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Fuzzy Graph Theory: A Survey

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Abstract. A fuzzy graph (f-graph) is a pair $G : (\sigma, \mu)$ where σ is a fuzzy subset of a set S and μ is a fuzzy relation on σ. A fuzzy graph $H : (\tau, \upsilon)$ is called a partial fuzzy subgraph of $G : (\sigma, \mu)$ if $\tau(u) \le \sigma(u)$ for every u and $\upsilon(u, v) \le \mu(u, v)$ for every u and v . In particular we call a partial fuzzy subgraph $H : (\tau, \upsilon)$ a fuzzy subgraph of $G : (\sigma, \mu)$ if $\tau(u) = \sigma(u)$ for every u in τ^* and $\upsilon(u, v) = \mu(u, v)$ for every arc (u, v) in υ^*. A connected f-graph $G : (\sigma, \mu)$ is a fuzzy tree(f-tree) if it has a fuzzy spanning subgraph $F : (\sigma, \upsilon)$, which is a tree, where for all arcs (x, y) not in F there exists a path from x to y in F whose strength is more than $\mu(x, y)$. A path P of length n is a sequence of distinct nodes $u_0, u_1, ..., u_n$ such that $\mu(u_{i-1}, u_i) > 0$, $i = 1, 2, ..., n$ and the degree of membership of a weakest arc is defined as its strength. If $u_0 = u_n$ and $n \ge 3$, then P is called a cycle and a cycle P is called a fuzzy cycle(f-cycle) if it contains more than one weakest arc . The strength of connectedness between two nodes x and y is defined as the maximum of the strengths of all paths between x and y and is denoted by $CONN_G(x, y)$. An $x - y$ path P is called a strongest $x - y$ path if its strength equals $CONN_G(x, y)$. An f-graph $G : (\sigma, \mu)$ is connected if for every x,y in σ^*, $CONN_G(x, y) > 0$. In this paper, we offer a survey of selected recent results on fuzzy graphs.

Keywords: Fuzzy graph, Strongest path, strong path, fuzzy cycle, fuzzy tree

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1. Introduction

It is quite well known that graphs are simply models of relations. A graph is a convenient way of representing information involving relationship between objects. The objects are represented by vertices and relations by edges. When there is vagueness in the description of the objects or in its relationships or in both, it is natural that we need to design a 'Fuzzy Graph Model'.

We know that a graph [6] is a symmetric binary relation on a nonempty set V. Similarly, a fuzzy graph is a symmetric binary fuzzy relation on a fuzzy subset. The concept of fuzzy sets and fuzzy relations was introduced by L.A.Zadeh in 1965 [1] and further studied in [2]. It was Rosenfeld [5] who considered fuzzy relations on fuzzy sets and developed the theory of fuzzy graphs in 1975. The concepts of fuzzy trees, blocks, bridges and cut nodes in fuzzy graph has been studied in [5]. During the same time

R.T.Yeh and S.Y. Bang [7] have also introduced various connectedness concepts in fuzzy graphs.

Yeh and Bang's [7] approach for the study of fuzzy graphs were motivated by its applicability to pattern classification and clustering analysis. They worked more with the fuzzy matrix of a fuzzy graph, introduced concepts like vertex connectivity $\Omega(G)$, edge connectivity $\lambda(G)$ and established the fuzzy analogue of Whiteney's theorem. They also proved that for any three real numbers a, b, c such that $0 < a \leq b \leq c$, there exists a fuzzy graph G with $\Omega(G) = a$, $\lambda(G) = b$ and $\delta(G) = c$. Techniques of fuzzy clustering analysis can also be found in [7].

After the pioneering work of .Rosenfeld [5] and R.T.Yeh and S.Y. Bang [7] in 1975, where some basic fuzzy graph theoretic concepts and applications have been indicated, several authors have been finding deeper results, and fuzzy analogues of many other graph theoretic concepts. This include fuzzy trees [7,28], fuzzy line graphs[13], autmorphism of fuzzy graphs [9], fuzzy interval graphs [14,15], cycles and cocycles of fuzzy graphs [18] and also in [11,12,17,20,49].

The concepts of connectedness and acyclicity levels were introduced for fuzzy graphs [7] and several fuzzy tree definitions which are consistent with cut - level representations were given in [7]. Introducing the notion of fuzzy chordal graphs, W. L.Craine [16] has obtained the fuzzy analogue of the characterizations of interval graphs. The notion of fuzzy graphs is generalized to fuzzy hypergraphs also [3].

Bhattacharya [10] has extended the definitions of eccentricity and center based on the metric in fuzzy graphs defined in [5], and the inequality $r(G) \leq d(G) \leq 2r(G)$ also has been proved.

A. Somasundaram and S. Somasundaram [24] and A. Somasundaram [25] introduced the concepts of domination and total domination in fuzzy graphs and determined the domination number for several classes of fuzzy graphs and obtained bounds for the same. A Somasundaram [26] studied several operations on fuzzy graphs such as union, join, composition, cartesian product and obtained their domination parameters. Nair and Cheng [27] discussed the concepts of clique and fuzzy cliques in fuzzy graphs. Various properties of fuzzy cliques and a characterization of fuzzy cliques were also presented. Moderson and Yao [28] analyzed different connectedness levels in fuzzy graphs. The structural property of fuzzy finite graphs provided a tool that allowed for the solution of operations research problems. In the same paper the authors examined the properties of various types of fuzzy cycles, fuzzy trees, fuzzy bridges, and fuzzy cut nodes in fuzzy graphs. Nagoor Gani and Basheer Ahmed [29] examined the properties of various types of degree, order and size of fuzzy graphs and compared the relationship between degree, order and size of fuzzy graphs.

