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Some Characterization of Semiprime n-Ideals in Lattices

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Abstract. The concept of semi prime ideals in lattices was given by Y. Rav by generalizing the concept of 0-distributive lattices given by J. C. Varlet. For a neutral element $n \in L$, recently Ayub, Noor and Podder have introduced the concept of n-distributive lattices which is a generalization of both 0-distributive and 1-distributive lattices. In a very recent paper, M. Ayub Ali and others have generalized the concept of n-distributive lattices and given the notion of semi prime n-ideals. For an element n in a lattice L, any convex sublattice containing n is called an n-ideal. In this paper, we have included several characterizations of semi prime n-ideals in lattices. We have given a characterization of minimal prime n-ideals containing $\{a\}^{\perp^{n}_{J}}$ for $a \in L$. Finally we have included a prime Separation Theorem with the help of annihilator n-ideal.

Keywords. Neutral element, Semi prime n-ideal, Maximal convex sublattice, Minimal prime n-ideal.

AMS Subject Classifications (2010): 06A12, 06A99, 06B10

1. Introduction

In generalizing the notion of pseudo complemented lattice, J. C. Varlet [9] introduced the notion of 0-distributive lattices. Then [3] have given several characterizations of these lattices. On the other hand, [7] have studied them in meet semi lattices. A lattice L with 0 is called a *0-distributive* lattice if for all $a,b,c \in L$ with $a \wedge b = 0 = a \wedge c$ imply $a \wedge (b \vee c) = 0$. Let L be a lattice and $n \in L$. Any convex sublattice of L containing n is called an n-*ideal* of L. An element $n \in L$ is called a standard element if for all $a,b \in L$, $a \wedge (b \vee n) = (a \wedge b) \vee (a \wedge n)$; while n is called a neutral element if

(i) it is standard and

(ii) $n \wedge (a \vee b) = (n \wedge a) \vee (n \wedge b)$ for all $a,b \in L$.

Set of all n-ideals of a lattice L is denoted by $I_n(L)$ which is an algebraic lattice; where $\{n\}$ and L are the smallest and largest elements. For two n-ideals I and J, $I \cap J$ is the infimum and $I \vee J = \{x \in L : i_1 \wedge j_1 \le x \le i_2 \vee j_2, \textit{for some } i_1, i_2 \in I \textit{ and } j_1, j_2 \in J\}$. The n-ideal generated by a finite numbers of elements $a_1, a_2, ..., a_m$ is called a finitely generated n-ideal denoted by $< a_1, a_2, ..., a_m >_n$. Moreover, $< a_1, a_2, ..., a_m >_n = \{x \in L / a_1 \wedge a_2 \wedge ... \wedge a_m \wedge n \le x \le a_1 \vee a_2 \vee ... \vee a_m \vee n\}$ $= [a_1 \wedge a_2 \wedge .... \wedge a_m \wedge n, a_1 \vee a_2 \vee .... \vee a_m \vee n]$

Thus, every finitely generated n-ideal is an interval containing n. n-ideal generated by a single element $a \in L$ is called a principal n-ideal denoted by $< a >_n$ and $< a >_n = [a \wedge n, a \vee n]$. Moreover $[a,b] \cap [c,d] = [a \vee c, b \wedge d]$ and $[a,b] \vee [c,d] = [a \wedge c, b \vee d]$. If n is a neutral element, then by [6], $< a >_n \cap < b >_n = < m(a,n,b) >_n$, where $m(x,y,z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$.

A non-empty subset I of a lattice L is called a *down set* (*up set*) if for $a \in I$ and $x \le a$ ($x \ge a$), $(x \in L)$ imply $x \in I$. I is called an *ideal* if it is a down set and for all $a,b \in I$, $a \vee b \in I$. A non-empty subset F of L is called a filter of L if it is an *up set and for* $a,b \in F$, $a \wedge b \in F$. A subset T of L is called *convex* if for $a \le x \le b$ with $a,b \in T$ imply $x \in T$. Of course all the ideals and filters of a lattice are convex sublattices. Moreover, for every convex sublattice C of L, $C = (C] \cap [C)$.

