Some New Families of Divisor Cordial Graph

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Abstract. A divisor cordial labeling of a graph G with vertex set V vertex G is a bijection f from V to {1, 2, 3, . . . |V|} such that an edge uv is assigned the label 1 if f(u) divides f(v) or f(v) divides f(u) and 0 otherwise, then the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. If a graph has a divisor cordial labeling, then it is called divisor cordial graph. In this paper we prove that flower graphs and helm graphs are divisor cordial. We also prove some special graphs such as switching of a vertex of cycle, wheel, helm; duplication of arbitrary vertex of cycle, duplication of arbitrary edge of cycle; split graph of K_{1,n}, B_{n,n}, B_{n,n}^2 are divisor cordial graphs.

Keywords: Cordial labeling, divisor cordial labeling, divisor cordial graph, split graph, Bistar

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1. Introduction

By a graph, we mean a finite undirected graph without loops and multiple edges. For terms not defined here we refer to Harary [2].

Definition 1.1. [8] A binary vertex labeling of graph is called a cordial labeling if |v_f(0) - v_f(1)| ≤ 1 and |e_f(0) - e_f(1)| ≤ 1, where v_f(i) denote the number of vertices labeled with i under f and e_f(i) denote the number of edges labeled with i, where i = 0, 1. A graph G is called cordial if it admits cordial labeling.

The concept of cordial labeling was introduced by Cahit. Many researchers have studied cordiality of graphs. Cahit [8] proved that tree is cordial. In the same paper, he proved that K_n is cordial if and only if n ≤ 3. Ho et al. [3] proved that unicycle graph is cordial unless it is C_{4k+2}. Vaidya et. Al. [6] has also discussed the cordiality of various graphs.

Definition 1.2. [8] Let G be a graph with vertex set v(G) and edge set E(G) and let f: E(G) → {0,1}. Define f* on V(G) by f*(v) = ∑ ( f(uv), uv E(G) ) ( mod2 ). The
function $f$ is called an **E-cordial labeling** of $G$ if $|v_f(0)-v_f(1)| \leq 1$ and $|e_f(0)-e_f(1)| \leq 1$. A graph is called **E-cordial** if it admits E-cordial labeling.

In 1997 Yılmaz and Cahit [2] have introduced E-cordial labeling as a weaker version of edge – graceful labeling. They proved that the trees with $n$ vertices, $K_n$, $C_n$ are E-cordial if and only if $n \perp 2 \pmod{4}$ while $K_{m,n}$ admits E-cordial labeling if and only if $m+ n \perp 2 \pmod{4}$.

**Definition 1.3.** [4] A prime cordial labeling of a graph $G$ with vertex set $V$ is a bijection $f$ from $V$ to $\{1, 2, 3, \ldots, |V|\}$ such that if each edge $uv$ is assigned the label 1 if $\gcd(f(u), f(v)) = 1$ and 0 if $\gcd(f(u), f(v)) > 1$, then the number of edges with 1 differ by at most 1.

Sundaram et.al [4] has introduced the notion of prime cordial labeling. They proved the following graph as prime cordial labeling – $C_n$ if and only if $n \geq 6$; $P_n$ if and only if $n \not\equiv 3 \pmod{4}$; the graph obtained by subdividing each edge of $K_{1,n}$ if and only if $n \geq 3$; bi stars; dragons; crowns; triangular snakes if and only if the snake has at least three triangles; ladders. J. Babujee and L. Shobana [3] proved the existence of prime cordial labeling for sun graph, kite graph and coconut tree and $Y$ -tree, < $K_{1,n} : 2 > n \geq 1$. Hoffman tree, and $K_2 \Theta C_n (C_n)$.

**Definition 1.4.** [10] Let $G = (V, E)$ be a simple graph and $f : V \rightarrow \{1, 2, 3, \ldots, |V|\}$ be a bijection. For each edge $uv$, assign the label 1 if $f(u) \mid f(v)$ or $f(v) \mid f(u)$ and the label 0 if $f(u) \nmid f(v)$. Then $f$ is called a **divisor cordial labeling**. A graph with a divisor cordial labeling is called **divisor cordial graph**.

Varatharajan et al. [11], introduced the concept of divisor cordial and proved the graphs such as path, cycle, wheel, star and some complete bipartite graphs are divisor cordial graphs and in [12], they proved some special classes of graphs such as full binary tree, dragon, corona, $G \ast K_{2,n}$ and $G \ast K_{3,n}$ are divisor cordial.

