Intuitionistic Fuzzy Pushdown Automata and Intuitionistic Fuzzy Context-Free Languages

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Abstract. In this paper intuitionistic fuzzy pushdown automata, acceptance by empty stack and acceptance by final states and their equivalence is proved. It follows that intuitionistic fuzzy pushdown automata with empty stack and IFPDAs are equivalent. We propose the notions of intuitionistic fuzzy context-free grammars (IFCFGs), intuitionistic fuzzy languages generated by IFCFGs. Additionally, we introduce the concepts of intuitionistic Chomsky normal form grammar and Greibach normal form grammar.

Keywords: Intuitionistic fuzzy set; instantaneous description; IFCFG; IFCNF; derivation

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1. Introduction

Intuitionistic fuzzy set (IFS) introduced by Atanassov [1], which emerges from the simultaneous consideration of the degrees of membership and nonmembership with a degree of hesitancy has been found to be very useful in dealing with problems involving vagueness and uncertainty. IFS theory has been found to support a wealth of important applications in many fields such as fuzzy multiple attribute decision making, fuzzy pattern recognition, medical diagnosis, fuzzy control and fuzzy optimization [2]. Since formal languages are not powerful enough in processing human languages, Lee and Zadeh [3] introduced the notion of fuzzy languages and gave some characterizations. Fuzzy grammars, automata and languages have contributed to the development of lexical analysis and in simulating fuzzy discrete event dynamical systems and hybrid systems [4].

To enhance the processing ability of fuzzy automata, the membership grades were extended to several general algebraic structures. Primarily, Qiu has established automata theory based on complete residuated lattice-valued logic [5]. Li and Pedrycz [6] have studied automata theory with membership values in lattice-ordered monoids. Jin and Li [7] have established a fundamental framework of fuzzy grammars based on lattices. Fuzzy pushdown automata theory based on complete residuated lattice-valued logic has been established in recent years by Xing et al. [8]. This paper deals with the notions of intuitionistic fuzzy context-free grammars and intuitionistic fuzzy pushdown automata and some results concerning them. Intuitionistic fuzzy context-free languages are
expected to reduce the gap between formal languages and the imprecision associated with natural languages.

The remaining part of the paper is arranged as follows. Section 2 describes some basic concepts of IFSs. Section 3 gives the definitions of intuitionistic fuzzy pushdown automata and languages. In section 4, we establish that every intuitionistic Fuzzy PDA that accepts intuitionistic Fuzzy Context Free Language with empty stack has an equivalent intuitionistic Fuzzy PDA that accepts the same language with final state and vice-versa. It follows that intuitionistic Fuzzy PDA with final states and empty stack are equivalent. Section 5 is devoted to the study of intuitionistic fuzzy context-free grammars (IFCFGs) and intuitionistic fuzzy context-free languages (IFCFLs). The notions of intuitionistic fuzzy Chomsky normal form (IFCNF) and intuitionistic fuzzy Greibach normal form (IFGNF) have been proposed. Conclusions and directions for future work are presented in Section 6.

2. Basic concepts

Definition 1. Let $X$ be the universe of discourse, an Intuitionistic fuzzy set (IFS) $A$ in $X$ is defined as an object of the form

$$A = \{ (x, \mu_{A}(x), \nu_{A}(x)) \mid x \in X \}$$

(1)

here the functions $\mu_{A} : X \rightarrow [0, 1]$ and $\nu_{A} : X \rightarrow [0, 1]$ denote the degree of membership and degree of non-membership respectively. For every element $x \in X$

$$0 \leq \mu_{A}(x) + \nu_{A}(x) \leq 1.$$  

(2)

For the sake of simplicity, we use the notation $A = (\mu_{A}, \nu_{A})$ instead of $A = \{ (x, \mu_{A}(x), \nu_{A}(x)) \mid x \in X \}$. Note that if $\mu_{A}(x) + \nu_{A}(x) = 1$ for every $x \in X$, then IFS $A$ reduces to a fuzzy set in $X$.

Definition 2. Let $\{A_{i} \mid i \in I\}$ be a family of IFSs in $X$. Then the infimum and supremum operations of IFSs are defined as follows:

$$\bigwedge A_{i} = \{ < x, \land \mu_{A}(x), \lor \nu_{A}(x) > \mid x \in X, i \in I \}$$

$$\bigvee A_{i} = \{ < x, \lor \mu_{A}(x), \land \nu_{A}(x) > \mid x \in X, i \in I \}$$

(3)

here $\land$ and $\lor$ denote the infimum and supremum of real numbers respectively.

Definition 3. Two IFSs $A = (\mu_{A}, \nu_{A})$ and $B = (\mu_{B}, \nu_{B})$ are said to be equal if $\mu_{A} = \mu_{B}$ and $\nu_{A} = \nu_{B}$.