A proper convex sublattice M of a lattice L is called a maximal convex sublattice if for any convex sublattice Q with $Q \supseteq M$ implies either $Q = M$ or $Q = L$. A proper convex sublattice M is called a prime convex sublattice if for any $t \in M$, $m(a,t,b) \in M$ implies either $a \in M$ or $b \in M$. Similarly, an n-ideal P of L is called a prime n-ideal if $m(a,n,b) \in P$ implies either $a \in P$ or $b \in P$. Equivalently, P is prime if and only if $< a >_n \cap < b >_n \subseteq P$ implies either $< a >_n \subseteq P$ or $< b >_n \subseteq P$. Moreover, by

[4], we know that every prime convex sublattice P of L is either an n-ideal or a filter.

By [2] L is called an n-distributive lattice if for all $a,b,c \in L$, $<a>_n \cap <b>_n = \{n\}$ and $<a>_n \cap <c>_n = \{n\}$ imply $<a>_n \cap [<b>_n \vee <c>_n] = \{n\}$. Equivalently, L is called n-distributive if $a \wedge b \leq n \leq a \vee b$ and $a \wedge c \leq n \leq a \vee c$ imply $a \wedge (b \vee c) \leq n \leq a \vee (b \wedge c)$.

By Y. Rav [8] an ideal I of a lattice is called a *semi prime ideal* if for all $x,y,z \in L$, $x \wedge y \in I$ and $x \wedge z \in I$ imply $x \wedge (y \vee z) \in I$. Thus, a lattice L with 0, is called *0-distributive* if and only if $(0]$ is a semi prime ideal. Let n be a neutral element of a lattice L. An n-ideal J of L is called a semi prime n-ideal if for all $a,b,c \in L$, $<a>_n \cap <b>_n \subseteq J$ and $<a>_n \cap <c>_n \subseteq J$ imply $<a>_n \cap (<b>_n \vee <c>_n) \subseteq J$. In a distributive lattice every n-ideal is semi prime. Moreover, every prime n-ideal is semi prime. Lattice itself with an element n is of course a semi prime n-ideal. It is easy to see that a lattice with the element n is n-distributive if $\{n\}$ is a semi prime n-ideal. In the pentagonal lattice $\{0,a,b,c,n; a<b, a \vee c = b \vee c = n, a \wedge c = b \wedge c = 0\}$, n is neutral. Here $\{n\}$ and $<0>_n = L$ are semi prime but not prime. Moreover, $<a>_n, <c>_n$ are prime but $<b>_n$ is not even semi prime. Again in $M_3 = \{0,a,b,c,n; a \wedge b = b \wedge c = c \wedge a = 0;\ a \vee b = b \vee c = c \vee a = n\}$, $<0>_n = L$ is semi prime. But $\{n\}$, $<a>_n$, $<b>_n$, $<c>_n$ are not semi prime.

Throughout the paper we will consider n as a neutral element.

Following result is easy to prove.

Lemma 1. *Let F be a filter (ideal) of L disjoint to an n-ideal J. Then F is a maximal filter (ideal) if and only if L-F minimal prime down set (up set) containing J.* ■

Following results are due to [1]

Lemma 2. *Every convex sublattice disjoint from an n-ideal I is contained in a maximal convex sub lattice disjoint from I.* ■

Lemma 3. *Let I be an n- ideal of a lattice L. A convex sublattice M disjoint from I is a maximal convex sublattice disjoint from I if and only if for all $a \notin M$, there exists $b \in M$ such that $m(a,n,b) \in I$.* ■

Let L be a lattice with neutral element n. For $A \subseteq L$, we define $A^{\perp_n} = \{x \in L : m(x,n,a) = n \text{ for all } a \in A\}$. $A^{\perp_n}$ is always a convex sub set containing n but it is not necessarily an n - ideal.

