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A Note on the Diophantine Equation $2^a + 7^b = c^2$ a, b are Odd Integers

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Abstract. In [6], the authors discuss the Diophantine equation $4^x + 7^y = z^2$ i.e., $2^{2x} + 7^y = z^2$. They show that the equation has no solutions in non-negative integers. The equation in [6] is a particular case of the equation $2^a + 7^b = c^2$, and the author has respectively shown in [3, 2]: When $a \geq 1$ and $b = 1$, the unique solution is $(a, b, c) = (1, 1, 3)$, whereas for all odd values a with all even values b, the unique solution is $(a, b, c) = (5, 2, 9)$. The purpose of this Note is to complete the set of all solutions of $2^a + 7^b = c^2$ by considering all odd values a with all odd values b. We show that no solutions exist in this case. The equation $2^a + 7^b = c^2$ has therefore only the above two solutions.

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AMS Mathematics Subject Classification (2010): 11D61

1. Introduction

The field of Diophantine equations is ancient, vast, and no general method exists to decide whether a given Diophantine equation has any solutions, or how many solutions. In most cases, we are reduced to study individual equations, rather than classes of equations.

The literature contains a very large number of articles on non-linear such individual equations involving primes and powers of all kinds. Among them are for example [1, 2, 3, 6].

The general equation

$$p^x + q^y = z^2$$

has many forms. For the equation $4^x + 7^y = z^2$ it has been shown [6] that it has no solutions in positive integers. The equation

$$2^a + 7^b = c^2 \qquad (1)$$

when $a = 2x$ is even, yields $4^x + 7^y = z^2$ as in [6]. In [2], we investigated equation (1) when $a = 2x + 1$ is odd and $b = 2n$ is even. In this Note, we consider the odd values $a = 2x + 1$ and $b = 2n + 1$ in order to obtain the complete set of solutions of equation (1).

2. The equation $2^{2x+1} + 7^{2n+1} = z^2$

In Theorem 2.1, we establish that the equation $2^{2x+1} + 7^{2n+1} = z^2$ has no solutions.

Theorem 2.1. The equation

$$2^{2x+1} + 7^{2n+1} = z^2 \quad (2)$$

has no solutions in positive integers x, n and z.

Proof: For all integers $x \geq 1$, $n \geq 1$ and z we now show that equation (2) is impossible.

From (2), the integer z^2 is odd. Each odd integer z^2 is clearly of the form $4T + 1$. It is easily verified for every integer $n \geq 1$, that 7^{2n+1} has the form $4M + 3$. For all $x \geq 1$, $2^{2x+1} = 4 \cdot 2^{2x-1}$.

In equation (2), the left-hand side is equal to

$$2^{2x+1} + 7^{2n+1} = 4 \cdot 2^{2x-1} + (4M + 3) = 4(2^{2x-1} + M) + 3,$$

whereas the right-hand side of equation (2) is

$$z^2 = 4T + 1.$$

The two sides of equation (2) contradict each other. Therefore, there do not exist integers x, n and z which satisfy equation (2).

The assertion then follows. □

Remark 2.1. The complete set of solutions to the equation $2^a + 7^b = c^2$ consists of only two solutions. These were respectively obtained in [3, 2] and are as mentioned earlier: $(a, b, c) = (1, 1, 3)$ and $(a, b, c) = (5, 2, 9)$.

Final Remark. In [2], the author raised two questions concerning the solutions of $2^a + 7^b = c^2$ when a and b are both odd. He conjectured that the answer to these questions is negative. The result of this Note confirms that the answer to both questions is indeed negative.

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