Definition 4. Let $X$ and $Y$ be any two sets then an Intuitionistic Fuzzy Relation (IFR) from $X$ to $Y$ is an Intuitionistic fuzzy subset of $X \times Y$. The expression $R$ is given by

$$R = \{ < (x, y), \mu_{R}(x, y), \nu_{R}(x, y) > \mid x \in X, y \in Y \},$$

here the mappings $\mu_{R} : X \times Y \rightarrow [0, 1]$ and $\nu_{R} : X \times Y \rightarrow [0, 1]$ satisfy

$$0 \leq \mu_{R}(x, y) + \nu_{R}(x, y) \leq 1,$$  

for all $(x, y) \in X \times Y$.

(4)

Definition 5. Intuitionistic Fuzzy Binary Relation (IFBR) from $X$ to $X$ is an Intuitionistic fuzzy subset of $X \times X$. It is given by the relation

$$R = \{ < (x, y), \mu_{R}(x, y), \nu_{R}(x, y) > \mid (x, y) \in X \times X \},$$

where the mappings $\mu_{R} : X \times X \rightarrow [0, 1]$ and $\nu_{R} : X \times X \rightarrow [0, 1]$ satisfy
Intuitionistic Fuzzy Pushdown Automata and Intuitionistic Fuzzy Context-Free Languages

\[ 0 \leq \mu_R(x, y) + \nu_R(x, y) \leq 1, \text{ for all } (x, y) \in X \times X. \]

We will again use the notation \( R = (\mu_R, \nu_R) \) instead of \( R = \{(x, y), \mu_R(x, y), \nu_R(x, y) \} \) for \( (x, y) \in X \times X \). The reflexive and transitive closure of an IFBR \( R \) over \( X \) is written as \( R^* = R^\circ \) for \( n = 0 \) to \( \infty \). Here \( R^\circ = R \circ R \) for \( n = 0 \) and \( R^0 = (\mu_{id}, \nu_{id}) \) defined as \( \mu_{id}(x, y) = 1 \) if \( x = y \), \( 0 \) otherwise and \( \nu_{id}(x, y) = 0 \) if \( x = y \), \( 1 \) otherwise. For all \( (x, y) \in X \times X \),

**Definition 6.** Let \( R = (\mu_R, \nu_R) \) be an IFR from \( X \) to \( Y \) and \( S = (\mu_S, \nu_S) \) be an IFR from \( Y \) to \( Z \), the composition of IFRs \( R \) and \( S \) is an IFS \( R \circ S = (\mu_{R \circ S}, \nu_{R \circ S}) \) from \( X \) to \( Z \) given by

\[
\mu_{R \circ S}(x, z) = \vee (\mu_R(x, y) \wedge \mu_S(y, z)) \quad \forall y \in Y \\
\nu_{R \circ S}(x, z) = \land (\nu_R(x, y) \lor \nu_S(y, z)) \quad \forall y \in Y,
\]

for all \( (x, z) \in X \times Z \). \( (5) \)

**Definition 7.** Let \( A = (\mu_A, \nu_A) \) be an IFS from \( X \) to \( X \). Then the image set of \( A \) denoted by \( \text{Im}(A) \) is defined as \( \text{Im}(A) = \text{Im}(\mu_A) \cup \text{Im}(\nu_A) = \{ \mu_A(x) \mid x \in X \} \cup \{ \nu_A(x) \mid x \in X \} \).

**Definition 8.** For \( \lambda_1, \lambda_2 \in [0, 1] \) where \( \lambda_1 + \lambda_2 \leq 1 \), \( (\lambda_1, \lambda_2) \)-cut of IFS \( A \) is defined as

\[ A_{(\lambda_1, \lambda_2)} = \{ x \in X \mid \mu_A(x) \geq \lambda_1 \text{ and } \nu_A(x) \leq \lambda_2 \}. \]

And support set of \( A \) is defined by \( \text{supp}(A) = \{ x \in X \mid \mu_A(x) > 0, \nu_A(x) < 1 \} \). If \( \text{supp}(A) \) is finite, then \( A \) is called finite IFS.

3. Intuitionistic fuzzy pushdown automata (IFPDA)

**Definition 9.** An intuitionistic fuzzy pushdown automaton (IFPDA) is a seven tuple \( M = (Q, \Sigma, \Gamma, \delta, I, Z_0, F) \), where

- \( Q \) is a finite nonempty set of states;
- \( \Sigma \) is a finite nonempty set of input symbols;
- \( \Gamma \) is a finite nonempty set of stack symbols

\( \delta = (\mu_\delta, \nu_\delta) \) is a finite IF subset of \( Q \times (\Sigma \cup \{ \varepsilon \}) \times \Gamma \times (Q \times \Gamma^*) \) defined by

\[
\mu_\delta : Q \times (\Sigma \cup \{ \varepsilon \}) \times \Gamma \times (Q \times \Gamma^*) \to [0, 1] \text{ and } \\
\nu_\delta : Q \times (\Sigma \cup \{ \varepsilon \}) \times \Gamma \times (Q \times \Gamma^*) \to [0, 1]
\]

\( Z_0 \in \Gamma \) is the start stack symbol;

\( I = (\mu_I, \nu_I) \) and \( F = (\mu_F, \nu_F) \) are intuitionistic fuzzy subsets of \( Q \), which are called the intuitionistic fuzzy subsets of initial and final states respectively.