Theorem 4. *J is a semi prime n-idealof a lattice L if and only if $(J]$ is a semi prime ideal and $[J)$ is a semi prime filter.*

Proof. Let $x \vee y \in [J)$ and $x \vee z \in [J)$. Then $x \vee y \geq j_1$ and $x \vee z \geq j_2$ for some $j_1, j_2 \in J$. Thus $j_1 \wedge n \leq (x \vee y) \wedge n \leq n$ implies $(x \vee y) \wedge n \in J$ by convexity. So $m(x,n,y \wedge n) = (x \vee y) \wedge n \in J$ *implies* $< x >_n \cap < y \wedge n >_n \subseteq J$. Similarly, $< x >_n \cap < z \wedge n >_n \subseteq J$. Since J is semi prime, so $< x >_n \cap (< y \wedge n >_n \vee < z \wedge n >_n) = [x \wedge n, x \vee n] \cap [y \wedge z \wedge n, n] =$ $[(x \vee (y \wedge z)) \wedge n, n] \subseteq J$ implies $(x \vee (y \wedge z)) \wedge n \in J$, and so $x \vee (y \wedge z) \in [J)$. Therefore, $[J)$ is a semi prime filter. Similarly, we can prove that $(J]$ is a semi prime ideal.

Conversely, let $< x >_n \cap < y >_n \subseteq J$ and $< x >_n \cap < z >_n \subseteq J$. That is $[(x \vee y) \wedge n, (x \wedge y) \vee n] \subseteq J$ and $[(x \vee z) \wedge n, (x \wedge z) \vee n] \subseteq J$. It follows that $(x \wedge y) \vee n \in J$ and $(x \wedge z) \vee n \in J$. Thus $(x \vee n) \wedge (y \vee n) \in J$ and $(x \vee n) \wedge (z \vee n) \in J$ as n is neutral. Then $x \wedge (y \vee n) \in (J]$ and $x \wedge (z \vee n) \in (J]$. *So* $x \wedge (y \vee z \vee n) \in (J]$ as $(J]$ is a semi prime ideal. This implies $(x \wedge (y \vee z)) \vee (x \wedge n) \in (J]$ and so $(x \wedge (y \vee z)) \vee (x \wedge n) \leq j_1$ for some $j_1 \in J$. Then $n \leq (x \wedge (y \vee z)) \vee n \leq j_1 \vee n$ implies $(x \wedge (y \vee z)) \vee n \in J$. Similarly, we can prove that $(x \vee (y \wedge z)) \wedge n \in J$ *as* $[J)$ is a semi prime filter. Therefore, $< x >_n \cap (< y >_n \vee < z >_n) \subseteq J$ $= [(x \vee (y \wedge z)) \wedge n, (x \wedge (y \vee z)) \vee n] \subseteq J$, and so J is semi prime. ■

Theorem 5. *If the intersection of all prime n-ideals of a lattice L is equal to J, then J is a semi prime n-ideal.*

Proof. Let $< a >_n \cap < b >_n \subseteq J$, $< a >_n \cap < c >_n \subseteq J$. This implies $[(a \vee b) \wedge n, (a \wedge b) \vee n] \subseteq J$, $[(a \vee c) \wedge n, (a \wedge c) \vee n] \subseteq J$.

