Spectral Conditions for a Graph to be $k$-Connected

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Abstract. Using spectral radius and signless Laplacian spectral radius, we in this note present sufficient conditions for a graph to be $k$-connected.

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1. Introduction

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [2]. For a graph $G = (V, E)$, we use $n$ and $e$ to denote its order $|V|$ and size $|E|$, respectively. We use $\delta = d_1 \leq d_2 \leq \cdots \leq d_n = \Delta$ to denote the degree sequence of a graph. The eigenvalues of a graph $G$ are defined as the eigenvalues of its adjacency matrix $A(G)$. The largest eigenvalue, denoted $\rho(G)$, of a graph $G$ is called the spectral radius of $G$. The signless Laplacian eigenvalues of a graph $G$ are defined as the eigenvalues of the matrix $Q(G) = D(G) + A(G)$, where $D(G)$ is the diagonal matrix $\text{diag}(d_1, d_2, \ldots, d_n)$ and $A(G)$ is the adjacency matrix of $G$. The largest signless Laplacian eigenvalue, denoted $q(G)$, of a graph $G$ is called the signless Laplacian spectral radius of $G$.

2. Main results

In [4], Li obtained sufficient conditions which are based on the spectral radius for some Hamiltonian properties of graphs. In [5], Li obtained sufficient conditions which are based on the signless Laplacian spectral radius for some Hamiltonian properties of graphs. Using similar ideas as the ones in [4] and [5], we will present sufficient conditions which are based on the spectral radius or the signless Laplacian spectral radius for a graph to be $k$-connected. The results are as follows.

Theorem 1. Let $G$ be a connected graph of order $n \geq 2$ and let $1 \leq k \leq n - 1$. If

$$\rho > \frac{n+k-5+\sqrt{(n+k-5)^2+8(n+k-3+(n-k-1)(\Delta-1))}}{4},$$

then $G$ is $k$-connected.

Theorem 2. Let $G$ be a connected graph of order $n \geq 2$ and let $1 \leq k \leq n - 1$. If
Rao Li

\[ q > \frac{(\Delta - 1) + \sqrt{(\Delta - 1)^2 + 8((n-2)\Delta + k - 1)}}{2} \]

then \( G \) is \( k \)-connected.

Since \( \Delta \leq n - 1 \), Theorem 1 and Theorem 2 have the following Corollary 1 and Corollary 2, respectively.

**Corollary 1.** Let \( G \) be a connected graph of order \( n \geq 2 \) and let \( 1 \leq k \leq n - 1 \). If

\[ \rho > \frac{n + k - 5 + \sqrt{(n + k - 5)^2 + 8(n + k - 3 + (n - k - 1)(n - 2))}}{4} \]

then \( G \) is \( k \)-connected.

**Corollary 2.** Let \( G \) be a connected graph of order \( n \geq 2 \) and let \( 1 \leq k \leq n - 1 \). If

\[ q > \frac{(n - 2) + \sqrt{(n - 2)^2 + 8((n - 2)(n - 1) + k - 1)}}{2} \]

then \( G \) is \( k \)-connected.

In order to prove Theorem 1 and Theorem 2, we need the following results as our lemmas.

**Lemma 1.** ([11]) Let \( G \) be a graph of order \( n \geq 2 \) with degree sequence \( d_1 \leq d_2 \leq \cdots \leq d_n \) and let \( 1 \leq k \leq n - 1 \). If

\[ 1 \leq i \leq \left\lfloor \frac{n - k + 1}{2} \right\rfloor, d_i \leq i + k - 2 \Rightarrow d_{n-k+1} \geq n - i, \]

then \( G \) is \( k \)-connected.

**Lemma 2.** ([6]) Let \( G \) be a connected graph with degree sequence \( d_1 \leq d_2 \leq \cdots \leq d_n \). Then for each \( i \) with \( 1 \leq i \leq n \),

\[ \rho(G) \leq \frac{d_i - 1 + \sqrt{(d_i + 1)^2 + 4(i-1)(d_n - d_i)}}{2} \]

Moreover, if \( i = n \), the equality holds if and only if \( G \) is a regular graph. If \( 1 \leq i \leq n - 1 \), the equality holds if and only if \( G \) is either a regular graph or bidegreed graph in which \( d_n = d_{n-1} = \cdots = d_{n-i+2} = n - 1 \) and \( d_{n-i+1} = d_{n-i} = \cdots = d_1 = \delta \).

