Open Distance Pattern Edge Coloring of a Graph

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Received 14 June 2014; accepted 26 June 2014

Abstract. Let $G$ be a connected graph with diameter $d(G)$, $X = \{1, 2, 3, ..., d(G)\}$ be a non-empty set of colors of cardinality $d(G)$ and let $\emptyset \neq M \subseteq V(G)$. Let $f^\circ_M$ be an assignment of subsets of $X$ to the vertices of $G$ such that $f^\circ_M(u) = \{d(u, v); v \in M, u \neq v\}$ where $d(u, v)$ is the distance between $u$ and $v$. We call $f^\circ_M$ an $M$-Open Distance Pattern Coloring of $G$, an $M$-Open Distance Pattern Edge Coloring of $G$, if no two incident edges have same $f^\circ_M$, where $f^\circ_M = f^\circ_M(u) \oplus f^\circ_M(v)$ for every $uv \in E(G)$; and if such an $M$ exists then $G$ is called an $M$-Open Distance Pattern Edge Colorable Graph (odpec graph). The minimum cardinality of such an $M$, if it exists, is the $M$-Open Distance Pattern Edge Coloring number of $G$ denoted by $\eta_E(G)$.

Keywords: open distance pattern edge coloring, distance pattern labeling of vertices, odpu-graphs, dpd-graphs.

AMS Mathematics Subject Classification (2010): 05C12, 05C15, 05C22

1. Introduction
For all terminologies which are not defined in this paper, we refer to [9]. All the graphs considered in this paper are finite, simple and connected. Beginning with the origin of the Four Color Problem in 1852, research on graph colorings has developed [4] into one of the most popular areas of graph theory. Historically, graph coloring involved in finding the minimum number of colors to be assigned to the vertices or edges or regions so that adjacent vertices or edges or regions must have different colors.

A proper vertex coloring of a graph $G$ is an assignment of colors to the vertices of $G$, one color to each vertex, so that adjacent vertices are colored differently. In a $k$-coloring, we may assume that it is the colors $1, 2, ..., k$ that are being used. While all $k$ colors are typically used in a $k$-coloring of a graph, there are occasions when only some of the $k$ colors are used.

Interest in edge colorings of graphs was likely inspired by Four Color Problem. The first paper dealing with the edge-coloring problem was written by Tait in 1880 [13]. In
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this paper, Tait proved that if the four-color conjecture is true, then the edges of every 3-connected planar graph can be properly colored using only three colors.

Let $G$ be a graph with no loops. [4] A $k$-edge-coloring of $G$ is an assignment of $k$ colors to the edges of $G$ in such a way that any two edges meeting at a common vertex are assigned different colors. If $G$ has a $k$-edge coloring, then $G$ is said to be $k$-edge colorable. The chromatic index of $G$, denoted by $\chi'(G)$, is the smallest $k$ for which $G$ is $k$-edge-colorable. Let there be given an edge coloring of a graph $G$ of order $n$. Any two edges of $G$ that are colored the same cannot be adjacent. Hence, one can never have more than $\frac{n}{2}$ edges of $G$ that are of same color.

Coloring of the vertices and edges of a graph $G$ which are required to posses certain conditions have often been motivated by their utility in various applied fields and their intrinsic mathematical interest. An enormous amount of literature has built up on several kinds of colorings of graphs. The classic $k$-coloring problem tries to assign a color from 1 to $k$ to each vertex in a graph such that no two adjacent vertices share the same color [6]. The $k$-coloring problem, along with many variations and generalizations, is well studied in both computer science and mathematics. Its applications range from frequency assignment and register allocation, to circuit board testing and timetable scheduling [See 3, 5, 14].

On 29th November 2006, B.D. Acharya [7] conveyed to the second author the following definitions for a detailed study.