Labeled graph have variety of application in coding theory, particularly for missile guidance codes, design of good radar type codes and convolution codes with optimal auto correlation properties. Labeled graph plays vital role in the study of X-ray crystallography, communication networks and to determine optimal circuit layouts.

In this paper, we prove that flower graph, Helm graph are divisor cordial. We also prove some special graphs such as switching of a vertex of cycle, wheel, helm; duplication of arbitrary vertex of cycle, duplication of arbitrary edge of cycle; split graph of $K_{1,n}$, $B_{n,n}$, $B_{n,n}$ are also divisor cordial graphs. Before we deal with the main results, we give some definitions that are useful to the ensuing sections.

**Definition 1.5.** [8] The **Helm**, $H_n$ is the graph on $2n + 1$ vertices, obtained from a wheel $W_n$ by attaching a pendant edge to each of $n$ rim vertices.

**Definition 1.6.** [8] The **closed helm** $CH_n$ is the graph obtained from a helm $H_n$ by joining each pendant vertex to form a cycle.

**Definition 1.7.** [9] The **flower graph** $Fl_n$ is the graph obtained from a helm $H_n$ by joining each pendant vertex to the apex vertex of the helm.
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2. Main Results

Theorem 2.1. $F_n$ is divisor cordial for $n \geq 3$.

**Proof:** Let $H_n$ be a helm with $v$ as the apex vertex; $v_1, v_2, \ldots, v_n$ as the vertices of the cycle and $u_1, u_2, u_3, \ldots, u_n$ be the pendant vertices. Let $F_n$ be the flower graph obtained from helm $H_n$ then $|V(F_n)| = 2n+1$ and $|E(F_n)| = 4n$. We define $f$: $V(F_n) \rightarrow \{1, 2, 3, \ldots, |V|\}$ as,

$$
\begin{align*}
\text{f (v)} &= 1 \\
\text{f (v)} &= 2 \\
\text{f (u)} &= 3 \\
\text{f (v)} &= 2i+1 ; 2 \leq i \leq n \\
\text{f (u)} &= 2i ; 2 \leq i \leq n
\end{align*}
$$

From the above, we observe that since the apex vertex $v$ is labeled with 1, so the edges incident with $v$ receive label 1 and there are $2n$ edges incident to $v$. Therefore $e_1(1) = 2n$. The edges $u_iv_i$ receive the label 0 for every $i = 1, 2, 3, \ldots, n$. Since $f(u_i)$ and $f(v_i)$ are consecutive labels. Similarly, all the edges in the cycle of the base wheel also receive the label 0 since the vertices of the cycle are labeled with consecutive odd integers except for $v_1$, where $f(v_1) = 2$. Thus $e_1(0) = 2n$.

Therefore $|e_1(0) - e_1(1)| = 0$

Hence $F_n$ is divisor cordial graph

![Figure 1: $F_6$ is a divisor cordial helm graph](image)

Theorem 2.2. Helm graph $H_n$ is divisor cordial, for $n > 3$.

**Proof:** Let $v$ be the apex vertex; $v_1, v_2, \ldots, v_n$ are the vertices of cycle and $u_1, u_2, \ldots, u_n$ be the pendant vertices for $n > 3$. Then $|V(H_n)| = 2n+1$ and $|E(H_n)| = 3n$. We define $f$ as follows,

$$
\begin{align*}
\text{f (v)} &= 1 \\
\text{f (u)} &= 2i + 1 ; 1 \leq i \leq n-1 \\
\text{f (u)} &= f(v_{n-1}) + 4 \\
\text{f (u)} &= f(u_{n-1}) + 2
\end{align*}
$$

The vertices $v_i$ are labeled in the following order,

$$
\begin{align*}
2, 2 \times 2, 2 \times 2^2, \ldots, 2 \times 2^k \\
6, 6 \times 2, 6 \times 2^2, \ldots, 6 \times 2^k \\
10, 10 \times 2, 10 \times 2^2, \ldots, 10 \times 2^k \\
\ldots \ldots \ldots \ldots
\end{align*}
$$

(1)
where \((4m - 2)^{2k_m} \leq n\) and \(m \geq 1, k_m \geq 0\). We observe that \((4m - 2)^a\) divides \((4m - 2)^b\); \((a < b)\) and \((4m - 2)^{2k_i}\) does not divide \((4m + 2)\).