Bhutani and Rosenfeld [30] defined fuzzy end nodes in fuzzy graphs and showed that no node can be both a cut node and a fuzzy end node. The authors studied some properties of fuzzy end nodes in fuzzy trees, and characterized fuzzy cycles that have no cut nodes or fuzzy end nodes. They introduced the concept of strong arcs and strong paths in fuzzy graphs [31] and proved that a fuzzy bridge is strong, but a strong arc need not be a fuzzy bridge. Also the authors obtained a characterization of fuzzy trees in terms of strong paths and proved some properties of strong arcs in fuzzy trees. They introduced [32] the concepts of closure, cover and basis and studied these properties in fuzzy trees. In a connected fuzzy graph G, there is a strong path, and hence a geodesic between any two nodes u, and v of G. The length of a geodesic between u and v is called the g-distance dg(u, v). Using this concept of distance, it is proved that the center of a fuzzy tree consists of either a single node or two nodes joined by a strong arc. A node is called a median of (u, v,w) if it lies on geodesics between u and v, v and w and w and u . Also it is proved that in a fuzzy tree, every triple of nodes has a unique median, but the converse is not true. Also in [33], two simple dissimilarity measures between fuzzy subsets of a finite set S are defined.

Bhutani and Batton [34] studied the operations of fuzzy graphs under which M-Strong property is preserved. Bhutani et al. [35] discussed degrees of fuzzy end nodes such as weak fuzzy end node, partial fuzzy end node and full fuzzy end node and further studied some properties of fuzzy end nodes and fuzzy cut nodes. Mordeson and Nair [36] defined arc disjoint fuzzy graphs and studied some of their properties.

Aymeric Perchant and Isabella Bloch [37] introduced a generic definition of fuzzy morphism between graphs that includes classical graph related definitions as subcases such as graph and subgraph isomorphism. Blue et.al [38] presented a taxonomy of fuzzy graphs that treats fuzziness in vertex existence, edge existence, edge connectivity and edge weight. Within that frame work they formulated some standard graph theoretic problems for fuzzy graphs using a unified approach distinguished by its uniform application of guiding principles such as the construction of memberships grades via the ranking of fuzzy numbers, the preservation of membership grade normalization, and the collapsing of fuzzy sets of graphs into fuzzy graphs. Bershtein and Dziouba [39] introduced the bipartite degree of fuzzy graphs and suggested an algorithm for the maximal bipartite part construction. Assia Alaoui [40] extended the concepts of internal stability, external stability, external domination and some of their combinations to fuzzy graphs.

The concept of strong arc in maximum spanning trees [41] and its applications in cluster analysis and neural networks [42,60] were studied by Sameena and Sunitha. Geodetic distance (g-distance) in fuzzy trees, strong degree of a node and g-self centered fuzzy graphs were also studied by the same authors in [42, 43]. Applications of fuzzy graphs to database theory, to problem concerning the group structure and also to chemical structures are found in literature [3,51].

2. Theory of fuzzy graphs – definitions and basic concepts

For basic concepts in fuzzy sets we refer [45,46,47,52] and for concepts in graph theory we refer [6,48, 50].

A fuzzy graph (f-graph) [5] is a pair $G : (\sigma, \mu)$ where σ is a fuzzy subset of a set S and μ is a fuzzy relation on σ. We assume that S is finite and nonempty, μ is reflexive and symmetric [5]. In all the examples σ is chosen suitably. Also, we denote the underlying graph by $G^* : (\sigma^*, \mu^*\_)$ where $\sigma^* = \{u \in S : \sigma(u) > 0\}$ and $\mu^* = \{(u, v) \in S\times S : \mu(u, v) > 0\}$. A fuzzy graph $H : (\tau, \upsilon)$ is called a partial fuzzy subgraph of $G : (\sigma, \mu)$ if $\tau(u) \le \sigma(u)$ for every u and $\upsilon(u, v) \le \mu(u, v)$ for every u and v [4]. In particular we call a partial fuzzy subgraph $H : (\tau, \upsilon)$ a fuzzy subgraph of $G : (\sigma, \mu)$ if $\tau(u) = \sigma(u)$ for every u in τ^* and $\upsilon(u, v) = \mu(u, v)$ for every arc (u, v) in υ^*. Now a fuzzy subgraph $H : (\tau, \upsilon)$ spans the fuzzy graph $G : (\sigma, \mu)$ if $\tau = \sigma$. A connected f-graph $G : (\sigma, \mu)$ is a fuzzy tree(f-tree) if it has a fuzzy spanning subgraph $F : (\sigma, \upsilon)$, which is a tree, where for all arcs (x, y) not in F there exists a path from x to y in F whose strength is more than $\mu(x, y)$

[5]. Note that here F is a tree which contains all nodes of G and hence is a spanning tree of G. Also note that F is the unique maximum spanning tree(MST) of G [21]. A path P of length n is a sequence of distinct nodes u_0, u_1, ...u_n such that $\mu(u_{i-1}, u_i) > 0$, i = 1, 2, ..., n and the degree of membership of a weakest arc is defined as its strength. If $u_0 = u_n$ and $n \geq 3$, then P is called a cycle and a cycle P is called a fuzzy cycle(f-cycle) if it contains more than one weakest arc [4]. The strength of connectedness between two nodes x and y is defined as the maximum of the strengths of all paths between x and y and is denoted by CONNG(x, y). An x − y path P is called a strongest x − y path if its strength equals CONNG(x, y) [5]. An f-graph G : (σ, μ) is connected if for every x,y in σ^* ,CONNG(x, y) > 0. Through out this, we assume that G is connected. An arc of a f-graph is called strong if its weight is at least as great as the connectedness of its end nodes when it is deleted and an x−y path P is called a strong path if P contains only strong arcs [1]. An arc is called an f-bridge of G if its removal reduces the strength of connectedness between some pair of nodes in G [5]. Similarly an f-cutnode w is a node in G whose removal from G reduces the strength of connectedness between some pair of nodes other than w. A complete fuzzy graph (CFG) is an f-graph G : (σ, μ) such that $\mu(x, y) = \sigma(x) \wedge \sigma(y)$ for all x and y. The complement [22] of a fuzzy graph *G : (σ,μ)* is the fuzzy graph $\overline{G} : (\overline{\sigma}, \overline{\mu})$ where $\overline{\sigma} \equiv \sigma$ *and* $\overline{\mu}\ (u,v) = \sigma(u) \wedge \sigma(v) - \mu(u,v)$ for all *u*, *v* in *V*. A fuzzy graph *G* is self complementary if $G \approx \overline{G}$.

