$. The problem of determining the chromatic index of an arbitrary graph is a difficult task. Holyer [5] has proved that edge coloring problem is *NP*-complete. Edge coloring problems have wide variety of applications and are well studied in both computer science and mathematics [3]. Edge coloring problems are used to model various scheduling problems. In Cellular communication, frequency reusing is done by modeling it as an edge coloring problem to avoid co-channel interference [9]. In this paper, we consider skew edge coloring problems that are inspired from the study of skew Room squares by Brualdi [4]. The concept of skew chromatic index was introduced by Foregger and better upper bounds for *s*(*G*) was discussedwhen *G* is cyclic, cubic or bipartite [4].

A skew edge coloring of *G* is an assignment of an ordered pair of colors (a_i, b_i) to each edge e_i of *G* such that

(i) the a_i ’s form an edge coloring of *G*,

(ii) the b_i 's form an edge coloring of *G*, and

(iii) the pairs $\{a_i, b_i\}$ are all distinct.

The two edge colorings are referred to as component colorings of the skew edge coloring. The skew chromatic index *s*(*G*) is the minimum number of colors required for a skew edge coloring of *G*. For example, $s(C_4) = 3$. The first component colorings of the edges of C_4 are 1, 2, 1, 3 and the second component colorings of the edges of C_4are1, 2, 3, 2. It is observed that not less than three colors can be used for skew edge coloring of C_4. See Figure1.

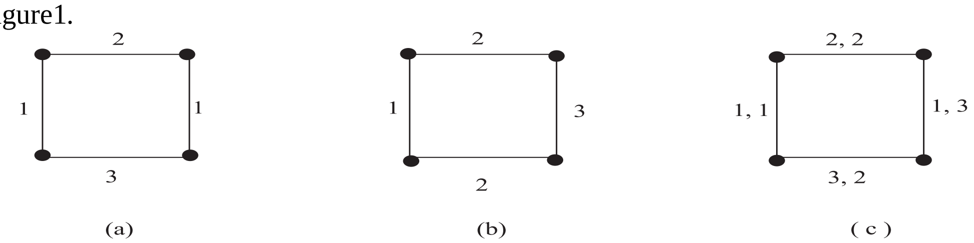

Figure 1: (a) First component coloring ofC_4, (b) Second component coloring of C_4, (c) Skew edge coloring of C_4

We see that $s(K_3) = 3$ where the two component colorings of K_3 are 1, 2, 3 and 2, 3, 1 and not less than three colors can be used for skew edge coloring ofK_3Theterminology used in this paper are as in [1]. In this paper, an algorithm is determined for skew edge coloring of circular ladder graphs. Also,we have solved the skew chromatic index of circular ladder graphs in polynomial time.

2. Lower bound on *s(G)*

Skew chromatic index, *s*(*G*) is defined as the minimum number of colors used in two edge colorings of *G* such that no two edges are assigned the same unordered pair of colors. Each component coloring of a skew edge coloring is itself an edge coloring. Therefore $s(G) \geq \chi'(G)$. Since $\Delta(G) \leq \chi'(G) \leq \Delta(G)+1$, we have $s(G) \geq \Delta(G)$. If '*k*' colors are used for skew edge coloring, then there are $\binom{k+1}{2}$ unordered pairsof colors [2] and this number must be at least as large as the number of edges in *G*. Let *k*(*m*) denote the smallest integer '*k*' satisfying $\binom{k+1}{2} \geq m$ where '*m*' denotes the number of edges in *G*. Thus the best lower bound for *s*(*G*) is $s(G) \geq \max\{\Delta(G), k(|E(G)|)\}$ [4].

3. Circular ladder graph

Definition 3.1.A circular ladder graph [11] is defined as the Cartesian product $C_n \times K_2$ where K_2 is the complete graph on two vertices and C_n is the cycle graph on *n* vertices.

