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Some Connectivity Concepts in Bipolar Fuzzy Graphs

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Abstract. In this article, we introduce some connectivity concepts in bipolar fuzzy graphs. Analogous to fuzzy cutvertices and fuzzy bridges in fuzzy graphs, bipolar fuzzy cutvertices and bipolar fuzzy bridges are introduced and characterized. We also propose the concepts of gain and loss for paths and pairs of vertices. Connectivity in complete bipolar fuzzy graphs is also discussed.

Keywords: Fuzzy relation; bipolar fuzzy graph; gain; loss; bipolar fuzzy cutvertex; bipolar fuzzy bridge.

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1. Introduction

The introduction of fuzzy sets by Zadeh [25] in 1965 changed the face of science and technology to a great extent. Fuzzy sets paved the way for a new philosophical thinking of ‘Fuzzy Logic’ which now, is an essential concept in artificial intelligence. This logic is also used in the production of a large number of electronic and other household items with ‘partial’ thinking ability. Fuzzy logic and the theory of fuzzy sets have been applied widely in areas like information theory, pattern recognition, clustering, expert systems, database theory, control theory, robotics, networks and nanotechnology [13, 22]. As a consequence, Rosenfeld [15], Yeh and Bang [24] introduced fuzzy graphs independently in 1975 and Akram [1] introduced bipolar fuzzy graphs in 2010.

Rosenfeld [15] considered fuzzy relations on fuzzy sets and developed the structure of fuzzy graphs, obtaining the analogues of several graph-theoretical concepts while, Yeh and Bang [24] introduced various connectedness concepts of graphs and digraphs into fuzzy graphs. Several authors found deeper results and fuzzy analogues of many other graph-theoretical concepts. Bhattacharya and Suraweera [5, 6] studied connectivity algorithms in fuzzy graphs. Zadeh [25,26] analyzed more fuzzy relations. Bhutani and Rosenfeld [7, 8] introduced strong arcs, fuzzy end nodes and automorphisms in fuzzy graphs. Mordeson and Nair [12, 13] discussed many connectivity concepts while Sunitha and Vijayakumar [19, 20] studied fuzzy trees, complement of a fuzzy graph and blocks in fuzzy graphs. Mathew and Sunitha [10, 11] studied different types of arcs in fuzzy graphs and connectivity in fuzzy graphs. The important definitions and results in fuzzy graph theory are listed in [21]. Akram et.al [1, 2] introduced many concepts like bipolar fuzzy graph, interval-valued fuzzy graphs, intuitionistic fuzzy graphs, etc.

Analogous to fuzzy hypergraph, the concept of bipolar fuzzy hypergraph was introduced in [16]. Further studies in bipolar fuzzy graphs can be seen in [14, 18].

Fuzzy graphs found an increasing number of applications in modeling real time systems where the information inherent in the system varies with different levels of precision. Fuzzy models are becoming useful because of their capability of reducing the difference between the traditional numerical models used in engineering and science and symbolic models used in expert systems.

In this article, we discuss connectivity in bipolar fuzzy graphs. Bipolar fuzzy graphs can be used to model many problems in economics, operations research, etc. involving two similar, but opposite type of qualitative variables like success and failure, gain and loss, etc. Hence, the study of connectivity in this structure is of utmost importance. We discuss concepts like gain and loss of a pair of vertices, maximum gain and minimum loss paths, bipolar cutvertices, bipolar bridges, etc.

2. Preliminaries

This section contains a quick review of the basic definitions and results in graph theory and fuzzy graph theory which is required for this article. First, we recollect some basic ideas from undirected graphs [9].

Recall that, a graph is an ordered pair $G = (V, E)$, where V is the set of vertices of G and E is the set of edges of G. A subgraph of a graph $G = (V, E)$is a graph $H = (W, F)$, where $W \subseteq V$ and $F \subseteq E$. A simple graph is an undirected graph that has no loops (edges starting and ending at the same vertex) and not more than one edge between any two different vertices. A simple graph with a single vertex is called trivial graph and one with no edges is called an empty graph.