**Definition 10.** The state (configuration) of IFPDA is an IF subset of \( Q \times \Sigma^* \times \Gamma^* \) given by \( (q, w, u, \mu, \nu) \) which indicates that IFPDA is currently in state \( q \) with \( w \) as unread part of input string, \( u \) on the top of the stack with the degree of membership and nonmembership \( \mu, \nu \in [0, 1] \) respectively.

**Definition 11.** The move of an IFPDA \( M \) denoted by \( \overset{\rightarrow}{M} = (\mu_{M^-}, \nu_{M^-}) \) is an IFBR on \( Q \times \Sigma^* \times \Gamma^* \) to \( (Q \times \Sigma^* \times \Gamma^*) \) defined as \( (q, aw, Zy) \overset{\rightarrow}{M} (p, w, xy) = (\mu_{M^-}, \nu_{M^-}) \) where

\[
\mu_{M^-}((q, aw, Zy), (p, w, xy)) = \mu_\delta(q, a, Z, p, x) \text{ and } \\
\nu_{M^-}((q, aw, Zy), (p, w, xy)) = \nu_\delta(q, a, Z, p, x).
\]

Here \( p, q \in Q, a \in \Sigma \cup \{ \varepsilon \}, w \in \Sigma^* \text{ and } x, y \in \Gamma^* \). \( (6) \)
Chatrapathy K and V Ramaswamy

$\vdash_M^*$ is the reflexive and transitive closure of $\vdash_M$. When no confusion arises, we denote $\vdash_M$ by $\vdash$ and $\vdash_M^*$ by $\vdash^*$, respectively. Note that $\vdash^* = (\mu_\ast, v_\ast)$ is an intuitionistic fuzzy subset defined as follows:

If $(q_1, w_1, Y_1, \mu_1, v_1) \vdash (q_2, w_2, Y_2, \mu_2, v_2) \vdash \ldots \vdash (q_k, w_k, Y_k, \mu_k, v_k)$ is the sequence of moves in IFPDA then $(q_1, w_1, Y_1, \mu_1, v_1) \vdash^* (q_k, w_k, Y_k, \mu_k, v_k)$. Here $q_i \in Q$, $w_i \in \Sigma^*$, $Y_i \in \Gamma^*$ and $\mu_i, v_i \in [0, 1]$ for $i = 1$ to $k$.

**Definition 12.** Let $M = (Q, \Sigma, \Gamma, \delta, I, Z_0, F)$ be an IFPDA. The IF language accepted by $M$ can be defined in two ways:

i. Language accepted by $M$ with final states denoted by $L(M) = (\mu_{\text{LM}}, v_{\text{LM}})$ where $\mu_{\text{LM}}$ and $v_{\text{LM}}$ are fuzzy subsets of $\Sigma^*$ and are given by

$$
\mu_{\text{LM}}(\omega) = \vee \{ \mu_i(\alpha) \land (q_0, \omega, z_0), (p, e, r) \land \mu_F(p) | q_0, p \in Q, r \in \Gamma^* \}
$$

and

$$
v_{\text{LM}}(\omega) = \wedge \{ v_i(q_0) \lor (q_0, \omega, z_0), (p, e, r) \lor v_F(p) | q_0, p \in Q, r \in \Gamma^* \}
$$

for all $\omega \in \Sigma^*$.

ii. Language accepted by $M$ with empty stack denoted by $N(M) = (\mu_{\text{LN}}, v_{\text{LN}})$ where $\mu_{\text{LN}}$ and $v_{\text{LN}}$ are fuzzy subsets and are given by

$$
\mu_{\text{LN}}(\omega) = \vee \{ \mu_i(\alpha) \land (q_0, \omega, z_0), (p, e, r) \land \mu_F(p) | q_0, p \in F, r \in \Gamma^* \}
$$

and

$$
v_{\text{LN}}(\omega) = \wedge \{ v_i(q_0) \lor (q_0, \omega, z_0), (p, e, r) \lor v_F(p) | q_0, p \in F, r \in \Gamma^* \}
$$

for all $\omega \in \Sigma^*$.