**Lemma 3.** ([7]) Let \( G \) be a connected graph with degree sequence \( d_1 \leq d_2 \leq \cdots \leq d_n \). Then for each \( i \) with \( 1 \leq i \leq n \),

\[ q(G) \leq \frac{\left(d_n + 2d_i - 1 + \sqrt{(2d_i - d_n + 1)^2 + 8(i-1)(d_n - d_i)}\right)}{2} \]

Moreover, if \( i = n \), the equality holds if and only if \( G \) is a regular graph. If \( 1 \leq i \leq n - 1 \), the equality holds if and only if \( G \) is either a regular graph or bidegreed
Spectral Conditions for a Graph to be $k$ Connected

Let $G$ be a graph of order $n$ with maximum degree $\Delta$. Then $q(G) \leq 2\Delta$.

Moreover, if $G$ is connected, then equality holds if and only if $G$ is regular.

Proof of Theorem 1. Let $G$ be a graph satisfying the conditions in Theorem 1. Suppose that $G$ is not $k$-connected. Then, from Lemma 1, there exists an integer $j$ such that $1 \leq j \leq \left\lfloor \frac{n-k}{2} \right\rfloor \leq \frac{n-k}{2}$, $d_j \leq j + k - 2$, and $d_{n-k+1} \leq n - j - 1$. Obviously, $d_j \geq 1$. Let $i = j$ in Lemma 2. Then we have that

$$\rho \leq \frac{d_j - 1 + \sqrt{(d_j + 1)^2 + 4(j-1)(d_n - d_j)}}{2}.$$

Thus

$$\rho^2 \leq \rho(d_j - 1) + d_j + (j-1)(d_n - d_j).$$

Therefore

$$\rho^2 \leq \rho(j + k - 3) + j + k - 2 + (j-1)(\Delta - 1).$$

Hence

$$\rho^2 \leq \rho \left( \frac{n-k+1}{2} + k - 3 \right) + \frac{n-k+1}{2} + k - 2 + \left( \frac{n-k+1}{2} - 1 \right)(\Delta - 1).$$

By solving the inequality, we have that

$$\rho \leq n + k - 5 + \sqrt{(n + k - 5)^2 + 8(n + k - 3 + (n-k-1)(\Delta - 1))},$$

which is a contradiction.

This completes the proof of Theorem 1.

Proof of Theorem 2. Let $G$ be a graph satisfying the conditions in Theorem 2. Suppose that $G$ is not $k$-connected. Then, from Lemma 1, there exists an integer $j$ such that $1 \leq j \leq \left\lfloor \frac{n-k+1}{2} \right\rfloor \leq \frac{n-k+1}{2}$, $d_j \leq j + k - 2$, and $d_{n-k+1} \leq n - j - 1$. Obviously, $d_j \geq 1$. Let $i = j$ in Lemma 3. Then we have that

$$q \leq \frac{d_n + 2d_j - 1 + \sqrt{(2d_j - d_n + 1)^2 + 8(j-1)(d_n - d_j)}}{2}.$$

Thus

$$q^2 \leq q(d_n + 2d_j - 1) + 2d_j(1-d_n) + 2(j-1)(d_n - d_j).$$

Therefore,

$$q^2 \leq q(d_n - 1) + 2d_j(q - d_n + 1) + 2(j-1)(d_n - d_j).$$

By Lemma 4, we have that
Rao Li

\[ q^2 \leq q(\Delta - 1) + 2(j + k - 2)(\Delta + 1) + (n - k - 1)(\Delta - 1). \]

Hence

\[ q^2 \leq q(\Delta - 1) + (n + k - 3)(\Delta + 1) + (n - k - 1)(\Delta - 1). \]

By solving the inequality, we have that

\[ q \leq \frac{(\Delta - 1) + \sqrt{(\Delta - 1)^2 + 8((n - 2)\Delta + k - 1)}}{2}, \]

which is a contradiction.

This completes the proof of Theorem 2. \qed

REFERENCES

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