In other words, acircular ladder graph*CL*(*n*)[8] is defined as the union of an outer cycle $\Gamma_O : u_1u_2 \ldots u_nu_1$ and inner cycle $\Gamma_I : v_1v_2 \ldots v_nv_1$ with additional edges $(u_i, v_i), i = 1, 2, \ldots, n$ called spokes. See Figure 2.

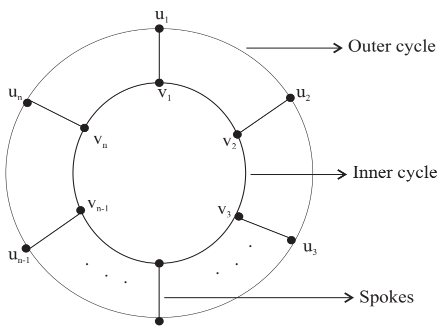

Figure2: Generalized circular ladder graph

Different labeling methods of cycles are available in the literature [7]. Here the inner cycle vertices are labeled as $v_1, v_2, ..., v_n$ and the outer cycle vertices are labeled as $u_{1,} u_2, .., u_n$.

We now present an algorithm for skew edge coloring of circular ladder graphs for $n \geq 5$.

3.1. Algorithm for skew edge coloring of circular ladder graphs

***Input*:** A circular ladder graphwith *2n* vertices and *m* = *3n* edges, $n \geq 5$.

***Step 1*:** Find the smallest positive integer '*k*' such that $\binom{k+1}{2} \geq m$.

***Step 2*:**Let {1, 2, 3,…,*k*} be the set of colors available to color the edgeswith $\binom{k+1}{2}$ unordered pairs of colors $\{a_i, b_i\}$ where a_i 's form the first component coloring and b_i 's form the second component coloring.

***Step 3*:** Compute the difference $d = |n-k|$.

Depending on the value of '*d*' the following cases arise.

Case (i):d=0

In this case, first assign colors from the set {(*j*, *j*)}, $j = 1, 2, \ldots, k$ taken in orderto the inner cycle $\Gamma_I : v_1v_2 ... v_nv_1$. Next assign colors from the set {(*j*, *j*+ 1)}, $j = 1, 2, \ldots, k-1$ taken in orderto the edges $(u_i, u_{i+1}), i = 1, 2, ..., n-1$ in the outer cycle Γ_O followed by the coloring of the remaining edge (u_n, u_1) using {*k*, 1}. Then assign colors from the set {(*j*, *j*+ 2)}, $j = 1, 2, \ldots, k-2$ taken in orderto the spokes $(u_i, v_i), i = n, 1, 2, ..., n-3$ followed by the coloring of the remaining spokes viz., (u_{n-2}, v_{n-2}) and (u_{n-1}, v_{n-1}) using the colors {*k* - 1, 1} and {*k*, 2}.See Figure3

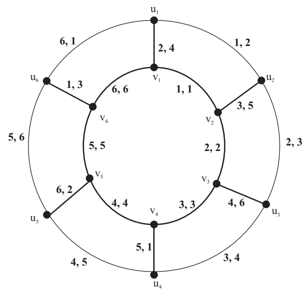

Figure 3. Skew edge coloring of *CL*(6) with 12 vertices and 18 edges

***Case (ii)*: *d*=1**

In this case, first color the spoke (u_1, v_1). Then every subsequent path of length three viz., $u_i u_{i+1} v_{i+1} v_i$, $i = 1, 2, ..., n-1$ consisting of one outer cycle edge, one spoke and one inner cycle edge are colored. The remaining two edges viz., (v_n, v_1) and (u_n, u_1) are colored taken in order.

In the aforementioned method, the first '*k*' edges are assigned colors from the set $\{(j, j)\}$, $j = 1, 2, \ldots, k$ taken in order. The second set of '*k*' edges are assigned colors from the set $\{(j, j+1)\}$, $j = 1, 2, \ldots, k-1$ and $\{k, 1\}$ taken in order. The third set of '*k*' edges are assigned colors from the set $\{(j, j+2)\}$, $j = 1, 2, \ldots, k-2$ and $\{k-1, 1\}, \{k, 2\}$ taken in order. This process is continued till all the edges are colored. See Figure 4.