**4. Equivalence of IFPDA with final states and IFPDA with empty stack**

**Proposition 1.** If $f$ is an intuitionistic fuzzy language accepted with final states by an IFPDA $M = (Q, \Sigma, \Gamma, \delta, I, Z_0, F)$, then $f$ is an IFS in $\Sigma^*$, and the image set of $f$ is finite.

**Proof.** First we will prove that $f = (\mu_f, v_f)$ is an IFS in $\Sigma^*$ by showing that $0 \leq \mu_f(\omega) + v_f(\omega) \leq 1$, for any $\omega = x_1 \cdots x_n, x_i \in \Sigma \cup \{e\}, i = 1 \ldots n$. Clearly,

$$
\mu_f(\omega) = \vee \{ \mu_i(\omega) \land (q_0, \omega, z_0), (p, e, r) \land \mu_F(p) | q_0, p \in F, r \in \Gamma^* \}
$$

and

$$
v_f(\omega) = \wedge \{ v_i(q_0) \lor (q_0, \omega, z_0), (p, e, r) \lor v_F(p) | q_0, p \in F, r \in \Gamma^* \}
$$

On the one hand, $0 \leq \mu_f(\omega) + v_f(\omega)$; on the other hand, there exists a sequence $q_0, q_1, \ldots, q_n \in Q, z_1 \ldots z_n \in \Gamma, r_1 \ldots r_n \in \Gamma^*$ such that

$$
\mu_f(\omega) = \mu_i(\omega) \land (q_0, \omega, z_0), (q_1, u_2 \cdots u_n, z_1, r_1)
$$

and

$$
v_f(\omega) \leq v_i(q_0) \lor v_F((q_0, \omega, z_0), (q_1, x_2 \cdots x_n, z_1, r_1)) \lor v_F((q_2, x_3 \cdots x_n, z_2, r_2)) \cdots
$$

Therefore, $\mu_f(\omega) + v_f(\omega) \leq (\mu_i(q_0) + v_F((q_0, \omega, z_0), (q_1, x_2 \cdots x_n, z_1, r_1))) + v_F((q_2, x_3 \cdots x_n, z_2, r_2)) \cdots
$$

and

$$
\mu_f(\omega) + v_f(\omega) \leq (\mu_i(q_0) + v_F((q_0, \omega, z_0), (q_1, x_2 \cdots x_n, z_1, r_1))) + v_F((q_2, x_3 \cdots x_n, z_2, r_2)) \cdots
$$

18
Intuitionistic Fuzzy Pushdown Automata and Intuitionistic Fuzzy Context-Free Languages

To prove that $\text{Im}(f)$ is finite, let $X = \text{Im}(\mu_1) \cup \text{Im}(\mu_2) \cup \text{Im}(\nu_1) \cup \text{Im}(\nu_2)$ and $Y = \text{Im}(\nu_1) \cup \text{Im}(\nu_2)$. Since $\delta = (\mu_5, \nu_3)$ and $F = (\mu_F, \nu_F)$ are finite IFS, $\mu_f(\omega) \in X$ and $\nu_f(\omega) \in Y$ for any $\omega = x_1 \cdots x_n$. Therefore, $\text{Im}(f) = \text{Im}(\mu_f) \cup \text{Im}(\nu_f)$ is finite.

**Proposition 2.** If $f$ is a fuzzy language accepted with empty states by some IFPDA $M = (Q', \Sigma, \Gamma, \delta', I', Z_0, F')$, then $f$ is an IFS in $\Sigma^*$, and the image set of $f$ is finite.

**Proof.** Similar to the Proposition 1

**Proposition 3.** Let $f$ be IFS in $\Sigma^*$, then the following statements are equivalent:

i. $f$ can be accepted by some IFPDA $M = (Q, \Sigma, \Gamma, \delta, I, Z_0, F)$,

ii. $f$ can be accepted by some IFPDM $M' = (Q', \Sigma, \Gamma, \delta', I', Z_0, F')$ where $q_0 \not\in Q'$

**Proof.** (i) $\rightarrow$ (ii).

Construct an IFPDA $M' = (Q', \Sigma, \Gamma, \delta', I', X_0, F')$ as follows:

$Q' = Q \cup \left\{ q_0 \right\}$, $\Gamma = \Gamma \cup \left\{ X_0 \right\}$, where $q_0 \not\in Q, X_0 \not\in \Gamma$.

Define an IFS $I'$ in $Q'$ by

$\mu'_i(q) = 1$ if $q = q_0, \mu_i(q)$ if $q \neq q_0$ and $\nu'_i(q) = 0$ if $q = q_0, \nu_i(q)$ if $q \neq q_0$.

Define an IFS $I'$ in $Q'$ by

$\mu_i'(q) = 0$ if $q = q_0, \mu_i(q)$ if $q \neq q_0$ and $\nu_i'(q) = 1$ if $q = q_0, \nu_i(q)$ if $q \neq q_0$.