**Case 1. **\(n\) **is even**

The apex vertex \(v\) is labeled as 1, so the \(n\) edges incident to \(v\) receive label 1. Now the cyclic vertices \(v_i\) being labeled as in (1), and so \((n/2)\) edges receive the label 1 and \((n/2)\) edges receive 0. Since \(f(u_i) = 2i+1\); \(1 \leq i \leq n-1\) and \(v_i's\) are labeled as in (1), they does not divide each other, and receives label 0.

Therefore \(e_f(1) = e_f(0) = n + n/2 = 3n / 2\)

Thus \(|e_f(0) - e_f(1)| = 0\)

**Case 2. **\(n\) **is odd**

The edges \(vv_i\) receive the label 1, as given in the above case, and so \(n\) edges get the label 1. The cyclic vertices \(v_i\) are labeled as in (1), so that \((n-1)/2\) edges receive label 1 and \((n + 1) / 2\) edges receive label 0. The rim edges \(v_iv_j\) receive label 0 (as in the above case).

Therefore \(e_f(1) = n + (n + 1) / 2 = (3n + 1) / 2\)

\(e_f(0) = n + (n - 1) / 2 = (3n - 1) / 2\)

Thus from the two cases \(|e_f(0) - e_f(1)| \leq 0\)

Hence \(H_n\) is divisor cordial graph

**Figure 2.** \(H_9\) is divisor cordial

3. **Divisor Cordial graphs obtained by switching of a vertex.**

**Definition 3.1.** [8] A vertex switching \(G_v\) of a graph \(G\) is the graph obtained by taking a vertex \(v\) of \(G\), removing the entire edges incident to \(v\) and adding edges joining \(v\) to every other vertex which are not adjacent to \(v\) in \(G\).

**Theorem 3.2.** The graph obtained by switching of an arbitrary vertex in cycle \(C_n\) is divisor cordial.

**Proof:** Let \(v_1, v_2, ..., v_n\) be the successive vertices of \(C_n\), and \(G_v\) denotes the graph obtained by switching of vertex \(v\) of \(G\). Without loss of generality let the switched vertex
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be \( v_1 \). We note that \(|V(G_{v_1})| = n\) and \(|E(G_{v_1})| = 2n - 5\). We define \( f: V(G_{v_1}) \rightarrow \{1, 2, 3, \ldots, n\} \) as follows:

\[
\begin{align*}
    f(v_1) &= 1 \\
    f(v_i) &= i \quad 2 \leq i \leq n
\end{align*}
\]

Since the switched vertex \( v_1 \) is labeled as 1, the \((n - 5)\) edges incident to 1 receive label 1 and other edges receive 0 as the consecutive integers does not divides each other. Therefore \( e_f(0) = n - 2 \) and \( e_f(1) = n - 3 \)

Thus \(| e_f(0) - e_f(1) | = 1 \)

Hence the graph obtained by switching of an arbitrary vertex in cycle \( C_n \) is a divisor cordial graph.

**Theorem 3.3.** The graph obtained by switching of a rim vertex in wheel \( W_n, n \geq 4 \) is a divisor cordial graph.

**Proof:** Let \( v \) be the apex vertex and \( v_1, v_2, \ldots, v_n \) be the rim vertices of wheel \( W_n \). Let \( G_{v_1} \) denote graph obtained by switching of a rim vertex \( v_1 \) of \( G = W_n \). We note that \(|V(G_{v_1})| = n + 1\) and \(|E(G_{v_1})| = 3(n - 2)\). We define \( f \) as follows,

\[
\begin{align*}
    f(v) &= 1; \quad f(v_1) = 2; \quad f(v_2) = 4; \\
    f(v_i) &= 3; \quad f(v_i) = i+1 \quad 4 \leq i \leq n.
\end{align*}
\]

**Case 1. \( n \) is odd**

The apex vertex \( v \) is labeled with 1. Trivially the \((n - 1)\) edges adjacent to it receive label 1. Also \( f(v_1) = 2 \) and \( f(v_i) = i+1 \quad 4 \leq i \leq n \). Out of \((n-3)\) edges incident to \( v_1 \), \((n-5) / 2\) edges receive label 1 and other \((n-1) / 2\) receive 0. Also the consecutive numbers does not divide each other, so the \((n - 2)\) edges \( v_1v_i \) receive label 0. Therefore, \( e_f(1) = (n-1 )+ (n-5) / 2 = (3n -7)/2 \) and \( e_f(0) = (n-2) + (n-1) / 2 = (3n-5)/2 \).