3. Types of arcs in a fuzzy graph

Depending on the CONNG(x, y) of an arc (x, y) in a fuzzy graph G Sunil Mathew and Sunitha [44] have defined the following three different types of arcs. Note that CONNG−(x,y)(x, y) is the strength of connectedness between x and y in the fuzzy graph obtained from G by deleting the arc (x, y).

Definition 3.1. *An arc (x, y) in G is called α- strong if μ(x, y) > CONNG−(x,y)(x, y)*

Definition 3. 2. *An arc (x, y) in G is called β- strong if μ(x, y) = CONNG−(x,y)(x, y).*

Definition 3.3. *An arc (x, y) in G is called a δ- arc if μ(x, y) < CONNG−(x,y)(x, y).*

Remark 3.4. *A strong arc is either α- strong or β- strong by definition 3.1 and definition 3.2 respectively.*

Definition 3.5. *A δ- arc (x, y) is called a δ^* - arc if μ(x, y) > μ(u, v) where (u, v) is a weakest arc of G.*

Definition 3.6. *A path in an f-graph G : (σ, μ) is called an α-strong path if all its arcs are α- strong and is called a β- strong path if all its arcs are β- strong.*

Example 3.7. Let G : (σ, μ) be with σ^* = {u, v,w, x} and μ(u, v) = 0.2 = μ(x, u), μ(v,w) = 1 = μ(w, x), μ(v, x) = 0.3. Here, (v,w) and (w, x) are α-strong arcs, (u, v) and (x, u) are β-strong arcs and (v, x) is a δ-arc. Also (v, x) is a δ^* - arc since μ(v, x) > μ(u, v), where (u,

v) is a weakest arc of G. Here P1 : x,w, v is an α- strong x − v path whereas P2 : x, u, v is a β - strong x − v path.

Note that in an f-graph G, the types of arcs cannot be determined by simply examining the arc weights; for, the membership value of a δ-arc can exceed membership values of α -strong and β- strong arcs. Also membership value of a β- strong arc can exceed that of an α - strong arc [44].

4. Cut nodes, bridges, bonds and cut bonds in fuzzy graphs

The notion of strength of connectedness plays a significant role in the structure of fuzzy graphs. When a fuzzy bridge (fuzzy cutnode) [Definitions 4.1 and 4.2] is removed from a fuzzy graph, the strength of connectedness between some pair of nodes is reduced rather than a disconnection as in the crisp case.

The notions of bridge and cutnode are extended to fuzzy graphs as follows [5,21,44,53].

Definition 4.1. *An arc (u, v) is a fuzzy bridge of $G : (\sigma, \mu)$ if the deletion of (u, v) reduces the strength of connectedness between some pair of nodes.*

Equivalently, (u, v) is a fuzzy bridge if and only if there are nodes x , y such that (u, v) is an arc of every strongest x - y path.

Definition 4.2. *A node is a fuzzy cutnode of $G : (\sigma, \mu)$ if removal of it reduces the strength of connectedness between some other pair of nodes.*

Equivalently, w is a fuzzy cutnode if and only if there exist u, v distinct from w such that w is on every strongest $u – v$ path.

Note that weakest arcs of cycles cannot be fuzzy bridges [Theorem 4.4] and it follows that if G is a fuzzy graph such that G^* is a cycle, then all arcs except the weakest are fuzzy bridges. Moreover we have the following theorem.

Theorem 4.3. [21] *Let $G : (\sigma, \mu)$ be a fuzzy graph and let (u, v) be a fuzzy bridge of G. Then $Conn_G (u, v) = \mu(u, v)$.*

Theorem 4.4. [5] *The following statements are equivalent for an arc (u, v) of a fuzzy graph $G : (\sigma, \mu)$.*

(1) (u,v) is a fuzzy bridge

(2) (u,v) is not a weakest arc of any cycle in G.

Theorem 4.5. [21] *Let $G : (\sigma, \mu)$ be a fuzzy graph such that its underlying crisp graph G^* is a cycle. Then, a node is a fuzzy cutnode of G if and only if it is a common node of two fuzzy bridges.*

Theorem 4.6. [21] *Let $G : (\sigma, \mu)$ be a fuzzy graph and let w be a common node of at least two fuzzy bridges, then w is a fuzzy cutnode.*

Theorem 4.7. [21] *If $G : (\sigma, \mu)$ is a fuzzy graph with $\sigma^* = S$ and $|S| = p$, then G has at most $p − 1$ fuzzy bridges.*

Lemma 4.8. [8] *If G is a complete fuzzy graph, then $CONNG(u, v) = \mu(u, v)$.*

Lemma 4.9. [8] *A complete fuzzy graph has no fuzzy cutnodes.*

Theorem 4.10. [21] *If G is a fuzzy tree then G is not complete.*

Theorem 4.11. [23] *A complete fuzzy graph has at most one fuzzy bridge.*

Proposition 4.12. [31] *Every fuzzy bridge is strong, but a strong arc need not be a fuzzy bridge.*

Proposition 4.13. [31] *An arc (x, y) of a fuzzy graph G is strong if and only if $\mu(x, y) = CONNG(x, y)$.*

It follows from Lemma 4.8 and Proposition 4.13 that all arcs of a complete fuzzy graph are strong.