Then for any $\omega = x_1 \cdots x_n \in \Sigma^*$, $i = 1 \ldots n$, we have

$\mu_{LM}^i(\omega) = \text{Im}(\mu_i(q) \wedge \mu_{\Sigma}((q, \omega, X_0), (p_0, \omega, Z_0), (q_1, x_2 \cdots x_n, z_1 r_1)) \wedge \cdots)
\wedge \mu_{\Sigma}((q_n, x_n, z_{n-1} r_{n-1}), (q_n, q, r_n)) \wedge \mu_{\Gamma}(q_n))
\mid q \in Q', p_0, q_1, \ldots, q_n \in Q', z_1, \ldots, z_{n-1} \in \Gamma, r_1, \ldots, r_n \in \Gamma^*\}
= \text{Im}(1 \wedge \mu_1(p_0) \wedge \mu_{\Sigma}((p_0, \omega, Z_0), (q_1, x_2 \cdots x_n, z_1 r_1)) \wedge \cdots)
\wedge \mu_{\Sigma}((q_n, x_n, z_{n-1} r_{n-1}), (q_n, q, r_n)) \wedge \mu_{\Gamma}(q_n))
\mid p_0, q_1, q_n \in Q, z_1, \ldots, z_{n-1} \in \Gamma, r_1, \ldots, r_n \in \Gamma^*\}
= \mu_{LM}^i(\omega)$, and

$\nu_{LM}^i(\omega) = \wedge \{ \nu_i'(q) \mid \nu_i'(q) \wedge \nu_{\Sigma}'((p_0, \omega, X_0), (p_0, \omega, 0_0), (q_1, x_2 \cdots x_n, z_1 r_1)) \wedge \cdots
\wedge \nu_{\Sigma}'((p_n, x_n, z_{n-1} r_{n-1}), (q_n, q, r_n)) \wedge \nu_{\Gamma}(q_n))
\mid q \in Q', (p_0, q_1, \ldots, q_n) \in Q', z_1, \ldots, z_{n-1} \in \Gamma, r_1, \ldots, r_n \in \Gamma^*\}$

19
There exists an IFPDA \( M' = (Q', \Sigma, \Gamma, \delta, q_0, Z_0, F') \) as follows:

\[ \begin{align*}
\mu(x) &= 1 \text{ if } q = q_0, 0 \text{ if } q \neq q_0 \\
\nu(x) &= 0 \text{ if } q = q_0, 1 \text{ if } q \neq q_0
\end{align*} \]

Then, it follows that \( M \) accepts \( f \).

**Proposition 4.** Let \( f \) be IFS in a nonempty set \( \Sigma' \). Then the following statements are equivalent:

(i) \( f \) can be accepted by an IFPDA \( M = (Q, \Sigma, \Gamma, \delta, I, Z_0, \phi) \) by empty state;

(ii) There exists an IFPDA \( M' = (Q', \Sigma, \Gamma, \delta', q_0, X_0, \phi) \) recognizing \( f \).

**Proof.** Similar to Proposition 3

**Proposition 5.** Let \( f \) be IFS in a nonempty set \( \Sigma' \). Then the following statements are equivalent:

(i) \( f \) can be accepted by an IFPDA \( M = (Q, \Sigma, \Gamma, \delta, I, Z_0, F) \) by final state;

(ii) There exists an IFPDA \( M' = (Q', \Sigma, \Gamma, \delta', q_0, X_0, \phi) \) recognizing \( f \) with empty stack, where \( q_0 \in Q' \)

**Proof.** (i) \( \rightarrow \) (ii).

Let \( M = (Q, \Sigma, \Gamma, \delta, I, Z_0, F) \) be a IFPDA that accepts \( f = (\mu, \nu) \) by final state.

Construct an IFPDA \( M' = (Q', \Sigma, \Gamma, \delta, I', X_0, F') \) as follows:

\[ Q' = Q \cup \{ q_0' \}, \Gamma = \Gamma \cup \{ X_0 \}, \text{ where } q_0' \notin Q, X_0 \notin \Gamma. \]

Define an IFS \( I' \) in \( Q' \) by

\[ \mu'(q) = 1 \text{ if } q = q_0', \mu(q) \text{ if } q \neq q_0', \nu'(q) = 0 \text{ if } q = q_0', \nu(q) \text{ if } q \neq q_0' \]

Define an IFS \( F' \) in \( Q' \) by

\[ \mu'(q) = 0 \text{ if } q = q_0', \mu(q) \text{ if } q \neq q_0', \nu'(q) = 1 \text{ if } q = q_0', \nu(q) \text{ if } q \neq q_0' \]