**Case 2. \( n \) is even**
The apex vertex is 1, trivially the \((n - 1)\) edges receive 1. Since \(f(v_1) = 2\), out of \((n - 2)\) edges, \((n - 4)/2\) edges receive 1 and \((n - 2)/2\) edges get label 0. Also, since the consecutive numbers does not divide each other the \((n - 2)\) edges \(v_i v_j\) contribute 0. Therefore \(e_t(1) = (n-1) + (n-4)/2 = 3(n-2)/2\) and 
\(e_t(0) = (n-2) + (n-4)/2 = 3(n-2)/2\).

Thus \(|e_t(0) - e_t(1)| \leq 0\).

Hence \(G\) is a divisor cordial graph.

**Theorem 3.4.** The graph obtained by switching of an apex vertex in \(H_n\) admits divisor cordial labeling.

**Proof:** Let \(H_n\) be a helm with \(v\) as the apex vertex; \(v_1, v_2, v_3, \ldots v_n\) be the vertices of cycle and \(u_1, u_2, u_3, \ldots u_n\) be the pendant vertices. Let \(G_v\) denote graph obtained by switching of an apex vertex \(V\) of \(G = H_n\). Here \(|V(H_n)| = 2n + 1\).

We define \(f\) as follows.

\[
\begin{align*}
  f(v) & = 1 \\
  f(u_j) & = 2j + 1; \ 1 \leq j \leq n \\
  f(v_n) & = 2n + 1 \\
  f(u_n) & = 2n
\end{align*}
\]

Label the vertices \(v_i\) are arranged as in (1) where \((4m - 2) 2^{km} \leq n\) and \(m \geq 1, k_m \geq 0\). We observe that \((4m - 2) 2^a\) divides \((4m - 2)2^b\); \((a < b)\) and \((4m - 2) 2^{ki}\) does not divide \((4m + 2)\).

**Case 1.** \(n\) is odd

The apex vertex \(v\) is labeled with 1, trivially edges incident to \(v\) receive label 1, and there are \(n\) such 1’s. Since the vertices \(v_i\) are arranged as in (1), the adjacent vertices which are their multiplier’s divide each other, and so the edges incident to it receive 1, and there are \((n - 1)/2\) such 1’s and so other \((n + 1)/2\) edges receive 0. Now the edges \(u_i v_i\) contribute 0. Since \(u_i\) and \(v_i\) are consecutive numbers they does not divide each other. Therefore, \(e_t(1) = n + (n - 1)/2 = (3n - 1)/2\) and \(e_t(0) = n + (n + 1)/2 = (3n + 1)/2\).

**Case 2.** \(n\) is even

The vertices \(v_i\) are labeled as in (2), so that the outer cycle \(n/2\) edges will receive 1, all the other arguments are as in the above case. Therefore \(e_t(0) = e_t(1) = n + n/2 = 3n/2\).

Hence in both the cases, \(|e_t(0) - e_t(1)| \leq 1\).
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Hence the graph obtained by switching of $H_n$ is a divisor cordial graph.

Figure 5. Switching of apex vertex $v$ in graph $H_7$

4. Duplication of a vertex and duplication of an edge

**Definition 4.1.** [5] *Duplication of a vertex* $v_k$ graph $G$ produces a new graph $G'$, by adding a new vertex $v'_k$ in such a way that $N(v_k) = N(v'_k)$

**Definition 4.2.** [5] *Duplication of an edge* $e_k$ by an edge $e'_k$ produces a new graph $G$, in such a way that $N(v_k) \cap N(v'_k) = \{v_{k-1}\}$ and $N(v_{k+1}) \cap N(v'_{k+1}) = \{v_{k+2}\}$.

**Definition 4.3.** [5] For a graph $G$, the split graph is obtained by duplicating its vertices altogether.

**Theorem 4.4.** The graph obtained by duplication of an arbitrary vertex of $C_n$ admits divisor cordial labeling.

**Proof:** Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices of the cycle $C_n$.

Let $G$ be the graph obtained by duplicating an arbitrary vertex of $C_n$. Without loss of generality, let this vertex be $v_1$. Then $E(G) = \{E(C_n); e', e''\}$ where $e' = v_1v_2$ and $e'' = v_nv_1'$ and $V(G) = \{V(C_n), v'_1\}$. Hence $|V(G)| = n + 1, |E(G)| = n + 2$.

The other vertices are labeled in the following order.