Proposition 4.14. [31] *Let G be connected and let x, y be any two nodes in G. Then there exists a strong path from x to y.*

Corollary 4.15. [31] *If G is a nontrivial connected fuzzy graph then every node of G has at least one strong neighbor.*

Proposition 4.16. [30] *A cutnode has at least two strong neighbors.*

In fuzzy graphs, it is observed that reduction of strength of connectedness between pair of nodes on removal of fuzzy bridges depends on the particular choice of nodes. This idea has been explored in terms of bonds and cut bonds [44,53,61, 63].

Definition 4.17. [53] *An arc (x, y) is called a fuzzy bond if $CONNG-(x, y)(u, v) < CONNG(u, v)$ for some pair of nodes u and v with at least one of them different from x and y.*

Proposition 4.18. [61] *At least one of the end nodes of an f-bond is an f-cutnode.*

Remark 4.19. *In graphs, a minimal cut is a bond. Hence all bridges are bonds. But in fuzzy graphs this is not true. For example, a complete fuzzy graph can contain a fuzzy bridge and this fuzzy bridge cannot be a fuzzy bond since it has no fuzzy cut nodes..*

Definition 4.20. [63] *An arc (u, v) in a fuzzy graph is called a fuzzy cut-bond(f cut- bond) if $CONNG-(u,v)(x, y) < CONNG(x, y)$ for some pair of nodes x, y in G such that $x \neq u \neq v \neq y$.*

Remark 4.21. *All fuzzy cut-bonds are f-bonds and hence are fuzzy bridges. An arc is a fuzzy bridge if and only if it is α-strong [44]. Thus both f-bonds and f- cut-bonds are special type of α-strong arcs.*

5. Fuzzy trees

Rosenfeld [5] has proved that if there exists a unique strongest path joining any two nodes in *G* then *G* is a fuzzy tree and the converse does not hold. As in crisp graph, it is not obvious from the drawing that a fuzzy graph is a fuzzy tree. The study of fuzzy trees is explored in [5,7,21,28,44].

Definition 5.1. [21,41] *A maximum spanning tree of a connected fuzzy graph G : (σ, μ) is a fuzzy spanning subgraph T: (σ,ν) such that its underlying crisp graph T* is a tree, and for which* $\sum_{u \neq v} \nu(u,v)$ *is maximum.*

Analogous to minimum spanning tree algorithm for crisp graphs, an algorithm to obtain a maximum spanning tree of a connected fuzzy graph is given in [10]. Note that the strength of the unique *u-v* path in *T* gives the strength of connectedness between *u* and *v* for all *u,v.*

Theorem 5.2. [23] *A node w is a fuzzy cutnode of a connected fuzzy graph G : (σ, μ) if and only if w is an internal node of every maximum spanning tree of G.*

Theorem 5.3. [21] *Let G : (σ, μ) be a connected fuzzy graph and let T be a maximum spanning tree of G. Then end nodes of T are not fuzzy cutnodes.*

Corollary 5.4. [21] *Every fuzzy graph has at least two nodes which are not fuzzy cutnodes of G.*

Theorem 5.5. [23] *An arc (u, v) is a fuzzy bridge of a connected fuzzy graph G : (σ, μ) if and only if (u, v) is in every maximum spanning tree of G.*

Theorem 5.6. [41] *An arc in a fuzzy graph G is strong if and only if it is an arc of at least one maximum spanning tree of G.*

Definition 5.7. [5] *A connected f-graph G : (σ, μ) is a fuzzy tree(f-tree) if it has a fuzzy spanning subgraph F : (σ, ν) which is a tree, where for all arcs (x, y) not in F there exists a path from x to y in F whose strength is more than μ(x, y). Thus, for all arcs (x, y) which are not in F, $\mu(x, y) < CONN_F(x, y)$.*

Note that here F is a tree which contains all nodes of G and hence is a spanning tree of G. If G is not connected and if the components are fuzzy trees then, G is called a fuzzy forest. Also note that
$CONN_G(x, y) = CONN_F(x, y)$ if G is a fuzzy tree and F is the unique MST of G.

Theorem 5.8. [21] *A connected fuzzy graph is a fuzzy tree if and only if it has a unique maximum spanning tree.*

Proposition 5.9. [5] *If there is at most one strongest path between any two nodes of G, then G is a fuzzy forest.*

Theorem 5.10. [3] *Let G : (σ, μ) be a cycle. Then G (σ, μ) is a fuzzy cycle if and only if G : (σ, μ) is not a fuzzy tree.*

Theorem 5.11. [23**]** *Let G : (σ, μ) be a connected fuzzy graph with no fuzzy cycles. Then G is a fuzzy tree.*

Proposition 5.12. [5] *If G is a fuzzy tree, then the arcs of F are just the fuzzy bridges of G.*

Theorem 5.13. [23] *If G is a fuzzy tree then the removal of any fuzzy bridge reduces the strength of connectedness between its end nodes and also between some other pair of nodes.*

It follows from Theorem 5.13 that all fuzzy bridges in a fuzzy tree are fuzzy bonds.

Theorem 5.14. [21] *A connected fuzzy graph G : (σ, μ) is a fuzzy tree if and only if the following are equivalent.*
(1) *(u, v) is a fuzzy bridge.*
(2) $CONN_G(u, v) = \mu(u, v)$.