Define an IFS \( \delta' \) in \( Q' \times (\Sigma \cup \{ \varepsilon \}) \times \Gamma' \) by mappings \( \mu', \nu' \): \( Q' \times (\Sigma \cup \{ \varepsilon \}) \times \Gamma' \times Q' \times \Gamma' \) \( \rightarrow \) \([0, 1]\) as

\[ \mu'(q, e, X_0, p, Z_0) = \mu(p), \]

\[ \nu'(q, e, X_0, p, Z_0) = \nu(p), \]

\[ \mu'(q, e, \text{any}, q_0, \varepsilon) = \mu(q), \forall q \in F, \]

\[ \nu'(q, e, \text{any}, q_0, \varepsilon) = \nu(q), \forall q \in F, \]

\[ \mu'(q_0, e, \text{any}, q_0, \varepsilon) = 1 \]

\[ \nu'(q_0, e, \text{any}, q_0, \varepsilon) = 0 \]
Intuitionistic Fuzzy Pushdown Automata and Intuitionistic Fuzzy Context-Free Languages

Proposition 6. Construct an IFPDA $M' = (Q', \Sigma, \Gamma, \delta, \iota, Z_0, F)$ recognizing $f$ with final state $F$.

Therefore $L(M') = L(M)$.

Proposition 6. Let $f$ be IFS in a nonempty set $\Sigma^*$. Then the following statements are equivalent:
(i) $f$ can be accepted by an IFPDA $M = (Q, \Sigma, \Gamma, \delta, \iota, Z_0, \phi)$ by empty stack;
(ii) There exists an IFPDA $M' = (Q', \Sigma, \Gamma', \delta', q_0, X_0, F)$ recognizing $f$ with final state $F$.

Proof. (i) $\Rightarrow$ (ii).

Let $M = (Q, \Sigma, \Gamma, \delta, \iota, Z_0, \phi)$ be an IFPDA that accepts $f = (\mu, \nu)$ by empty stack.

Construct an IFPDA $M' = (Q', \Sigma, \Gamma', \delta', \iota', X_0, F)$ as follows:

$q' = Q \cup \{q'_0\}, \Gamma' = \Gamma \cup \{X_0\}$, where $q_0 \notin Q, X_0 \notin \Gamma$.

Define an IFS $f'$ in $Q'$ by

$\mu' (q) = 1$ if $q = q'_0, \mu'(q)$ if $q \neq q'_0$ and $\nu' (q) = 0$ if $q = q'_0, \nu' (q)$ if $q \neq q'_0$.
Define an IFS $F'$ in $Q'$ by

$\mu_{F'}(q) = 0$ if $q = q_0$, $\mu_F(q)$ if $q \neq q_0$ and

$\nu_{F'}(q) = 1$ if $q = q_0$, $\nu_F(q)$ if $q \neq q_0$

Define an IFS $\delta'$ in $Q' \times (\Sigma \cup \{\varepsilon\}) \times \Gamma' \times Q' \times \Gamma''$ by mappings $\mu_{\delta}'$, $\nu_{\delta}'$; $Q' \times (\Sigma \cup \{\varepsilon\}) \times \Gamma' \times Q' \times \Gamma'' \rightarrow [0, 1]$ defined as

$\mu_{\delta}(q_0', \varepsilon, X_0, p, Z_0) = \mu_F(p)$,

$\nu_{\delta}(q_0', \varepsilon, X_0, p, Z_0) = \nu_F(p)$,

$\mu_{\delta}(q, \varepsilon, X_0, q_0, \varepsilon) = 1$,

$\nu_{\delta}(q, \varepsilon, a, z, p, \gamma) = \mu_{\delta}(q, a, z, p, \gamma)$,

Otherwise, $\nu_{\delta}(q_0', a, z, p, \gamma) = 0$ and $\nu_{\delta}(q_0', a, z, p, \gamma) = 1$.

Then for any $\omega = x_1 \cdot \cdot \cdot x_n \in \Sigma^*$, $x_i \in \Sigma \cup \{\varepsilon\}$, $i = 1 \ldots n$, we have

$\mu_{LM'}(\omega) = V \{ \mu_i'(q) \forall_{\mu_{LM}}(\{(q, \omega, X_0), (p_0, \omega, Z_0), (q_1, x_2 \cdot \cdot \cdot x_n, z_0 r_1') \})$

$\wedge \forall_{\mu_{LM}}(\{(q_1, x_2 \cdot \cdot \cdot x_n, z_1 r_1'), (q_2, x_3 \cdot \cdot \cdot x_n, z_2 r_2') \}) \wedge \ldots$

$\wedge \forall_{\mu_{LM}}(\{(q_{n-1}, x_n, z_{n-1} r_{n-1}'), (q_n, \varepsilon, r_n) \}) \wedge 1 \}$