Let $G = (V, E)$ be a given connected simple $(p, q)$-graph with diameter $d_G, \emptyset \neq M \subseteq V(G)$ and $u \in V(G)$. Then, the $M$-distance-pattern of $u$ is the set $f_M(u) = \{d(u, v) : v \in M\}$. Clearly, $f_M(u) = \{j : N_j^M[u] \neq \emptyset\}$, where $N_j^M[u] = \{v \in M : d(u, v) = j\}$. If $f_M : u \mapsto f_M(u)$ is an injective function, then the set $M$ is a distance-pattern distinguishing set (or, a ‘dpd-set’ in short) of $G$. A graph $G$ with a dpd-set is called a distance-pattern distinguishing graph (dpd-graph) [7]. We associate with each vertex $u$ of a graph $G = (V, E)$ its open $M$-distance pattern (or, ‘odp’ in short), $f_M^O(u) = \{d(u, v) : v \in M, u \neq v\}$, and the graphs in which every vertex has the same open distance pattern are called odp-uniform graphs (or, simply, ‘odpu-graphs’), where the set-valued function (or, set-valuation) [7] $f_M^O$ is called the open distance pattern uniform (or, a odpu)-function and $M$ is called an odpu-set of $G$. The minimum cardinality of a dpd-set (odpu-set) in $G$, if it exists, is the dpd-number (odpu-number) of $G$.

Motivated from the definition of dpd (odpu)-graphs and the classic $k$-coloring problem, we defined $M$-open distance pattern coloring of a graph $G$ in [8] as follows.

**Definition 1.** [8] Let $G$ be a connected graph with diameter $d(G, X) = \{1, 2, 3, \ldots, d(G)\}$ be a non-empty set of colors of cardinality $d(G)$ and let $\emptyset \neq M \subseteq V(G)$. Let $f_M^O$ be an assignment of subsets of $X$ to the vertices of $G$ such that $f_M^O(u) = \{d(u, v) : v \in M, u \neq v\}$ where $d(u, v)$ is the distance between $u$ and $v$. Given such a function $f_M^O$ for all vertices in $G$, an induced edge function $f_M^E$ of an edge $uv \in E(G)$, $f_M^E(uv) = f_M^O(u) \oplus f_M^O(v)$. We call $f_M^O$ an $M$-open distance pattern coloring of $G$, if no two adjacent vertices have same $f_M^O$ and if such an $M$ exists for a graph $G$, then $G$ is called an open distance pattern colorable graph(odpc-graph); the minimum cardinality of such an $M$ if it exists, is the open distance pattern coloring number of $G$, denoted by $\eta_M(G)$. 192
Lemma 3. Proof: We have Subcase 1: spokes of an number of define Definition 2. odpec number. Also odpc number of any graph is always greater than pattern colorable graphs. Labeling namely 2. An example of says /g183 , /g4666 /g1842 /g1858 /g3404 /g1858 /g1833 /g1544 /g182 - Open Distance Pattern Edge Colorable Graph (odpec graph). The minimum cardinality of such an M, if it exists, is the M-Open Distance Pattern Edge Coloring number of G denoted by $\eta_{\text{odpec}}(G)$. Complete graphs $K_n, \forall n \geq 1, C_n; n = 3, 5, 7$ are not M-odpec. An example of M-open distance pattern edge colorable graph is path $P_5$. For, let $V(P_5)$ be $\{v_1, v_2, v_3, v_4, v_5\}$. Then $M = \{v_1, v_2, v_3\}$ is an M-odpec set since the incident edges receive non-identical $f_M^{\odot}$. However, not all M-odpc graphs are M-Open Distance Pattern Edge Colorable. For example consider $K_{1,n}; K_{1,n}$ is M-odpc when M is chosen as any two vertices in the spokes of $K_{1,n}$ and this is the only choice of M. for $K_{1,n}$ to be odpc. With this choice of M, no two adjacent vertices have same $f_M^{\odot}$ whereas $f_M^{\odot}$ of all the edges receives the labeling namely {1,2}. Hence $K_{1,n}$ is M-odpc, but not M-odpec.