Theorem 5.15. [21] *If G is a fuzzy tree then internal nodes of F are the fuzzy cutnodes of G.*

Corollary 5.16. [21] *A fuzzy cutnode of a fuzzy tree is the common node of at least two fuzzy bridges.*

Proposition 5.17. [31] *G is a fuzzy tree if and only if there is a unique strong path in G between any two nodes of G.*

Proposition 5.18. [31] *In a fuzzy tree, a strong path between any two nodes u and v is a path of maximum strength between u and v.*

Definition 5.19. [30] *A node z is called a fuzzy end node(f- end node) of G : (σ, μ) if it has exactly one strong neighbor in G.*

The type of arcs and nodes in a fuzzy tree are studied in [21,30,44,53,62] . The following are the results based on this.

Theorem 5.20. [30] *Any non trivial fuzzy tree has at least two fuzzy end nodes.*

Theorem 5.21. [30] *A cycle C is a fuzzy tree if and only if every node of C is either a cutnode or a fuzzy end node.*

Theorem 5.22. [53] *Let G : (σ, μ) be an f-tree with $|\sigma^*| \geq 3$. An arc (x, y) in G is an f-bond if and only if (x, y) is an arc of the unique maximum spanning tree F : (σ, ν) of G.*

Theorem 5.23. [53] *Let G be an f-graph. Then G is an f-tree if and only if every strong arc of G is an f-bond of G.*

Theorem 5.24. [62] *Let T be a tree. An arc (u, v) of T is a cutbond if and only if u and v are cutnodes of G.*

Theorem 5.25. [62] *Let* $G : (\sigma, \mu)$ *be an f-tree. An arc (u, v) of G is a fuzzy cutbond if and only if u and v are f-cutnodes of G.*

Corollary 5.26. [62] *Let* $G : (\sigma, \mu)$ *be an f-tree with* $|\sigma^*| = n$ *and F its unique MST. Then the number of fuzzy cutbonds of G is* $(n - 1) - l$ *where l is the number of pendant arcs of F.*

Theorem 5.27. [30] *A cycle C is a fuzzy tree if and only if every node of C is either a fuzzy cut node or a fuzzy end node.*

6. Blocks in fuzzy graphs

The concept of block was introduced by A. Rosenfeld [5] and an excellent study on this can be found in [23,61,63]. Even then extraction of blocks from a fuzzy graph is still an open problem.

Definition 6.1. [5] *A fuzzy graph is said to be a block (also called non- separable) if it is connected and has no fuzzy cutnodes.*

Note that in a graph, a block cannot have bridges. But in fuzzy graphs, a block may have fuzzy bridges.

Theorem 6.2. [23] *The following statements are equivalent for a connected fuzzy graph* $G : (\sigma, \mu)$.
1. *G is a block.*
2. *Any two nodes u and v such that (u, v) is not a fuzzy bridge are joined by two internally disjoint strongest paths.*
3. *For every three distinct nodes of G, there is a strongest path joining any two of them not containing the third.*

Definition 6.3. [60] *A cycle C is called a strong cycle if all arcs of C are strong.*

Definition 6.4. [61] *The strength of a cycle C in an f-graph is defined as the weight of a weakest arc in C.*

In graphs, any two nodes of a block belong to a cycle and conversely [50]. In the following theorem it is shown that, when we replace cycles by strong cycles in fuzzy graphs, this condition is only necessary.

Theorem 6.5. [63] *If* $G : (\sigma, \mu)$ *is a block then the following conditions hold and are equivalent.*
(i) Any two nodes of G lie on a common strong cycle.
(ii) Each node and a strong arc of G lie on a common strong cycle.
(iii) Any two strong arcs of G lie on a common strong cycle.
(iv) For any two given nodes and a strong arc in G there exists a strong path joining the nodes containing the arc.

(v) For every three distinct nodes of G there exist strong paths joining any two of them containing the third.
(vi) For every three nodes of G there exist strong paths joining any two of them which does not contain the third.

Definition 6.6. [61] *A cycle C in an f-graph G is called a strongest strong cycle(SSC) if C is the union of two strongest strong u − v paths for every pair of nodes u and v in C except when (u, v) is an f-bridge of G in C.*

Definition 6.7. [30] *A cycle is called a locamin cycle if every node of the cycle lies on a weakest arc.*

Theorem 6.8. [61] *Let $G : (\sigma, \mu)$ be an f-graph such that G* is a cycle. Then the following are equivalent.*
(i) G is a block.
(ii) G is an SSC.
(iii) G is a locamin cycle.

Theorem 6.9. [61] *If any two nodes of an f-graph G lie on common SSC, then G is a block.*

Theorem 6.10. [23] *If G is a block with atleast one fuzzy bridge, then removal of any fuzzy bridge reduces the strength of connectedness only between its end nodes.*

Definition 6.11. [30] *A cycle is called multimin if it has more than one weakest arc.*
Note that a fuzzy cycle is nothing but a multimin cycle.

Theorem 6.12. [30] *A cycle is multimin if and only if it is not a fuzzy tree.*

Theorem 6.13. [30] *A cycle is multimin if and only if it has no fuzzy end nodes.*

Theorem 6.14. [30] *A multmin cycle is locamin if and only if it has no fuzzy cutnodes.*

Theorem 6.15 [61] *If G is a block,then no f-bridge of G is an f-bond of G and the converse is not true.*

7. Connectivity parameters in fuzzy graphs

Yeh and Bang [7] introduced two connectivity parameters of a fuzzy graph namely vertex connectivity (Ω) and edge connectivity (λ). Both these concepts are related with sets disconnecting the fuzzy graph. But in fuzzy graphs, only the reduction of strength of connectedness between some pair of nodes is relevant. Sunil Mathew and Sunitha [53,61] have generalized these definitions using the concepts of strong arcs.