Let P be any prime n-ideal. If $a \in P$, then $< a >_n \subseteq P$ and so $< a >_n \cap (< b >_n \vee < c >_n) \subseteq P$. If $< a >_n \not\subseteq P$ then $< b >_n \subseteq P$, $< c >_n \subseteq P$ as P is prime. Thus $< b >_n \vee < c >_n \subseteq P$. That is, in either case $< a >_n \cap (< b >_n \vee < c >_n) \subseteq P$ for all prime n-ideals P containing J. Therefore, $< a >_n \cap (< b >_n \vee < c >_n) \subseteq \cap P = J$. Hence J is semi prime. ■

Let $A \subseteq L$ and J be an n-ideal of L. We define $A^{\perp^n J} = \{x \in L : m(x,n,a) \in J \text{ for all } a \in A\}$. This is clearly a convex subset containing J. In presence of distributivity, this is an n- ideal. $A^{\perp^n J}$ is called an n- annihilator of A relative to J. We denote $I_J(L)$, the set of all n-ideals containing J. Of course, $I_J(L)$ is a bounded lattice with J and L as the smallest and the largest elements. If $A \in I_J(L)$, and $A^{\perp^n J}$ is an n- ideal, then $A^{\perp^n J}$ is called an annihilator n- ideal and it is the pseudo complement of A in $I_J(L)$.

Theorem 6. *Let A be a non-empty subset of a lattice L and J be an n-ideal of L. Then*
$A^{\perp^n J} = \bigcap(P : P$ is a minimal prime convex subset containing J but not containing A

Proof. Suppose $X = \bigcap(P : A \square P, P$ is a minimal prime convex set). Let $x \in A^{\perp_J}$. Then $m(x,n,a) \in J$ for all $a \in A$. Choose any P of right hand expression. Since $A \square P$, there exists $z \in A$ but $z \notin P$. Then $m(x,n,z) \in J \subseteq P$. So $x \in P$, as P is prime. Hence $x \in X$.

Conversely, let $x \in X$. If $x \notin A^{\perp^n J}$, then $m(x,n,b) \notin J$ for some $b \in A$. Thus $[(x \vee b) \wedge n, (x \wedge b) \vee n]\ P \square J$. This implies either $m(x \wedge n, n, b \wedge n) = (x \wedge n) \vee (b \wedge n) \notin J$ or $m(x \vee n, n, b \vee n) = (x \wedge b) \vee n \notin J$. This implies either $x \wedge n \notin \{b \wedge n\}^{\perp^n j}$ or $x \vee n \notin \{b \vee n\}^{\perp^n j}$. Suppose $(x \wedge b) \vee n \notin J$. Set $D = [(x \wedge b))$. Then D is a filter disjoint to J. Then by lemma 3, there exists a maximal filter $F \supseteq D$ and disjoint to J. Then $L - M$ is a maximal prime down set containing J. Now $x \notin L - M$ as $x \in D$ implies $x \in M$. Moreover, A $\square L - M$ as $b \in A$ but $b \in M$ implies $b \notin L - M$, which is a contradiction to $x \in X$. Therefore, $x \in A^{\perp^n J}$. ■

Following result is due to [1].

Theorem 7. *Let* L *be a lattice and* J *be an n-ideal of* L. *The following conditions are equivalent.*

(i) J *is semi prime.*

(ii) $\{a\}^{\perp^n J} = \{x \in L : x \wedge a \in J\}$ *is a semi prime n- ideal containing* J.

(iii) $A^{\perp^{n}J} = \{x \in L : x \wedge a \in J \text{ for all } a \in A\}$ *is a semi prime n-ideal containing* J.
(iv) $I_J(L)$ *is pseudo complemented*
(v) $I_J(L)$ *is a 0 –distributive lattice.*
(vi) *Every maximal convex sublattice disjoint from* J *is prime.* ■

Theorem 8. *Let L be a lattice and J be an n-ideal. Then the following conditions are equivalent.*
(i) *J is semi-prime.*
(ii) *Every maximal convex sublattice of L disjoint with J is prime.*
(iii) *Every minimal prime down set (up set) containing J is a minimal prime n-ideal containing J.*
(iv) *Every filter (ideal) disjoint with J is disjoint from a minimal prime n-ideal containing J.*
(v) *For each element* $a \notin J$, *there is a minimal prime n-ideal containing J but not containing a.*
(vi) *For each* $a \notin J$, $(a]$ *is contained in a prime ideal and* $[a)$ *is contained in a prime filter disjoint to J.*
Proof. (i)$\Leftrightarrow$(ii) follows from Theorem 7.
(ii)$\Rightarrow$(iii). Let A be a minimal prime down set (up set) containing J. Then $L - A$ is a maximal filter (ideal) disjoint with J. Then by (ii) $L - A$ is prime and so A is a minimal prime ideal (filter) containing J and so it is a minimal prime n-ideal.