Two vertices x and y in an undirected graph G are said to be adjacent in G if (x, y) is an edge of G. An edge may be also represented as xy or yx. The set of all vertices adjacent to a vertex x in G is called the neighbour set of x, denoted by $N(x)$. A $v_0 - v_n$ path P in G is an alternating sequence of vertices and edges $v_0, e_1, v_1, e_2, \cdots, e_n, v_n$ such that $v_i v_{i+1}$ is an edge for $i = 0, 1, 2, \cdots, n-1$. The number of edges in P is called the length of P and P is called a closed path or a cycle if $v_0 = v_n$. A graph G is called connected if there is a path joining any two vertices in G. A graph G is called a tree if it is connected and acyclic. The number of connected components in G is denoted by $\omega(G)$. A vertex v of G is said to be a cutvertex of G if $\omega(G - v) > \omega(G)$. Similarly, an edge e of G is called a cutedge if $\omega(G - e) > \omega(G)$. G is said to be a complete graph if all the vertices in G are pairwise adjacent.

Definition 2.1. [25, 26] A fuzzy subset μ on a set X is a map $\mu: X \rightarrow [0,1]$. A map $\nu: X \times X \rightarrow [0,1]$ is called a fuzzy relation on μ if $\nu(x, y) \leq min\{\mu(x), \mu(y)\}$ for all $x, y \in X$. A fuzzy relation ν is reflexive if $\nu(x, x) = \mu(x)$ for all $x \in X$. ν is called symmetric if $\nu(x, y) = \nu(y, x)$ for all $x, y \in X$. Rosenfeld [15] defined a fuzzy graph as follows.

Definition 2.2. [13] A fuzzy graph (f-graph) is a pair $G: (\sigma, \mu)$ where σ is a fuzzy subset on a set V and μ is a fuzzy relation on σ. It is assumed that V is finite and nonempty, μ is reflexive and symmetric. Thus, if $G: (\sigma, \mu)$ is a fuzzy graph, then $\sigma: V \rightarrow [0,1]$ and

$\mu: V \times V \to [0,1]$ is such that $\mu(u, v) \le \sigma(u) \wedge \sigma(v)$ for all $u, v \in V$ where $\wedge$ denotes the minimum.

We denote the underlying graph of a fuzzy graph $G: (\sigma, \mu)$ by $G^*: (\sigma^*, \mu^*)$ where $\sigma^* = \{u \in V : \sigma(u) > 0\}$ and $\mu^* = \{(u, v) \in V \times V : \mu(u, v) > 0\}$. In examples, if σ is not specified, it is chosen suitably. Also, $G: (\sigma, \mu)$ is called a trivial fuzzy graph if $G^*: (\sigma^*, \mu^*)$ is trivial. That is, $\sigma^*$is a singleton set.

Definition 2.3. [1] Let X be a nonempty set. A bipolar fuzzy set B in X is an object having the form, $B = \{(x, \mu^P(x), \mu^N(x)): x \in X\}$ or $\{(x, \mu^+(x), \mu^-(x)): x \in X\}$ where $\mu^+: X \to [0,1]$ and $\mu^-: X \to [-1,0]$ are mappings. $\mu^+(x)$ is said to be satisfaction degree of x and $\mu^-(x)$, nonsatisfaction degree.

Definition 2.4. A bipolar fuzzy set $B = ({\mu_B}^+, {\mu_B}^-)$ on $X \times X$ is called a bipolar fuzzy relation on X. B is symmetric if ${\mu_B}^+(x, y) = {\mu_B}^+(y, x)$ and ${\mu_B}^-(x, y) = {\mu_B}^-(y, x)$ for all $x, y \in X$.

Definition 2.5. A bipolar fuzzy graph is defined to be a pair $G = (A, B)$ where $A = ({\mu_A}^+, {\mu_A}^-)$ is a bipolar fuzzy set in a nonempty and finite set V and $B = ({\mu_B}^+, {\mu_B}^-)$ is a bipolar fuzzy set on V_2 satisfying ${\mu_B}^+(\{x, y\}) \le min\,\{{\mu_A}^+(x), {\mu_A}^+(y)\}$ and ${\mu_B}^-(\{x, y\}) \ge max\,\{{\mu_A}^-(x), {\mu_A}^-(y)\}$ for all $\{x, y\} \in V_2$. V_2 refers to the set of all $2-$element subsets of V.

This definition is a slightly modified form of the Definition 3.1 in [23]. In Definition 3.1, "bipolar fuzzy graph of a graph $G = (V, E)$" is defined. Since bipolar fuzzy graph is a generalization of graph, a definition independent of graphs is more appropriate. Also, in Definition 3.1, the memberships of elements in $V_2 - E$ are defined in two different ways.