Definition 7.1. [7] *A disconnection of a fuzzy graph $G : (\sigma, \mu)$ is a vertex set D whose removal results in a disconnected or a single vertex graph. The weight of D is defined to be $\Sigma_{v\in D} \{\min \mu(v, u) \mid \mu(v, u) \neq 0\}$.*

Definition 7.2. [7] *The vertex connectivity of a fuzzy graph G, denoted by Ω (G), is defined to be the minimum weight of a disconnection in G.*

Definition 7.3. [7] *Let G be a fuzzy graph and $\{V_1, V_2\}$ be a partition of its vertex set. The set of edges joining vertices of V_1 and vertices of V_2 is called a cut- set of G, denoted by (V_1, V_2) relative to the partition $\{V_1, V_2\}$. The weight of the cut-set (V_1, V_2) is defined as $\Sigma \mu(u, v)$ where $u \in V_1$ and $v \in V_2$.*

Definition 7.4. [7] *Let G be a fuzzy graph. The edge connectivity of G denoted by λ (G) is defined to be the minimum weight of cut-sets of G.*

The relations between vertex connectivity, edge connectivity and minimum degree is given as follows.

Theorem 7.5 [7] *Let G be a connected f-graph, then $\Omega(G) \le \lambda(G) \le \delta(G)$.*

Theorem 7.6. [7] *For any three real numbers a, b and c such that $0 < a \le b \le c$, there exists an f-graph G with $\Omega(G) = a$, $\lambda(G) = b$ and $\delta(G) = c$.*

Sunil Mathew and M S Sunitha introduced [53] two new connectivity parameters in a fuzzy graph namely, fuzzy node connectivity (κ) and fuzzy arc connectivity (κ') and the fuzzy analogue of Whitney's theorem is obtained.

Definition 7.7. [53] *Let G : (σ, μ) be a connected f-graph. A set of nodes $X = \{v_1, v_2, ..., v_m\}$ is said to be a fuzzy node cut (FNC) if either, $CONN_{G-X}(x, y) < CONN_G(x, y)$ for some pair of nodes $x, y \in \sigma^*$ such that both $x, y \neq v_i$ for $i = 1, 2, ..., m$ or $G - X$ is trivial.*

If there are *m* nodes in X, then X is called an *m*-FNC. Clearly a 1-FNC is a singleton set X = {u}, where u is an f-cutnode.

In [31], it is shown that there exists at least one strong arc incident on every node of a nontrivial connected fuzzy graph. Motivated by this, following definitions can be found in [53,61].

Definition 7.8. [53] *Let X be a fuzzy node cut in G. The strong weight of X, denoted by s(X), is defined as $s(X) = \sum_{x \in S} \mu(x, y)$, where $\mu(x, y)$ is the minimum of the weights of strong arcs incident on x.*

Definition 7.9. [53] *The fuzzy node connectivity of a connected fuzzy graph G is defined as the minimum strong weight of fuzzy node cuts of G. It is denoted by $\kappa(G)$.*

Definition 7.10. [53] *Let G : (σ, μ) be an f-graph. A set of strong arcs $E = \{e_1, e_2, ...e_n\}$ with $e_i = (u_i, v_i)$, $i = 1, 2, ..., n$ is said to be a fuzzy arc cut (FAC) if $CONN_{G-E}(x, y) < CONN_G(x, y)$ for some pair of nodes $x, y \in \boldsymbol{\sigma}^*$ with at least one of x and y is different from both u_i and v_i, $i = 1, 2, ...n$. or G - E is disconnected.*

Definition 7.11. [53] *The strong weight of a fuzzy arc cut E is defined as $S'(E) = \sum_{e_i \in E} \mu(e_i)$.*

Definition 7.12. [53] *The fuzzy arc connectivity* $\kappa'(G)$ *of a connected fuzzy graph G is defined as the minimum strong weight of fuzzy arc cuts of G.*

In a tree with at least three nodes, $\kappa(G) = \kappa'(G) = 1$. This is due to the fact that all arcs in a tree are strong with strength one and so we have the fuzzy analogue,

Theorem 7.13. [53] *In an f-tree* $G : (\sigma, \mu)$, $\kappa(G) = \kappa'(G) = \wedge\{\mu(x, y) : x, y \in \sigma^*$ *here* (x, y) *is a strong arc in G.*

Bhutani and Rosenfeld[31] introduced the concepts of strong arcs and strong paths in fuzzy graphs. These concepts motivated researchers to reformulate some of the concepts into fuzzy graph theory more effectively. Sameena and Sunitha [41] have defined the strong degree ds(v) of a node in an f-graph. The concept of strong degree is relevant in fuzzy graph applications especially problems related with flows as the flow through arcs, which are not strong, can be redirected through a different strongest path. Bhutani and Rosenfeld [31] have shown the existence of a strong path between any two nodes of a fuzzy graph. Thus there exists at least one strong arc incident at each node of a nontrivial connected f-graph.

Definition 7.14. [60] *Let* $G : (\sigma, \mu)$ *be a fuzzy graph. The strong degree of a node v is defined as the sum of membership values of all strong arcs incident at v and it is denoted by* $d_s(v)$. *Also, the minimum strong degree of G is* $\delta s(G) = Min\{d_s(v) | v \in \sigma^*\}$ *and maximum strong degree of G is* $\Delta s(G) = Max\{d_s(v), v \in \sigma^*\}$.

Theorem 7.15. [53] **(Fuzzy analogue of Whitney's Theorem)** *In a connected f-graph* $G : (\sigma, \mu)$, $\kappa(G) \le \kappa'(G) \le \delta_s(G)$.

7.1. Menger's Theorem for fuzzy graphs

The concept of the strongest path plays a crucial role in fuzzy graph theory. In classical graph theory, all paths in a graph are strongest with a strength value of one. In [64], Sunil Mathew and M S Sunitha, introduced Menger's theorem for fuzzy graphs and discuss the concepts of strength reducing sets and t-connected fuzzy graphs.