Lemma 3. M-open distance pattern edge coloring number of a graph G, is greater than 2. Proof: We have $|M| \geq 2$ for an M-odpc graph [8]. Hence it is enough to show that $|M| \neq 2$. If possible, $|M| = 2$. say $M = \{u, v\}$. Case 1: $uv \in E(G)$. This is not possible since $f_M^{\odot}(u) = f_M^{\odot}(v) = \{1\}$. Case 2: $uv \notin E(G)$. There exists a $u-v$ path say $u = v_1v_2 \ldots v_n = v$ of length $n - 1$. Subcase 1: $n$ is even. Then, for the two middle most vertices $v_{\frac{n}{2}}, v_{\frac{n}{2} + 1}$ of the $u-v$ path $f_M^{\odot}(v_{\frac{n}{2}}) = f_M^{\odot}(v_{\frac{n}{2} + 1}) = \{\frac{n}{2}, \frac{n}{2} - 1\}$. Subcase 2: $n$ is odd. $f_M^{\odot}(v_{\frac{n}{2} + 1}) = f_M^{\odot}(v_{\frac{n}{2} + 1}) = \{1, \frac{n}{2} - 2\}, f_M^{\odot}(v_{\frac{n}{2} - 1}, v_{\frac{n}{2}}) = f_M^{\odot}(v_{\frac{n}{2} - 1}, v_{\frac{n}{2} + 1}) = \{\frac{n}{2}\} - 2, \{\frac{n}{2} - 1, \frac{n}{2}\}\boxdot$ It is interesting to note that odpc number of a graph is always less than or equal to odpec number. Also odpc number of any graph is always greater than 2. Trees, complete bipartite graphs, even cycles have odpc number 2.

Proposition 4. An open distance pattern colorable graph G is Open Distance Pattern Edge Colorable if and only if there exists no vertex $v_i \in V(G)$ such that $f_M^{\odot}(v_{i+k}) = f_M^{\odot}(v_{i+j}) = f_M^{\odot}(v_{i+k}); j, k = 1, 2, \ldots, d(v_i)$, where M, is the odpec set of G. Proof: Let G be the odpc graph with odpc set M. Let $f_M^{\odot}(v_{i+j}) = f_M^{\odot}(v_{i+k})$ for some $j, k$. 193
Hence the different possibilities for the choices of a contradiction to the definition of Subcase 1. One may easily verify that no two adjacent vertices have the same open distance pattern, and hence the only possibility is Subcase 2: Case 1. Theorem 6. Comb $G \cong P_n^+$ is $M$-odpec if and only if diameter of $P_n^+$ is greater than 3.

Some Classes of $M$-odpec graphs

Theorem 5. Path $P_n$ is Open Distance Pattern Edge Colorable if and only if $n > 4$. Proof: Let $G$ be a graph isomorphic to $P_n$, $n > 4$. Let $v_1, v_2, \ldots, v_n$ be the vertices of $P_n$. In [8] we proved that $P_n$ is $M$-Open Distance Pattern Colorable. Choose $M = \{v_1, v_2, v_n\}$. For $i = 1, d(G); f_M^O(v_2) = \{1, d(G) - 1\}; f_M^O(v_1, v_2) = \{d(G), d(G) - 1\}$. For $i = 1, d(G); f_M^O(v_1) = \{d(G), d(G) - 1\}; f_M^O(v_1, v_2) = \{i - 2, i - 1, d(G) - (i - 1)\}$; for $i = 3, 4, \ldots, n$. f_M^O(v_1, v_2) = \{i, i - 2, d(G) - i, d(G) - (i - 1)\}; for $i = 3, 4, \ldots, n - 1$. One may easily verify that no two adjacent vertices have same $f_M^O$ and no two incident edges have same $f_M^O$. So the path $P_n$, $n > 4$, is an open distance pattern edge colorable graph. Conversely, assume $n \leq 4$.