V may be called the underlying set of $G = (A, B)$. A is said to be a bipolar fuzzy vertex set of G and B, bipolar fuzzy edge set of G. Let us denote $\{x, y\}$ by xy. V is assumed to be the underlying set of all bipolar fuzzy graphs in this paper.

Definition 2.6. [17] The underlying crisp graph of a bipolar fuzzy graph $G = (A, B)$, is the graph $G = (V', E')$ where $V' = \{v \in V: {\mu_A}^+(v) > 0 \text{ or } {\mu_A}^-(v) < 0\}$ and $E' = \{\{x, y\}: {\mu_B}^+(\{x, y\}) > 0 \text{ or } {\mu_B}^-(\{x, y\}) < 0\}$.

V' is called the vertex set and E' is called the edge set. A bipolar fuzzy graph may be also denoted as $G = (V', E')$.

Definition 2.7. [17] A bipolar fuzzy graph $G = (A, B)$ is connected if the underlying crisp graph $G = (V', E')$ is connected.

Definition 2.8. A partial bipolar fuzzy subgraph of a bipolar fuzzy graph $G = (A, B)$ is a bipolar fuzzy graph $H = (A', B')$ such that ${\mu_{A'}}^+(v_i) \le {\mu_A}^+(v_i)$ and ${\mu_{A'}}^-(v_i) \ge {\mu_A}^-(v_i)$ for all $v_i \in V$ and ${\mu_{B'}}^+(v_i v_j) \le {\mu_B}^-(v_i v_j)$ and ${\mu_{B'}}^-(v_i v_j) \ge {\mu_B}^-(v_i v_j)$ for every $v_i, v_j \in V$.

Definition 2.9. A bipolar fuzzy subgraph of a bipolar fuzzy graph $G = (A, B)$ is a bipolar fuzzy graph $H = (A', B')$ such that ${\mu_{A'}}^+(v_i) = {\mu_A}^+(v_i)$ and ${\mu_{A'}}^-(v_i) = {\mu_A}^-(v_i)$ for all

v_i in the vertex set of H and ${\mu_{B'}}^{+}(v_iv_j) = {\mu_B}^{+}\,(v_iv_j)$ and ${\mu_{B'}}^{-}(v_iv_j) = {\mu_B}^{-}\,(v_iv_j)$for every v_iv_j in the edge set of H.

Example 2.1. G_1 in Figure 2.1 is a bipolar fuzzy graph. H_1 in Figure 2.2 is a partial bipolar fuzzy subgraph and H_2 in Figure 2.3 is a bipolar fuzzy subgraph of G_1.

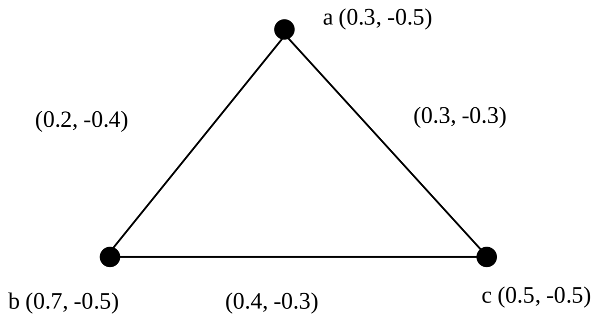

Figure 2.1. G_1, A bipolar fuzzy graph

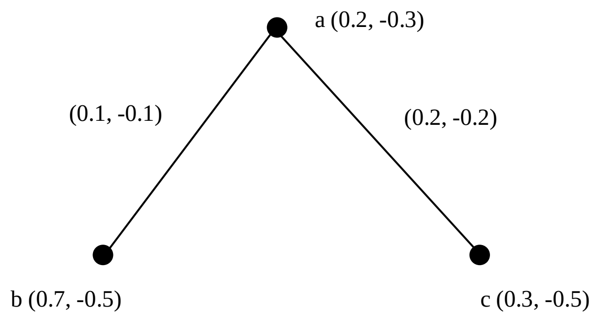

Figure 2.2. H_1, a partial bipolar fuzzy subgraph of G_1

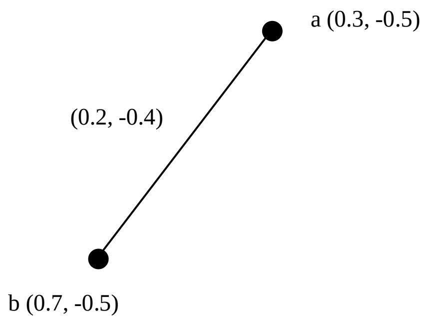

Figure 2.3. H_2, a bipolar fuzzy subgraph of G_1

Notation: We use the following notations to denote the conditions in the Definition 2.5.