In graph theory, a u - v separating set S of nodes is a collection of nodes in G whose removal disconnects the graph G and u and v belong to different components of G - S [50]. Similarly a u - v separating set of arcs is defined. Since the reduction in strength is more important and frequent in graphs and networks, the concept of strength reducing sets of nodes and arcs are defined as follows.

Definition 7.16. [64] *Let u and v be any two nodes in a fuzzy graph* $G : (\sigma, \mu)$ *such that the arc (u, v) is not strong. A set* $S \subseteq \sigma^*$ *of nodes is said to be a u - v strength reducing set of nodes if* $CONN_{G-S}(u, v) < CONN_G(u, v)$ *where G - S is the fuzzy subgraph of G obtained by removing all nodes in S.*

Definition 7.17. [64] *A set of arcs* $E \subseteq \mu^*$ *is said to be a u - v strength reducing set of arcs if* $CONN_{G-E}(u, v) < CONN_G(u, v)$, *where G - E is the fuzzy subgraph of G obtained by removing all arcs in E.*

Definition 7.18. [64] *A u - v strength reducing set of nodes (arcs) with n elements is said to be a minimum u - v strength reducing set of nodes (arcs) if there exist no u - v strength reducing set of nodes (arcs) with less than n elements. A minimum u - v strength reducing set of nodes is denoted by $S_G(u, v)$ and a minimum u - v strength reducing set of arcs is denoted by $E_G(u, v)$.*

Theorem 7.19. [64] *Let G : (σ, μ) be a connected fuzzy graph and u, v any two nodes in G such that (u, v) is not strong. Then a set S of nodes in G is a u - v strength reducing set if and only if every strongest path from u to v contains at least one node of S.*

Generalization of the node version of Menger's theorem is given in the following theorem.

Theorem 7.20. [64] *Let G : (σ, μ) be a fuzzy graph. For any two nodes u, v $\in$ σ* such that (u, v) is not strong, the maximum number of internally disjoint strongest u - v paths in G is equal to the number of nodes in a minimal u - v strength reducing set.*

Generalization of the arc version of Menger's theorem is given in the following theorem.

Theorem 7.21. [64] *Let G : (σ, μ) be a connected fuzzy graph and let u, v $\in$ σ*. Then the maximum number of arc disjoint strongest u - v paths in G is equal to the number of arcs in a minimum (with respect to the number of arcs) u - v strength reducing set.*

In [64] the concepts of t-connected fuzzy graphs and t-arc connected fuzzy graphs are discussed and characterized using the strongest paths.

8. Distance in fuzzy graphs

Now in fuzzy graph there are at least four metrics as given below.

1. μ-distance [5]
2. g-distance [32]
3. δ -distance [3]
4. ss-distance [60].

Definition 8.1. [5] *The μ- distance d_μ (u, v) is the smallest μ-length of any u - v path, where the μ - length of a path P : $u_0, u_1, \ldots\ldots\ldots, u_n$ is* $\ell(P) = \sum_{i=1}^{n} \frac{1}{\mu(u_{i-1}, u_i)}$.

If $n = 0$, then define $\ell(P) = 0$.

Definition 8.2. [3] *Let μ be a fuzzy relation on S . Then the δ - distance is defined as $d_\delta(x, y) = 1 - CONN_G(x, y)$.*

Definition 8.3. [32] *A strong path P from x to y is an x - y geodesic if there is no shorter strong path from x to y and the length of an x - y geodesic is defined as the geodesic distance from x to y denoted by $d_g(x,y)$.*

The existence of a strongest x − y path and a strong x − y path between any two nodes in a connected fuzzy graph has been proved . Here it is established that there is a strongest strong path (ss-path) between any two nodes in a connected fuzzy graph [60].

An x − y path P in a fuzzy graph G is called a strongest strong path (ss - path) if P is a strongest x − y path as well as strong x − y path . Any connected fuzzy graph G has at least one maximum spanning tree T. Now for any nodes x, y in G, an x − y path P in T is strongest and contains only strong arcs.

Theorem 8.4. [60] *Let x, y be any two nodes in a connected fuzzy graph G. Then there exists a strongest strong path from x to y.*

Note that every strong arc (x, y) is a strongest x − y path and hence we have

Theorem 8.5. [60] *An arc (x, y) is strongest strong if and only if it is strong.*

Definition 8.6. [60] *For any two vertices u , v ,the ss-distance $d_{ss}(u, v)$ in a fuzzy graph $G: (\sigma, \mu)$ is the reciprocal of the connectivity between u and v.*

$$d_{ss}(u,v) = \begin{cases} \dfrac{1}{CONN_G(u,v)} & if \quad u \neq v \\ 0, & if \quad u = v \end{cases}$$

Note that if G : (σ, μ) is disconnected and two vertices (say) u and v of G are not connected by a path, then $CONN_G(u, v) = 0$ and $d_{ss}(u, v) = \infty$ by the definition.

The following concepts are defined for all types of distances in a fuzzy graph G.
The eccentricity *e(v)* of a node *v* is the distance to a node farthest from v. The radius *r(G)* is the minimum eccentricity of the nodes and the diameter *d(G)* is the maximum eccentricity. A node *v* is a central node if *e(v) = r(G)* and $\langle C(G)\rangle = H : (\tau, \nu)$, the fuzzy subgraph of $G : (\sigma, \mu)$ induced by the central nodes of *G* is called the center of *G*. A connected fuzzy graph is self centered if $\langle C(G)\rangle$ is isomorphic to *G*.