Case 1: $n = 4$
Let $P_4: v_1v_2v_3v_4$ be path with some $M$-odpec set say, $M$. By Lemma 3 $|M| > 2$. Hence the different possibilities for the choices of $M$ are

Subcase 1: $|M| = 3$
$M_1 = \{v_1, v_2, v_3\}$; $f_M^O(v_1) = \{1, 2\}, f_M^O(v_2) = \{1\}, f_M^O(v_1, v_2) = \{2\}$. This is not possible. Similar is the case when $M_2 = \{v_1, v_2, v_4\}$

Subcase 2: $|M| = 4$
$M = \{v_1, v_2, v_3, v_4\}$. Then $f_M^O(v_2) = \{1, 2\}, f_M^O(v_3) = \{1, 2\}, a$ contradiction.

Case 2: $n = 3$.
The only possibility is $M = \{v_1, v_2, v_3\}$, since by Lemma 3 $|M| > 2$. When $M = \{v_1, v_2, v_3\}, f_M^O(v_1, v_2) = \{1, 2\}, f_M^O(v_2) = \{1\}, f_M^O(v_1, v_2) = f_M^O(v_2, v_3) = \{2\}$. A contradiction. Hence $P_n$: $n \leq 4$ is not $M$-odpec. □

[2] Given a graph $G$ we denote by $G^+$ the graph obtained from $G$ by augmenting a new vertex $v'$ for each vertex $v$ of $G$ and augmenting a new edge $vv'$. In particular, $P_n^+$ is called a comb.
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**Proof:** Consider the comb $G \cong P_n^+$ of $2n$ vertices; with diameter is greater than 3. Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices of its stem $P_n$ and $u_1, u_2, u_3, \ldots, u_n$ the pendant vertices. Choose $M = \{v_1, u_1, u_n\}$. $f_M^o(v_1) = \{1, d(G)\}$. $f_M^o(v_i) = \{i - 1, i, d(G) - i\}; i = 1, 2, \ldots, n$. $f_M^o(v_1u_1) = \{d(G), d(G) - 1\}$. $f_M^o(v_iu_{i+1}) = \{i - 1, i + 1, d(G) - i, d(G) - (i + 1)\}; i = 1, 2, \ldots, n - 1$. $f_M^o(v_iu_i) = \{i - 1, i + 1, d(G) - i, d(G) - (i - 1)\}; i = 2, 3, \ldots, n$. Therefore, $f_M^o$ is distinct for incident edges.

Conversely, assume comb is odpec for diameter less than 4. Only possibility is comb with diameter 3, since cardinality of an odpec-set is greater than 2. Let $v_1, v_2$ be the vertices of its stem and $u_1, u_2$ be the pendant vertices. Then $P_n^+$ is isomorphic to path $P_4$, which is not odpec by Theorem 5. Hence, the comb with diameter greater than 3 is an open distance pattern edge colorable graph with $\eta_E(G) = 3$ since $\eta_E(G) > 2$ by Lemma 3. □

[10] An olive tree is a rooted tree consisting of $n$ branches, where the $i^{th}$ branch is a path of length $i$.

**Theorem 7.** Olive tree $T$ is open distance pattern edge colorable if and only if $n > 2$, where $n$ is the number of branches of the olive tree.