$$ {\mu_{2ij}}^{+} \;=\; {\mu_2}^{+}(v_iv_j) \;\le min\,\{{\mu_{1i}}^{+}, {\mu_{1j}}^{+}\} $$
$$ {\mu_{2ij}}^{-} \;=\; {\mu_2}^{-}(v_iv_j) \;\ge max\,\{{\mu_{1i}}^{-}, {\mu_{1j}}^{-}\} $$

Definition 2.10. [1] A bipolar fuzzy graph G is said to be strong if ${\mu_{2ij}}^{+} = min\,\{{\mu_{1i}}^{+}, {\mu_{1j}}^{+}\}$ and ${\mu_{2ij}}^{-} = max\,\{{\mu_{1i}}^{-}, {\mu_{1j}}^{-}\}$ for every edge $v_i v_j \in E'$.

Definition 2.11. [3] A bipolar fuzzy graph G is said to be complete if ${\mu_{2ij}}^{+} = min\,\{{\mu_{1i}}^{+}, {\mu_{1j}}^{+}\}$and ${\mu_{2ij}}^{-} = max\,\{{\mu_{1i}}^{-}, {\mu_{1j}}^{-}\}$ for all $v_i, v_j \in V$.
It is clear from the above definitions that a complete bipolar fuzzy graph G is strong, but the converse is not true.

Definition 2.12. [4] A path P in a bipolar fuzzy graph G is a sequence of distinct vertices $v_1, v_2, \cdots, v_n$ such that either one of the following conditions is satisfied.

i. ${\mu_{2ij}}^{+} > 0$ and ${\mu_{2ij}}^{-} = 0$ for some i and j.
ii. ${\mu_{2ij}}^{+} = 0$ and ${\mu_{2ij}}^{-} < 0$ for some i and j.

According to the above definition of path, the bipolar fuzzy graph in Figure 2.4 is not a path. So, we define b −path.

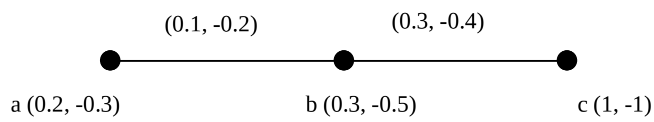

Figure 2.4. A bipolar fuzzy graph which is not a path

Definition 2.13. A sequence of distinct vertices $v_1, v_2, \cdots, v_n$ is called a bipolar path or b −path if atleast one of ${\mu_{2i(i+1)}}^{+}$ and ${\mu_{2i(i+1)}}^{-}$ is different from zero, for $i = 1,2,\cdots,n-1$. Clearly, a bipolar fuzzy graph is connected iff every pair of vertices is joined by a b −path.

Definition 2.14. A sequence of vertices $v_1, v_2, \cdots, v_n$, not necessarily distinct is called a bipolar walk or b −walk if atleast one of ${\mu_{2i(i+1)}}^{+}$ and ${\mu_{2i(i+1)}}^{-}$ is different from zero, for $i = 1,2,\cdots,n-1$. As in graphs where every walk contains a path, every b −walk contains a b −path. Hereafter, by a path we refer to a b −path and by a walk, we refer to a b −walk.

The concept of loss and gain are very important in many problems in economics, operations research and computer organization. We shall associate these concepts to a bipolar fuzzy graph in the definitions to follow.

Definition 2.15. Let $G = (V', E')$ be a bipolar fuzzy graph. For a $u - v$ path $P: u = u_1, u_2, \cdots, u_n = v$ in G, we define $min\,\{{\mu_2}^{+}(u_1u_2), {\mu_2}^{+}(u_2u_3), \cdots, {\mu_2}^{+}(u_{n-1}u_n)\}$ as the gain of P, denoted by $g(P)$ and $max\,\{|{\mu_2}^{-}(u_1u_2)|, |{\mu_2}^{-}(u_2u_3)|, \cdots, |{\mu_2}^{-}(u_{n-1}u_n)|\}$ as the loss of P, denoted by $l(P)$.