Theorem 8.7. [19] *A connected fuzzy graph $G : (\sigma, \mu)$ is μ - self centered if $CONN_G(u, v) = \mu(u, v)$ for all u, v in V and $r(G) = \dfrac{1}{\mu(u,v)}$ where $\mu(u, v)$ is least.*

Corollary 8.8. [19] *A complete fuzzy graph is μ - self centered and $r(G) = \dfrac{1}{\sigma(u)}$, where $\sigma(u)$ is least.*

As a consequence, there exists self centered fuzzy graph of radius c for each real
number *c > 0*. Also, for any two real numbers *a, b* such that $0 < a \leq b \leq 2a$, there exists a fuzzy graph *G* such that *r (G) = a* and *d (G) = b*.

An obvious necessary condition for a fuzzy graph to be μ - self centered is that each node is eccentric. The construction of a fuzzy graph *G* such that $\langle C(G)\rangle \approx H$ is carried out in [19].

Theorem 8.9. [19] *Let H = $(\sigma^{/},\mu^{/})$ be a fuzzy graph. Then there exist a fuzzy graph G : (σ, μ) such that for the μ - centre of G , $\langle C(G)\rangle$ is isomorphic to H. Also r(G) = d and d(G) = 2d.*

Theorem 8.10. [19] *Let H : $(\sigma^{/}, \mu^{/})$ be a fuzzy tree with diameter d. Then there exists a fuzzy tree G : (σ, μ) such that the μ - centre of G , $\langle C(G)\rangle$ is isomorphic to H.*

Theorem 8.11. [43,60] *A connected fuzzy graph G that is a fuzzy cycle is g-self centered.*

Theorem 8.12. [43,60] *A necessary condition for a fuzzy graph G to be g-self centered fuzzy graph is that each node of G is g-eccentric.*

Theorem 8.13. [43,60] *A connected fuzzy graph G such that G* is complete is g-self centered, if each arc in G is strong. Further $r_g(G) = 1$. Note that the converse need not be true.*

Corollary 8.14. [43,60] *A complete fuzzy graph is g-self centered and $r_g(G) = 1$.*

Theorem 8.15. [43,60] *Let H: $(\sigma^{/}, \mu^{/})$ be a fuzzy graph. Then there exists a connected fuzzy graph G : (σ, μ) such that the g – center of G is isomorphic to H. Also $r_g(G) = 2$ and $d_g(G) = 4$.*

Theorem 8.16. [42,60] *Let G: (σ, μ) be a fuzzy tree and T: (σ, ν) be the maximum spanning tree of G. Then for each node v in G, $e_g(v)$ in G is the same as $e_g(v)$ in T.*

Corollary 8.17. [42,60] *Let G be a fuzzy tree and T be the maximum spanning tree of G. Then the g - center of G is isomorphic to the g - center of T.*

Corollary 8.18. [42,60] *Let G be a fuzzy tree and T be the maximum spanning tree of G. Then G and T have same set of g-eccentric nodes.*

Theorem 8.19. [42,60] *Let G be a fuzzy tree. Then v is a g-eccentric node of G if and only if v is a g-peripheral node of G.*

Next, with respect to the δ - distance and ss – distance it is established that every connected fuzzy graph G : (σ, μ) is self centered [60].

9. Complement of a fuzzy graph

The study of complement of a fuzzy graph G is made in [22]. The properties of G and its complement $\overline{G}$ such as the automorphism group of *G* and $\overline{G}$ are identical is established. Distinct from crisp theory, it is also observed that a node can be a fuzzy cutnode of both *G* and $\overline{G}$.

If $G \approx \overline{G}$, then *G* is called a self complementary fuzzy graph and independent necessary and sufficient conditions for a fuzzy graph *G* to be self complementary are obtained.

Theorem 9.1. [22] *Let G : (σ, μ) be a selfcomplementary fuzzy graph. Then*

$$\sum_{u \neq v} \mu(u,v) = \frac{1}{2} \sum_{u \neq v} (\sigma(u) \wedge \sigma(v)).$$

Theorem 9.2. [22] *Let G :(σ, μ) be a fuzzy graph. If*

$\mu(u,v) = \frac{1}{2}(\sigma(u) \wedge \sigma(v)) \forall\ u,v \in V$, *then G is self complementary.*

Operations on fuzzy graphs such as union, join, cartesian product, composition have been studied with respect to the complement and established that complement of the union of two fuzzy graphs is the join of their complements and complement of the join of two fuzzy graphs is the union of their complements [22].

10. Other types of fuzzy graphs

Shannon and Atanassov, Akram, Ramakrishna, Samanta and Pal have studied about other types of fuzzy graphs such as vague fuzzy graphs, bipolar fuzzy graphs, interval valued fuzzy graphs, intutionistic fuzzy graphs, fuzzy k-competition graphs and p-competition fuzzy graphs etc. in [55 – 59] and in [65 – 71].

A vague set, as well as an intuitionistic fuzzy set is a further generalization of a fuzzy set. In the literature, the notions of intuitionistic fuzzy sets and vague sets are regarded as equivalent, in the sense that an intuitionistic fuzzy set is isomorphic to a vague set.

In a fuzzy set each element is associated with a point-value selected from the unit interval [0,1], which is termed the grade of membership in the set. Instead of using point-based membership as in fuzzy sets, interval-based membership is used in a vague set. The interval based membership in vague sets is more expressive in capturing vagueness of data.

11. Conclusion

The study of fuzzy graphs made in this report is far from being complete . We sincerely hope that the wide ranging applications of graph theory and the interdisciplinary nature of fuzzy set theory, if properly blended together could pave a way for a substantial growth of fuzzy graph theory. Research on the theory of fuzzy sets has been witnessing an exponential growth; both within mathematics and in its applications. This ranges from traditional mathematical subjects like logic, topology, algebra, analysis etc. to pattern recognition, information theory, artificial intelligence, operations research, neural networks, planning etc. Consequently, fuzzy set theory has emerged as a potential area of interdisciplinary research. We hope that the growth of fuzzy graph theory will be further accelerated by the development of fuzzy software and fuzzy hardware.

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