***Case (iii)*: $2 \le d \le 5$**

In this case, first the edges of the inner cycle $\Gamma_I : v_1 v_2 ... v_n v_1$ are colored. Next the spokes (u_i, v_i), $i = 1, 2, ..., n$ are colored in clockwise direction.Then the edges of the outer cycle $\Gamma_O : u_n u_1 u_2 ... u_{n-1} u_n$ are colored.

In the aforementioned method, the first '*k*' edges are assigned colors from the set $\{(j, j)\}$, $j = 1, 2, \ldots, k$ taken in order. The second set of '*k*' edges are assigned colors from the set $\{(j, j+1)\}$, $j = 1, 2, \ldots, k-1$ and $\{k, 1\}$ taken in order. The third set of '*k*' edges are assigned colors from the set $\{(j, j+2)\}$, $j = 1, 2, \ldots, k-2$ and $\{k-1, 1\}, \{k, 2\}$ taken in order. This process is continued till all the edges are colored. See Figure 5.

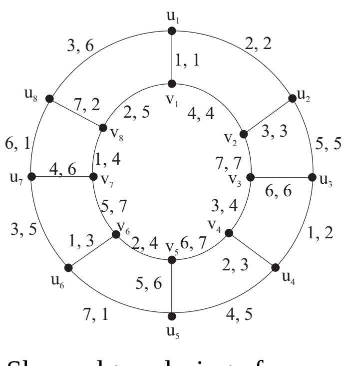

Figure 4: Skew edge coloring of *CL*(8) with 16 vertices and 24 edges

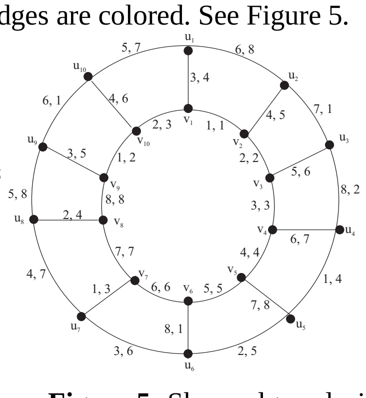

Figure 5: Skew edge coloring of *CL*(10)with 20 vertices and 30 edges

***Case (iv)*: *d* = 6**

In this case, first the edges of the inner cycle $\Gamma_I : v_1 v_2 \ldots v_n v_1$ are colored. Next all the spokes are colored in clockwise direction starting from (u_2, v_2). Then the edges of the outer cycle $\Gamma_O : u_2 u_3 u_4 \ldots u_n u_1 u_2$ are colored.

In the aforementioned method, the first 'k' edges are assigned colors from the set $\{(j, j)\}$, $j = 1, 2, \ldots, k$ taken in order. The second set of 'k' edges are assigned colors from the set $\{(j, j+1)\}$, $j = 1, 2, \ldots, k-1$ and $\{k, 1\}$ taken in order. The third set of 'k' edges are assigned colors from the set $\{(j, j+2)\}$, $j = 1, 2, \ldots, k-2$ and $\{k-1, 1\}, \{k, 2\}$ taken in order. This process is continued till all the edges are colored.

***Case (v)*: *d* > 6**

In this case, first color the spoke (u_1, v_1) followed by the coloring of edges (v_1, v_2) and (v_1, v_n) in the inner cycle.Next color the spokes (v_2, u_2) and (v_n, u_n).Then color the edges (u_1, u_2) and (u_n, u_1) in the outer cycle taken in order.All the remaining edges are colored in the following manner. Color the edges (v_i, v_{i+1}) followed by (v_j, v_{j-1}) in the inner cycle. Next color the spokes (v_{i+1}, u_{i+1}), (v_{j-1}, u_{j-1}). Then color the edges (u_i, u_{i+1}) followed by (u_j, u_{j-1}) in the outer cycle, where $i = 2, 3, 4, \ldots, \left\lceil \frac{n}{2} \right\rceil$ and $j = (n-i)+2$.