In Figure 2.4, gain of the path $P{:}\,abc = g(P) = min\,\{0.1,0.3\} = 0.1$ and loss of $P = l(P) = max\{0.2,0.4\} = 0.4$.

Note that if $e = uv$ is an edge, then its gain, denoted by $g(e) = {\mu_2}^+(uv)$ and loss of e, denoted by $l(e) = |{\mu_2}^-(uv)|$. In Figure 2.4, $g(ab) = 0.1,\ l(ab) = 0.2$.

Definition 2.16. A path P is said to be a gain path if $g(P) > l(P)$ and a loss path, otherwise.

Similarly, gain edges and loss edges can be defined. In Figure 2.4, path $P{:}\,abc$ is a loss path because $l(P) = 0.4 > 0.1 = g(P)$.

Definition 2.17. Let u, v be any two vertices in a connected bipolar fuzzy graph. Among all $u - v$ paths in G, a path whose gain is more than or equal to that of any other $u - v$ path in G, is said to be a maximum $u - v$ gain path (max $(u - v)\ g -$ path, in short). Similarly a $u - v$ path whose loss is less than or equal to that of any other $u - v$ path in G is said to be a minimum $u - v$ loss path (min $(u - v)\ l -$ path, in short). That is, a path P is a max $(u - v)\ g -$ path if $g(P) \geq g(P')$ and is a min $(u - v)\ l -$ path if $l(P) \leq l(P')$, where P' is any $u - v$ path in G.

Note that, a max $(u - v)\ g -$ path need not be a gain path and a min $(u - v)\ l -$ path need not be a loss path.

$BPFG$ is used as an abbreviation for a bipolar fuzzy graph in examples.

Example 2.2. Consider the following example of a $BPFG$ with four vertices.

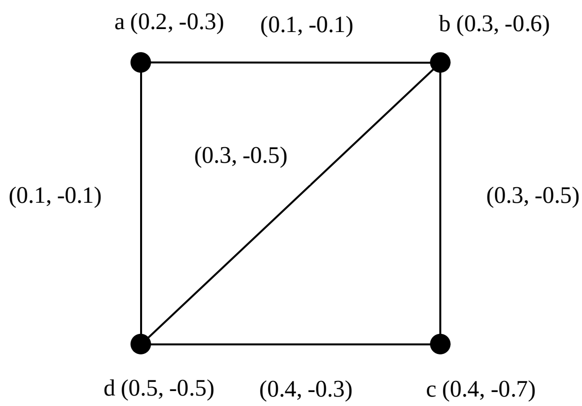

Figure 2.5. Gain paths and loss paths

Vertices	Max-gain	Max g-path	Min-loss	Min l-path
a-b	0.1	Any path	0.1	ab
a-c	0.1	Any path	0.3	adc
a-d	0.1	Any path	0.1	ad
b-c	0.3	bc,bdc	0.3	badc
b-d	0.3	bd, bcd	0.1	bad
c-d	0.4	cd	0.3	cd

Note that, $P_1 = abc$ is a loss path ($a - c$ loss path) because $l(P) = 0.5 > 0.1 = g(P)$ and edge cd is a $c - d$ gain path since $g(cd) = 0.4 > 0.3 = l(cd)$.

Definition 2.18. A $u-v$ path P in a bipolar fuzzy graph is said to be balanced if $g\,(P) = l(P)$. Also, P is said to be optimal if P is a max $(u-v)\; g-$ path and min $(u-v)\; l-$ path.

In the Example 2.2 (Figure 2.5), edge cd is an optimal $c-d$ path. adc is an optimal $a-c$ path. Also, there are many balanced paths in G. For example, bad is a balanced $b-d$ path.

Definition 2.19. Let $G = (V', E')$ be a bipolar fuzzy graph and let $u, v \in V'$. The gain of u and v, denoted by $G(u, v)$ is defined as the gain of a $max\; (u-v)\; g-$ path and loss of u and v, denoted as $L(u, v)$ is the loss of a min $(u-v)\; l-$ path. If H is a bipolar fuzzy subgraph of G, then the gain of u and v in H is the gain of a max $(u-v)\; g-$ path strictly belonging to H and is denoted by $G_H(u, v)$. Loss of u and v in H is similarly defined. If there exists no max $(u-v)\; g-$ path (or min $(u-v)\; l-$ path) completely in H, we define $G_H(u, v) = 0$ (or $L_H(u, v) = 0$).
Next, we have a trivial proposition.