**Proof:** Assume $n > 2$. Let the Olive tree $T$ be formed by joining $P_i$, $i = 1, 2, \ldots, n$ paths joined at $v$, where $P_i$ has length $i$. Olive tree $T$ have vertices $v_i0, v_i1, v_i2, \ldots, v_in$ where $1 \leq i \leq n$, and $v_i0$ is identified as $v$. Let $M = \{v_i\}, 1 \leq i \leq n$ the pendant vertices of $P_i$'s $i = 1, 2, \ldots, n$, of the olive tree. $f_M^o(v) = \{1, 2, \ldots, n\}$ and $f_M^o(v_{i1})$ do not contain $i + 1$ for $i = 1, 2, \ldots, n$. Hence all $f_M^o(v_{i1})$ is distinct for $i = 1, 2, \ldots, n$. $f_M^o(v_{ij}) = \{j + 1, j + 2, \ldots, j + n - 1, i - j\}; 1 \leq i \leq n; 1 \leq j \leq i$. For any vertex in the olive tree ($n > 2$), all the vertices adjacent to the vertex $v_{ij}$ have distinct $f_M^o$. Therefore, $f_M^o$ is distinct for all incident edges.

Conversely, assume $n \leq 2$. When $n = 2$, $T \cong P_4$; and when $n = 1$, $T \cong K_2$ which are not odpec. □

**Theorem 8.** The cycle $C_n$, is an open distance pattern edge colorable if and only if $n \geq 8$.

**Proof:** Assume cycle $C_n$ is open distance pattern edge colorable. Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$. Choose $M = \{v_1, v_2, v_3\}$.

**Case 1:** $n$ is odd. Let $n \geq 9$.

Choose $M = \{v_1, v_2, v_3\}$.

$f_M^o(v_1) = \{1, 4\}; f_M^o(v_2) = \{1, 3\}; f_M^o(v_3) = \{1, 2\}; f_M^o(v_4) = \{1, 2, 3\}$,

$f_M^o(v_i) = \{i - 1, i - 2, i - 5\};$ for $i = 5, 6, \ldots, \left\lceil \frac{n}{2} \right\rceil$,

$f_M^o(v_i) = \{n - (i - 1), (i - 2), i - 5\};$ for $i = \left\lceil \frac{n}{2} \right\rceil + 1$,
Case 2: \( n \) is even and \( \geq 8 \).

\[ f_M^\oplus (v_1) = \{1, 4\}; \quad f_M^\oplus (v_2) = \{1, 3\}; \quad f_M^\oplus (v_3) = \{1, 2\}; \quad f_M^\oplus (v_4) = \{1, 2, 3\}; \]

\[ f_M^\oplus (v_i) = \{i - 1, i - 2, i - 5\}; \quad 5 \leq i \leq n \frac{n}{2} + 1, \]

\[ f_M^\oplus (v_i) = \{n - (i - 1), n - (i - 2), i - 5\}; \quad \frac{n}{2} + 1 < i \leq \frac{n}{2} + 4, \]

\[ f_M^\oplus (v_i) = \{n - (i - 1), n - (i - 2), n - (i - 5)\}; \quad \frac{n}{2} + 4 < i \leq n, \]

\[ f_M^\oplus (v_1v_2) = \{3, 4\}; \quad f_M^\oplus (v_2v_3) = \{2, 3\}; \quad f_M^\oplus (v_3v_4) = \{3\}; \quad f_M^\oplus (v_4v_5) = \{1, 2, 4\}; \]

\[ f_M^\oplus (v_iv_{i+1}) = \{i - 2, i - 4, i - 5\}; \quad \text{for} i = 5, 6, \ldots, n \frac{n}{2}. \]

Hence, no two incident edges have identical \( f_M^\oplus \), and hence, cycle \( C_n, n \geq 8 \) is an open distance pattern edge colorable.

Conversely, assume \( n < 8 \) is \( M \)-odpec. In [8] we proved that \( C_n; n = 3, 5, 7 \) is not odpc and hence not odpec. Hence, we consider \( C_4 \) and \( C_6 \). Let \( C_4; v_1v_2v_3v_4 \) be a cycle with open distance pattern edge colorable set \( M, |M| \geq 3 \). The different possibilities for the choice of \( M \) are

Case 1: \(|M| = 3\).