In the aforementioned method, the first 'k' edges are assigned colors from the set $\{(j, j)\}$, $j = 1, 2, \ldots, k$ taken in order. The second set of 'k' edges are assigned colors from the set $\{(j, j+1)\}$, $j = 1, 2, \ldots, k-1$ and $\{k, 1\}$ taken in order. The third set of 'k' edges are assigned colors from the set $\{(j, j+2)\}$, $j = 1, 2, \ldots, k-2$ and $\{k-1, 1\}, \{k, 2\}$ taken in order. This process is continued till all the edges are colored. See Figure 6.

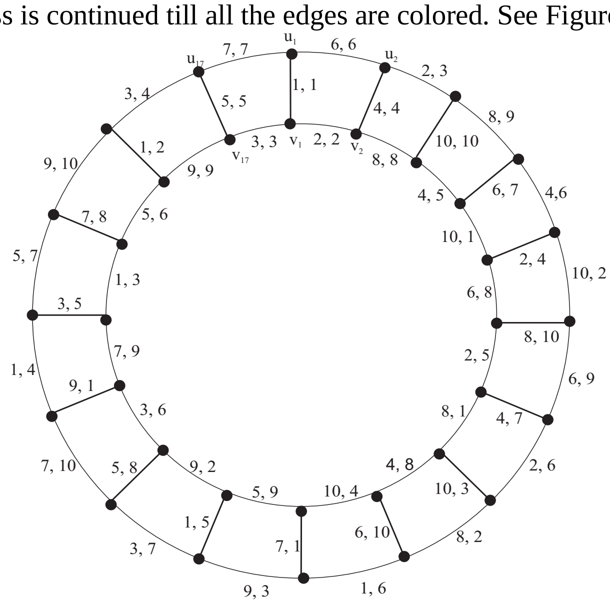

Figure 6: Skew edge coloring of $CL(17)$ with 34 vertices and 51 edges

***Output*:**Skew edge coloring of the given circular ladder graph.

Remark 3.1. For*n*< 5, skew edge coloring of circular ladder graphs exist and can be verified manually. It is observed that $s(CL(1)) = 3, s(CL(2)) = 4, s(CL(3)) = 4, s(CL(4)) = 5.$

Theorem 3.1. Let *G* be the circular ladder graph with *2n* vertices and *m* = *3n* edges, $n \geq 5$. Then $s(G) = k = \left\lceil \frac{-1+\sqrt{1+8m}}{2} \right\rceil$.

Proof: As there are *m* edges, at least *m*distinct unordered pairs of colors are required for skew edge coloring of *G*. If '*k*' colors are used, then there are $\binom{k+1}{2}$unordered pairs. This $\binom{k+1}{2}$must be at least as large as the number of edges in *G*. Therefore fix '*k'* in such a way that $\binom{k+1}{2} \geq m$. i.e, $\frac{(k+1)k}{2} \geq m$. It follows that $k^2 + k \geq 2m$. ie., $k^2 + k - 2m \geq 0$. Thus solving for '*k*' and taking the positive root, we obtain $k = \frac{-1+\sqrt{1+8m}}{2} \geq 0$, and its greatest integer $k = \left\lceil \frac{-1+\sqrt{1+8m}}{2} \right\rceil$ will be the minimum number of colors used in skew edge coloring. Therefore $s(G) = k = \left\lceil \frac{-1+\sqrt{1+8m}}{2} \right\rceil$.

Thus based on the lower bound for skew edge coloring, we obtain an optimal solution for skew chromatic index for circular ladder graphs.

4. Conclusion

In this paper, we have designed an algorithm for skew edge coloring of circular ladder graphs and also obtained an optimal solution for *s*(*G*). It would be interesting to identify the skew chromatic index for various networks.

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