(iii)$\Rightarrow$(ii). Let F be a maximal convex sublattice of L disjoint to J. Then by [4], F is either an ideal or a filter. Suppose F is a filter. Then $L - F$ is a minimal prime down set containing J. Thus by (iii), $L - F$ is a minimal prime ideal and so F is a prime filter and so is a prime convex sublattice.

(i)$\Rightarrow$(iv). Let F a filter of S disjoint from J. Then by Theorem 7, there is a prime (maximal) convex sublattice $Q \supseteq F$ and disjoint to J. Then $L - Q$ is a minimal prime ideal containing J and disjoint to F. Since $n \in J$ so $L - Q$ is a minimal prime n-ideal.

(iv)$\Rightarrow$(v). Let $a \in L$, $a \notin J$. Then by (iv) there exists a minimal prime n-ideal A containing J disjoint from $[a)$. Thus $a \notin A$.
(v)$\Rightarrow$(vi). Let $a \in L$ and $a \notin J$. Then by (v) there exists a minimal prime n-ideal containing J such that $a \notin P$ this implies $a \in L - P$. By [4], we know

that P is either an ideal or a filter. Thus $L-P$ is a prime ideal or a prime filter disjoint to J.

(vi)$\Rightarrow$(i). Suppose J is not semi prime. Then there exist $a,b,c \in L$ such that
$<a>_n \cap <b>_n \subseteq J$, $<a>_n \cap <c>_n \subseteq J$ but $<a>_n \cap (<b>_n \vee <c>_n) \not\subseteq J$

Then $[(a\vee b)\wedge n,(a\wedge b)\vee n]\subseteq J$, $[(a\vee c)\wedge n,(a\wedge c)\vee n]\subseteq J$ and $[(a\vee(b\wedge c))\wedge n,(a\wedge(b\vee c))\vee n]\not\subseteq J$. Then either $(a\vee(b\wedge c))\wedge n\notin J$ or $(a\wedge(b\vee c))\vee n\notin J$. Suppose $(a\vee(b\wedge c))\wedge n\notin J$, which implies $a\wedge(b\vee c)\notin J$. Then by (vi) there exists prime filter Q disjoint to J such that $[a\wedge(b\vee c))\subseteq Q$. Then $a\in Q$ and $b\vee c\in Q$. Since Q is prime, so either $b\in Q$ or $c\in Q$. Thus either $a\wedge b\in Q$ or $a\wedge c\in Q$ which implies $(a\wedge b)\vee n\in Q$ or $(a\wedge c)\vee n\in Q$ gives a contradiction to the fact that $Q\cap J=\varphi$. Therefore, J must be semi prime. ■

Theorem 9. *Let J be a semi-prime n-ideal of a lattice L and $x\in L$. Then a prime - ideal P containing $\{x\}^{\perp_J^n}$ is a minimal prime n-ideal containing $\{x\}^{\perp_J^n}$ if and only if for $p\in P$, there exists $q\in L-P$ such that $m(p,n,q)\in\{x\}^{\perp_J^n}$.*

Proof: Let P be a prime ideal containing $\{x\}^{\perp_J^n}$ such that the given condition holds. Let K be a prime n-ideal containing $\{x\}^{\perp_J^n}$ such that $K\subseteq P$. Let $p\in P$. Then there is $q\in L-P$ such that $m(p,n,q)\in\{x\}^{\perp_J^n}$. Hence $m(p,n,q)\in K$. Since K is prime and $q\notin K$, so $p\in K$. Thus, $P\subseteq K$ and so $K=P$. Therefore, P must be a minimal prime n-ideal containing $\{x\}^{\perp_J^n}$.