Proposition 2.1. If H be a subgraph of a bipolar fuzzy graph $= (V', E')$, then $G_H(u, v) \leq G(u, v)$ and $L_H(u, v) \leq L(u, v)$ for all pairs of vertices u and v.

Next, we introduce an important concept called the Gain-Loss Matrix (GLM) in bipolar fuzzy graphs.

Definition 2.20. Let $G = (V', E')$ be a bipolar fuzzy graph with n vertices, $\{a_1, a_2, \cdots, a_n\}$. The Gain-Loss Matrix ($GLM$) of G is defined as $M = [(G_{ij}, L_{ij})]$ where $G_{ij} = G(a_i, a_j)$ and $L_{ij} = L(a_i, a_j))\; for\; i \neq j$ and $({\mu_1}^+(a_i), |{\mu_1}^-(a_i)|)$, if $i = j$.
Consider the following example.

Example 2.3. GLM of the $BPFG$ in Example 2.1 (Figure 2.1) is given below.

$$GLM(G_1) = \begin{bmatrix} (0.3,0.5) & (0.3,0.3) & (0.3,0.3) \\ (0.3,0.3) & (0.7,0.5) & (0.4,0.3) \\ (0.3,0.3) & (0.4,0.3) & (0.5,0.5) \end{bmatrix}$$

Clearly, GLM of a $BPFG$ is a symmetric matrix.

Theorem 2.1. In a complete bipolar fuzzy graph ($CBPFG$), $G = (V', E')$, $G(u, v) = {\mu_2}^+(u, v)$ for all $u, v \in V$.
Proof: Consider a $CBPFG, G = (V', E')$ with vertices $v_1, v_2, \cdots, v_n$. By definition, for all $v_i, v_j \in V$, we have,

$${\mu_2}^+(v_i v_j) = min\{{\mu_1}^+(v_i), {\mu_1}^+(v_j)\}$$

Let $u, v \in V$ and let $P: u = u_1, u_2, \cdots, u_n = v$ be a $u-v$ path in G.

$$\begin{aligned} \text{Then, } g(P) &= min\{{\mu_2}^+(u_1u_2), {\mu_2}^+(u_2u_3), \cdots, {\mu_2}^+(u_{n-1}u_n)\} \\ &\leq min\{{\mu_2}^+(u_1u_2), {\mu_2}^+(u_{n-1}u_n)\} \\ &= min\{\, min\{{\mu_1}^+(u_1), {\mu_1}^+(u_2),\}, min\{{\mu_1}^+(u_{n-1}), {\mu_1}^+(u_n)\}\} \\ &\leq min\{{\mu_1}^+(u_1), {\mu_1}^+(u_n)\} \end{aligned}$$

$= \ min\{\mu_1{}^+(u), \mu_1{}^+(v)\}$
$= \ \mu_2{}^+(uv)$

Thus, $g(P) \ \leq \mu_2{}^+(uv)$ for any $u - v$ path P. In particular, gain of edge uv is $\mu_2{}^+(uv)$ and hence, $G(u, v) \ = \ \mu_2{}^+(uv)$.

Note:
In a complete bipolar fuzzy graph**,** $L(u, v)$ need not be equal to $|\ \mu_2{}^- \ (u, v)|$ for all $u, v \in V$. For example in G_2, $L(b, c) = 0.3$ and $|\ \mu_2{}^- \ (a, b)| = 0.4$.

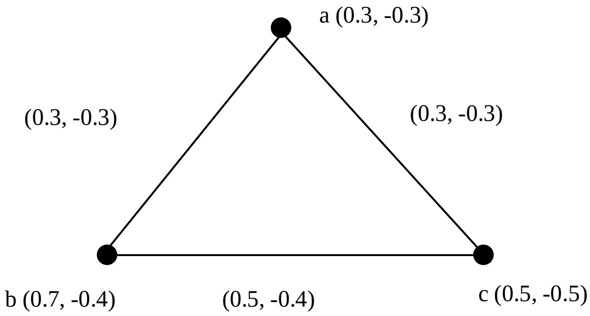

Figure 2.6. G_2, a complete bipolar fuzzy graph

3. Bipolar fuzzy cutvertices and bridges

In this section, we shall introduce and characterize bipolar fuzzy cutvertices and bridges. Three types of cutvertices are possible in a bipolar fuzzy graph, which are given below.