\( M = \{v_1, v_2, v_3\} \) (or \( \{v_2, v_3, v_4\} \) or \( \{v_3, v_4, v_1\} \) or \( \{v_4, v_1, v_2\} \) ). Then, \( f_M^\oplus (v_1) = f_M^\ominus (v_3) = \{2\}; \quad f_M^\oplus (v_2) = \{1\}. \) But \( f_M^\oplus (v_1v_2) = f_M^\oplus (v_2v_3) = \{2\}. \)

Case 2: \(|M| = 4\).

Then, \( M = \{v_1, v_2, v_3, v_4\} \) and \( f_M^\oplus (v_1) = f_M^\ominus (v_2) = f_M^\ominus (v_3) = f_M^\ominus (v_4) = \{1, 2\}. \)

Consider \( C_6; v_1v_2v_3v_4v_5v_6 \) be a cycle with open distance pattern edge colorable set \( M, |M| \geq 3 \). The different possibilities for the choice of \( M \) are
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Case 2: \(|M| = 3.\)

Choose \(M_1 = \{v_1, v_2, v_3\} \) or \(\{v_2, v_3, v_4\}\) or \(\{v_3, v_4, v_5\}\) or \(\{v_4, v_5, v_6\}\) or \(\{v_5, v_6, v_1\}\) or \(\{v_6, v_1, v_2\}\). Then

\[ f^0_{M_1}(v_1) = f^0_{M_1}(v_3) = 1,2; \quad f^0_{M_1}(v_2) = 1; \quad f^0_{M_1}(v_1v_2) = f^0_{M_1}(v_2v_3) = 2. \]

Next, choose \(M_2 = \{v_1, v_2, v_4\}\) or \(\{v_2, v_3, v_5\}\) or \(\{v_3, v_4, v_6\}\) or \(\{v_4, v_5, v_1\}\) or \(\{v_5, v_6, v_2\}\) or \(\{v_6, v_1, v_3\}\). Then,

\[ f^0_{M_2}(v_2) = f^0_{M_2}(v_3) = 1,2; \quad M_3 = \{v_1, v_3, v_4\} \text{ or } \{v_2, v_4, v_6\}. \]

\[ f^0_{M_3}(v_1) = f^0_{M_3}(v_3) = 2; \quad f^0_{M_3}(v_2) = 1,3; \quad f^0_{M_3}(v_1v_2) = f^0_{M_3}(v_2v_3) = 1,2,3. \]

Case 3: \(|M| = 4.\)

Choose \(M = \{v_1, v_2, v_3, v_4\}\) or \(\{v_2, v_3, v_4, v_5\}\) or \(\{v_3, v_4, v_5, v_6\}\) or \(\{v_4, v_5, v_6, v_1\}\) or \(\{v_5, v_6, v_1, v_2\}\) or \(\{v_6, v_1, v_2, v_3\}\). Then,

\[ f^0_{M_2}(v_1) = f^0_{M_2}(v_3) = 1,2; \quad f^0_{M_2}(v_2) = f^0_{M_2}(v_4) = 1,3. \]

Now, choose \(M_3 = \{v_1, v_2, v_4, v_5\}\) or \(\{v_2, v_3, v_5, v_6\}\) or \(\{v_3, v_4, v_6, v_1\}\). Then \(f^0_{M_3}(v_1) = f^0_{M_3}(v_2) = 1,2,3.\)

Case 4: \(|M| = 6.\)

Choose \(M = \{v_1, v_2, v_3, v_4, v_5, v_6\}\) so that \(f^0_{M}(v_1) = f^0_{M}(v_2) = 1,2,3.\) Hence, no choice of \(M\) admit an \(M\)-odpec. Hence, \(C_n; n < 8\) is not \(M\)-odpec. \(\square\)

Acknowledgement

The first author is indebted to the University Grants Commission (UGC) for granting her Teacher Fellowship under UGC’s faculty development programme under XI plan.

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