Conversely, let P be a minimal prime n-ideal containing $\{x\}^{\perp_J^n}$. Let $p\in P$. Suppose $m(p,n,q)\notin\{x\}^{\perp_J^n}$ for all $q\in L-P$. Thus $[(p\vee q)\wedge n,(p\wedge q)\vee n]\not\subseteq\{x\}^{\perp_J^n}$. So either $(p\vee q)\wedge n\notin\{x\}^{\perp_J^n}$ or $(p\wedge q)\vee n\notin\{x\}^{\perp_J^n}$. Suppose $(p\vee q)\wedge n\notin\{x\}^{\perp_J^n}$. Let $D=(L-P)\vee[p)$. We claim that $\{x\}^{\perp_J^n}\cap D=\varphi$. If not, let $y\in\{x\}^{\perp_J^n}\cap D$. Then $p\wedge q\le y\in\{x\}^{\perp_J^n}$ for some $q\in L-P$. Thus $n\le(p\wedge q)\vee n\le y\vee n$ implies $(p\wedge q)\vee n\in\{x\}^{\perp_J^n}$ gives a contradiction. Then by Theorem [7], there exists a maximal (prime) convex sublattice $Q\supseteq D$ and disjoint to $\{x\}^{\perp_J^n}$. We prove that $x \in Q$. If $x \notin Q$ then $(Q \vee [x)) \cap \{x\}^{\perp^n J} \neq \varphi$. Suppose $t \in (Q \vee [x)) \cap \{x\}^{\perp^n J}$. This implies $t \geq q_1 \wedge x$ and $m(t,n,x) \in J$ for some $q_1 \in Q$. Thus $q_1 \wedge x \leq t \wedge x$ and $(x \wedge t) \vee n \in J$. It follows that $(q_1 \wedge x) \vee n \in J$. So $q_1 \vee n \in Q$ as Q is a filter. Again $m(q_1 \vee n, n, x) = (q_1 \wedge x) \vee n \in J$ implies $q_1 \vee n \in \{x\}^{\perp^n J}$, which is again a contradiction. Hence $x \in Q$. Let $M = L - Q$. Then M is a prime ideal, infact prime n-ideal. Since $x \in Q$, so $x \notin M$. Let $r \in \{x\}^{\perp^n J}$. Then $m(r,n,x) \in J \subseteq M$. This implies $r \in M$ as M is prime. Thus, $\{x\}^{\perp^n J} \subseteq M$. Now $M \cap D = \varphi$. This implies $M \cap (L - P) = \varphi$ and hence $M \subseteq P$. Also $M \neq P$, because $p \in D$ implies $p \notin M$ but $p \in P$. Hence M is a prime n-ideal containing $\{x\}^{\perp^n J}$ which is properly contained in P. This gives a contradiction to the minimal property of P. Therefore the given condition holds. ■

We conclude the paper with the following prime Separation theorem for semi prime n-ideals

Theorem 10. Let J *be an n-ideal of a lattice* L. *Then the following conditions are equivalent:*

(i) *J is semi prime*

(ii) *For any proper convex sublattice F disjoint to J there is a prime convex sublattice Q* containing *F such that* $Q \cap J = \phi$.

Proof. (i)$\Rightarrow$(ii). Since $F \cap J = \phi$, so by Lemma 2, there exists a maximal convex sublattice $Q \supseteq F$ such that $Q \cap J = \phi$. Then by Theorem 7, Q is prime.

(ii)$\Rightarrow$(i). Let F be a maximal convex sublattice disjoint to J. Then by (ii) there exists a prime convex sublattice $Q \supseteq F$ such that $Q \cap J = \phi$. Since F is maximal, so $Q = F$. This implies F is prime and so by Theorem 7, J must be semi prime. ■

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