Definition 3.1. Let $G = (V', E')$ be a bipolar fuzzy graph with bipolar functions, μ_1 and μ_2. A vertex $u \in V'$ is said to be a bipolar fuzzy cutvertex (bf-cutvertex, in short) if there exist two vertices $x, y \in V', x \neq y \neq u$ such that $G_{G-u}(x, y) < G_G(x, y)$ and $L_{G-u}(x, y) > L_G(x, y)$. A vertex in a bipolar fuzzy graph is called a gain cutvertex if the first condition is satisfied and a loss cutvertex if the second condition is satisfied.

Now, we characterize bipolar fuzzy cutvertices in the following theorem.

Theorem 3.1. A vertex u in a bipolar fuzzy graph $G = (V', E')$ is a bipolar fuzzy cutvertex if and only if u is a vertex in every max $(x - y)$ gain path and is in every min $(x - y)$ loss path for some x and y in V'.

Proof: Let $G = (V', E')$ be a bipolar fuzzy graph with bipolar functions, μ_1 and μ_2. Suppose that u is a bipolar fuzzy cutvertex. By definition, there exist vertices x and y in G such that $x \neq y \neq u$ and

1. $G_{G-u}(x, y) < G_G(x, y)$
2. $L_{G-u}(x, y) > L_G(x, y)$

(1) implies that the removal of u from G removes all max $(x - y)$ gain paths and (2) implies that the removal of u removes all min $(x - y)$ loss paths. Thus, u is in every max $(x - y)$ gain path and in every min $(x - y)$ loss path.

Conversely, suppose that u is in every max $(x-y)$ gain path and in every min $(x-y)$ loss path. Then, the removal of u from G results in the removal of all max $(x-y)$ gain paths and min $(x-y)$ loss paths. Hence, the gain will decrease and loss will increase between x and y. So, $G_{G-u}(x,y) < G_G(x,y)$ and $L_{G-u}(x,y) > L_G(x,y)$. That is, u is a bipolar fuzzy cutvertex.

Now, we state a characterization theorem for other two types of cutvertices.

Theorem 3.2. Let $G = (V', E')$ be a bipolar fuzzy graph. A vertex u is a gain cutvertex (g-cutvertex) if and only if u is in every max $(x-y)$ gain path for some vertices x and y such that $x \neq y \neq u$ and is a loss cutvertex (l-cutvertex) if and only if u is in every min $(s-t)$ loss path for some vertices s and t such that $s \neq t \neq u$.

Definition 3.2. Let $G = (V', E')$ be a bipolar fuzzy graph with bipolar functions, μ_1 and μ_2. Let $e = xy$ be an edge in G. e is said to be a bipolar fuzzy bridge (bf-bridge, in short) if $G_{G-e}(x', y') < G_G(x', y')$ and $L_{G-e}(x', y') > L_G(x', y')$ for some $x', y' \in V'$. If atleast one of x' or y' is different from x and y, e is said to be a bipolar fuzzy bond and a bipolar fuzzy cutbond if both x' and y' are different from x and y.

Also, we can define gain bridges and loss bridges similar to their counterparts in vertices. Similar to bipolar fuzzy cutvertices, we have a characterization for bipolar fuzzy bridges, which is stated below without proof.

Theorem 3.3. An edge $e \in E$ of a bipolar fuzzy graph $G = (V', E')$ is a bf-bridge if and only if it is in every max $(u-v)$ gain path and in every min $(u-v)$ loss path for some vertices u and v in V'.

Next, we have an easy theorem to verify whether a particular edge is a bf-bridge or not.

Theorem 3.4. An edge xy is a bf-bridge if and only if $G_{G-xy}(x,y) < \mu_2{}^{+}(xy)$ and $L_{G-xy}(x,y) > |\mu_2{}^{-}(xy)|$.