$\nu_{LM'}(\omega) = V \{ \nu_i'(q) \forall_{\nu_{LM}}(\{(q, \omega, X_0), (p_0, \omega, Z_0), (q_1, x_2 \cdot \cdot \cdot x_n, z_0 r_1') \})$

$\forall_{\nu_{LM}}(\{(q_1, x_2 \cdot \cdot \cdot x_n, z_1 r_1'), (q_2, x_3 \cdot \cdot \cdot x_n, z_2 r_2') \}) \wedge \ldots$

$\forall_{\nu_{LM}}(\{(q_{n-1}, x_n, z_{n-1} r_{n-1}'), (q_n, \varepsilon, r_n) \}) \wedge 1 \}$

Therefore $L(M') = L(M)$.
Intuitionistic Fuzzy Pushdown Automata and Intuitionistic Fuzzy Context-Free Languages

**Theorem 1.** If $L$ is an intuitionistic fuzzy language (IFL) accepted by an IFPDA $M$ with final states, there exists an IFPDA $M'$ that accepts $L$ with empty stack.

**Proof:** By Proposition 5 and Proposition 6, it follows that the two IFLs accepted by empty stack and final state are equivalent.

5. Intuitionistic fuzzy context-free grammars and languages

**Definition 12.** An Intuitionistic Fuzzy Grammar (IFG) is a system $G = (N, T, I, P)$, where

i. $N$ is a finite nonempty set of variables;

ii. $T$ is a finite nonempty set of terminals, $T \cap N = \emptyset$;

iii. $I$ intuitionistic fuzzy set of start symbols (variables);

iv. $P$ is a finite set of productions over $T \cup N$, $P = \{ x \to \beta \mid x \in (N \cup T)^*; (N \cup T)^* \}$, $y \in (N \cup T)^*$ is an IFS over $(N \cup T)^*$ defined as $P = (\mu_P, \nu_P)$ where

$$
\mu_P(x, y) = \mu_P(x \to y) \quad \text{and} \quad \nu_P(x, y) = \nu_P(x \to y)
$$

are membership degree and nonmembership degree that $x$ will be replaced by $y$, respectively.

For $\alpha, \beta \in (N \cup T)^*$, if $x \to y \in P$, then $\alpha \beta\gamma$ is said to be directly derivable from $\alpha \beta \gamma$, denoted by $\alpha \beta \gamma \Rightarrow \alpha \beta\gamma$, and define $\mu_P(\alpha \beta \gamma \Rightarrow \alpha \beta\gamma) = \mu_P(x \to y), \nu_P(\alpha \beta \gamma \Rightarrow \alpha \beta\gamma) = \nu_P(x \to y)$. For $\alpha_1, \ldots, \alpha_m$ are strings in $(N \cup T)^*$, $\alpha_1 \to \alpha_2, \ldots, \alpha_m$, $\alpha_1$ is said to derive $\alpha_m$ in $G$, or equivalently, $\alpha_m$ is derivable from $\alpha_1$. This is expressed by $\alpha_1 \Rightarrow^* \alpha_m$ or simply $\alpha_1 \Rightarrow \alpha_m$. The expression $\alpha_1 \Rightarrow \alpha_2 \Rightarrow \cdots \Rightarrow \alpha_m$ is referred to as a derivation chain from $\alpha_1$ to $\alpha_m$.

**Proposition 7.** The language generated by IFG is an intuitionistic fuzzy language (IFL).

**Proof:** An intuitionistic fuzzy grammar $G$ generates an intuitionistic fuzzy language $L(G) = (\mu_G, \nu_G)$ in the following manner. For any string $\omega_n \in T^*$, $n \geq 1$, $\mu_G(\omega_n) = \bigvee \{ \mu_i(\omega_0) \land \mu_P(\omega_0 \Rightarrow \omega_1), \ldots, \mu_P(\omega_0 \Rightarrow \omega_n) \mid \omega_0 \in N, \omega_1, \ldots, \omega_n \in (N \cup T)^* \}$, and $\nu_G(\omega_n) = \bigwedge \{ \nu_P(\omega_0 \Rightarrow \omega_1), \ldots, \nu_P(\omega_0 \Rightarrow \omega_n) \mid \omega_0 \in N, \omega_1, \ldots, \omega_n \in (N \cup T)^* \}$. $\mu_G(\omega_n) \land \nu_G(\omega_n)$ express the membership degree and nonmembership degree of $\omega_n$ in the language generated by grammar $G$, respectively. Obviously, $L(G) = (\mu_G, \nu_G)$ is well defined. And also, for any string $\omega_n \in T^*$, $n \geq 1$, there is a derivation from $\omega_0$ to $\omega_n$, that is, $\omega_0 \Rightarrow \omega_1 \Rightarrow \cdots \Rightarrow \omega_{n-1} \Rightarrow \omega_n$. Therefore,

$$
\mu_G(\omega_n) + \nu_G(\omega_n) \leq \mu_G(\omega_0) + \nu_G(\omega_0) \land \nu_P(\omega_0 \Rightarrow \omega_1) \land \cdots \land \nu_P(\omega_n \Rightarrow \omega_n) = (\mu_G(\omega_0) + \nu_G(\omega_0)) \land (\mu_G(\omega_1) + \nu_G(\omega_1)) \land \cdots \land (\mu_G(\omega_n) + \nu_G(\omega_n)) \leq 1.
$$