1, 2 x 2, 2 x 2^2, \ldots, 2 x 2^{k1}
3, 3 x 2, 3 x 2^2, \ldots, 3 x 2^{k2}
5, 5 x 2, 5 x 2^2, \ldots, 5 x 2^{k3}

where $(2m - 1) 2^{k_m} \leq n$ and $m \geq 1, k_m > 0$. We observe that $(2m - 1) 2^a$ divides $(2m - 1) 2^b$; $(a < b)$ and $(2m - 1) 2^{k_m}$ does not divide $(2m + 1)$.

**Case 1. n is even**

We label the vertices as follows.
The cycle is labeled as in (2), so that \((n + 2)/2\) edges receive the label 1 and \((n - 2)/2\) edges receive the label 0. The two edges \(v_i'v_2\) and \(v_i'v_n\) contribute 0. Therefore \(e_f(1) = (n + 2)/2\) and \(e_f(0) = (n - 2)/2 + 2 = (n + 2)/2\).

**Case 2. \(n\) is odd**

We label the vertices as follows.

\[
f (v_1) = 1 \quad f(v_1') = n - 1 \quad f(v_n') = n + 1
\]

The vertices in the cycle are labeled as in (2). Here the \((n + 3)/2\) edges in the cycle receive 1 and \((n - 3)/2\) edges receive 0; the edges \(v_i'v_2\) and \(v_i'v_n\) contribute 0. Therefore \(e_f(1) = (n + 3)/2\) and \(e_f(0) = (n - 3)/2 + 2 = (n + 1)/2\)

Hence in the above two cases, \(|e_f(0) - e_f(1)| \leq 1\).

Thus duplication of an arbitrary vertex from \(C_n\) forms a divisor cordial graph.

**Case 1. \(n\) is even**

We label \(f(v_1) = 1\); \(f(v_1') = n\); \(f(v'') = n + 1\)

The vertices in the cycle are labeled as in (2). Here \((n + 2)/2\) edges receive label 1 and \((n - 2)/2\) edges receive label 0. The vertices \(v_i'\) and \(v_2'\) have consecutive odd labels and they do not divide, so that the edges \(v_1'v_2'\) contribute 0.

The vertex \(v_3\) has label 4 and 4 does not divide an odd number, so that the edge \(v_2'v_3\) contributes 0. Similarly \(v_3v_1'\) contributes 0. Therefore, \(e_f(0) = (n+2)/2\) and \(e_f(1) = (n-2)/2 + 3 = (n + 4)/2\). Hence \(|e_f(0) - e_f(1)| \leq 1\).

**Case 2. \(n\) is odd**

Label the vertices \(v_1'\) and \(v_2'\) as follows: \(f(v_1') = n\) and \(f(v_2') = n + 2\). Since the vertices in the cycle are labeled as in (2), \((n + 3)/2\) receive label 1, and \((n - 3)/2\) edges receive label 0. As said in the above case, edges \(v_3v_1'\), \(v_1'v_2'\) and \(v_2'v_3\) receive 0. Therefore \(e_f(1) = (n + 3)/2\) and \(e_f(0) = (n-3)/2 + 3 = (n + 3)/2\). Hence \(|e_f(0) - e_f(1)| = 0\).
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Thus the graph obtained by duplication of arbitrary edge in $C_n$ admits divisor cordial labeling.

**Figure 7.** Duplication of $v_1v_2$ from $C_6$

**Definition 5.1.** [5] *Bistar* is the graph obtained by joining the apex vertices of two copies of star $K_{1,n}$.

**Theorem 5.2.** $B_{n,n}$ is a divisor cordial graph.

**Proof:** Let $B_{n,n}$ be a bistar with vertex set $V(G) = \{u, v, u_i, v_i, \ldots, u_n, v_n\}$ where $u_i, v_i$ pendant vertices are and $u, v$ are the apex vertices. Then $|V(G)| = 2n + 2$ and $|E(G)| = 2n + 1$.

We define $f$ as,

\[
\begin{align*}
  f(u) &= 2 \\
  f(v) &= 1 \\
  f(v_i) &= 2i + 2; \ 1 \leq i \leq n \\
  f(u_j) &= 2j + 1; \ 1 \leq j \leq n
\end{align*}
\]

**Figure 8.** $B_{6,6}$ is divisor cordial
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The vertex v is labeled as 1. Trivially the (n + 1) edges adjacent to v receive 1. Next, u has been given label 2 and it is adjacent to n number of u’s which has odd integer label, therefore, \( e_f(1) = n + 1 \) and \( e_f(0) = n \). Hence \( |e_f(0) – e_f(1)| = 1 \)

Thus \( B_{n,n} \) is a divisor cordial graph.

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