Proof: Suppose $G = (V', E')$ is a bipolar fuzzy graph and xy, an edge in G such that $G_{G-xy}(x,y) < \mu_2{}^{+}(xy)$ and $L_{G-xy}(x,y) > |\mu_2{}^{-}(xy)|$. Since $\mu_2{}^{+}(x,y) \leq G(x,y)$and $|\mu_2{}^{-}(x,y)| \geq L(x,y)$ we have

$$G_{G-xy}(x,y) < G_G(x,y)$$
$$L_{G-xy}(x,y) > L_G(x,y)$$

It follows, xy is a bipolar fuzzy bridge.

Assume, xy is a bipolar fuzzy bridge. By Theorem 3.3, there exists a pair of vertices s and t in V' such that xy is present on every max $(s-t)$ $g-$ path and every min $(s-t)$ $l-$ path.

Suppose, $G_{G-xy}(x,y) \geq \mu_2{}^{+}(xy)$. Then, $G_{G-xy}(x,y) = G_G(x,y)$. It follows, there is a max $(x-y)$ $g-$path in G (say, P) which is different from xy. Let Q be a max $(s-t)$ $g-$ path in G. Replace xy in Q by P to obtain an $s-t$ walk. This walk contains an $s-t$ path. The gain of this path is greater than or equal to $G_G(s,t)$ which is not possible. Therefore, $G_{G-xy}(x,y) < \mu_2{}^{+}(xy)$.

Assume, $L_{G-xy}(x,y) \leq |\mu_2{}^{-}(xy)|$. Then, $L_{G-xy}(x,y) = L_G(x,y)$. It implies, there is a min $(x-y)\ l-$ path in G (say, P') which is different from xy. Let Q' be a min $(s-t)\ l-$ path. Replace xy in Q' by P' to obtain a $s-t$ walk. This walk contains an $s-t$ path. The loss of this path is less than or equal to $L_G(x,y)$ which is not possible.
Therefore, $L_{G-xy}(x,y) > |\mu_2{}^{-}(xy)|$.

Theorem 3.5. An edge $e = xy$ of a bipolar fuzzy graph $G = (V', E')$, which is a cycle is a bf-bridge if and only if there exists edges st, $s't' \in E'$ such that $\mu_2{}^{+}(st) < \mu_2{}^{+}(xy)$ and $|\mu_2{}^{-}(s't')| > |\mu_2{}^{-}(xy)|$.

Proof: Let xy be a bipolar fuzzy bridge. By definition, there exist two distinct vertices x' and y'such that xy lies on every max $(x'-y')\ g-$ path and on every min $(x'-y')\ l-$ path. Since G is a cycle, exactly one of the two $x'-y'$ paths (say, P) in G contains xy and is both the max $(x'-y')\ g-$ path and min $(x'-y')\ l-$path. Let the other $x'-y'$ path be Q. Then,

$$g(Q) < g(P) \leq \mu_2{}^{+}(xy)$$
$$l(Q) > l(P) \geq |\mu_2{}^{-}(xy)|$$

If $g(Q) = \mu_2{}^{+}(st)$ and $l(Q)=|\mu_2{}^{-}(s't')|$, then

$$\mu_2{}^{+}(st) < \mu_2{}^{+}(xy)$$
$$|\mu_2{}^{-}(s't')| > |\mu_2{}^{-}(xy)|$$

Assume that, xy is not a bipolar fuzzy bridge. Then, atleast one of the below conditions holds according to Theorem 3.4.

1. $G_{G-xy}(x,y) \geq \mu_2{}^{+}(xy)$
2. $L_{G-xy}(x,y) \leq |\mu_2{}^{-}(xy)|$

If (1) is true, the path P in G from x to y other than edge xy, has gain greater than or equal to $\mu_2{}^{+}(xy)$. It follows that for every edge $e \in E'$,$\mu_2{}^{+}(e) \geq \mu_2{}^{+}(xy)$. If (2) is true, the path P in G from x to y other than edge xy has loss less than or equal to $|\mu_2{}^{-}(xy)|$. It implies that for every edge $e \in E'$, $|\mu_2{}^{-}(e)| \leq |\mu_2{}^{-}(xy)|$.

4. Concluding remarks

The concept of a bipolar fuzzy graph introduced by Akram can be used as a good model in many problems of real life where we deal with two essential but opposite attributes of a system. In this paper, the authors extended the basic connectivity concepts like strength of connectedness into bipolar fuzzy graphs and also introduced bipolar fuzzy cutvertices and bipolar fuzzy bridges in bipolar fuzzy graphs. More connectivity problems will be discussed in the forthcoming papers.

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