**Definition 13.** For any intuitionistic fuzzy grammars $G_1$ and $G_2$, if $L(G_1) = L(G_2)$ in the sense of equality of intuitionistic fuzzy sets, then the grammars $G_1$ and $G_2$ are said to be equivalent.

**Proposition 8.** Let $A$ be IFS over $T^*$. Then the following statements are equivalent:

(i) $A$ is generated by a certain IFG $G = (N, T, I, P)$

(ii) $A$ is generated by an IFG $G' = (N', T', P', S)$
Chatrapathy K and V Ramaswamy

**Proof:** (i) $\rightarrow$ (ii).

Let $A$ be generated by an intuitionistic fuzzy grammar $G = (N, T, P, I)$. Then we construct an IFG $G' = (N', T', P', S)$ as follows:

- $N' = N \cup \{ S \}$, $S \notin N'$; $T' = T$, $P' = P \cup P_1$, where
- $P_1 = \{ S \rightarrow q \mid q \in \omega \supp (I), \mu_P (S \rightarrow q) = \mu_I (q), \nu_P (S \rightarrow q) = \nu_I (q) \}$.

Next we show that $L(G') = L(G)$.

In fact, $G' = (N', T', P', I')$ where $I'$ is an IFS over $N'$ defined as $\mu'_I(S) = 1$, $\nu'_I(S) = 0$; $\mu'_I(q) = 0$ and $\nu'_I(q) = 1 \forall q \in N$. For any $\omega_n \in T'$, $n \geq 1$,

- $\mu'_I(\omega_n) = \bigvee \{ \mu'_I(\omega_0) \land \mu_P (\omega_0 \Rightarrow \omega_1) \land \cdots \land \mu_P (\omega_{n-1} \Rightarrow \omega_n) \mid \omega_0 \in N', \omega_1, \ldots, \omega_{n-1} \in (N' \cup T') \} \land \mu_P (\omega_{n-1} \Rightarrow \omega_n) \}(\omega_1, \ldots, \omega_{n-1} \in (N' \cup T'))$
- $\nu'_I(\omega_n) = \bigvee \{ \nu'_I(\omega_0) \lor \nu_P (\omega_0 \Rightarrow \omega_1) \lor \cdots \lor \nu_P (\omega_{n-1} \Rightarrow \omega_n) \mid \omega_0 \in N', \omega_1, \ldots, \omega_{n-1} \in (N' \cup T') \}$

Hence $L(G') = L(G)$.

(i) $\rightarrow$ (ii). The proof is obvious.

**Definition 14.**

1. An IFG $G = (N, T, P, I)$ is said to be intuitionistic context free grammar (IFCFG) if it has only productions of the form $A \rightarrow \omega \in P$ with $A \in N$ and $\omega \in (N \cup T')^*$. And the language $L(G)$, generated by the IFCFG $G$, is said to be an intuitionistic fuzzy context-free language (IFCFL).

2. An IFCFG $G = (N, T, P, S)$ is called an intuitionistic fuzzy Chomsky normal form (IFCNF) if it has only productions of the form $A \rightarrow BC/\alpha \in P$ or $S \rightarrow \varepsilon$, where $A, B, C \in N, B \neq S, C \neq S$ and $\alpha \in T$.

3. An IFCFG $G = (N, T, P, S)$ is called an intuitionistic fuzzy Greibach normal form (IFGNF) if all the productions are of the form $A \rightarrow ax \in P$ or $S \rightarrow \varepsilon$, where $A \in N$, $a \in T$, and $x \in (N - \{ S \})^*$

**6. Conclusions**

We have defined intuitionistic fuzzy pushdown automata and the two different ways accepting languages by empty stack and final states. We also established that the languages accepted by IFPDA (with final state) are equivalent to those accepted by IFPDA (with empty stack). Secondly, we have introduced the notions of IFCFGs, IFCNFs, and IFGNFs. Our future work will be on related concepts, such as the equivalence of IFCFG and IFPDA. Converting IFCFG to IFPDA and vice-versa, algebraic properties of IFCFLs.
Intuitionistic Fuzzy Pushdown Automata and Intuitionistic Fuzzy Context-Free Languages

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