Further Results on Vertex Odd Divisor Cordial Labeling of Some Graphs

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Abstract. In this paper we prove that the graphs \( < F_n, F_n >, \) Theta graph, Switching of a vertex in a Petersen graph, \( < K_{1,n}, K_{1,n}, K_{1,n} > \) and \( < K_{1,n}, K_{1,n}, K_{1,n}, K_{1,n} > \) are vertex odd divisor cordial graphs.

Keywords: Graph labeling, cordial labeling, divisor cordial labeling, vertex odd divisor cordial labeling.

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1. Introduction

Graph theory has several interesting applications in system analysis, operations research and economics. Since most of the time the aspects of graph problems are uncertain, it is nice to deal with these aspects via the methods of labeling. The concept of labeling of graphs is an active research area and it has been widely studied by several researchers. In a wide area network (WAN), several systems are connected to the main server, the labeling technique plays a vital role to label the cables. The labeling of graphs have been applied in the fields such as circuit design, communication network, coding theory, and crystallography.

A graph labeling, is a process in which each vertex is assigned a value from the given set of numbers, the labeling of edges depends on the labels of its end vertices. An excellent survey of various graph labeling problems, we refer to Gallian [2]. Two well known graph labeling methods are graceful labeling and harmonious labeling. These labelings are studied by Cahit [1].

Cordial labeling was introduced by Cahit [1]. Many labeling schemes were introduced with slight variations in cordial such as prime cordial labeling, divisor cordial labeling. Varatharajan et al. [10] have analyzed the divisor cordial labeling. The divisor cordial labeling of various types of graph is presented in [5,6,7,8,11] Muthaiyan et al. [4] introduce the concept of vertex odd divisor cordial graph. In section 2, we summarize the necessary definitions and basic results. In section 3, we proved that some standard graphs are vertex odd divisor cordial graph. We conclude in section 4.
2. Basic definitions

In this section, we provide a brief summary of the definitions and other results which are prerequisites for the present work.

All the graphs considered here are simple finite, undirected without loops and multiple edges. Let \( G = (V, E) \) be a graph and as usual we denote \( p = |V| \) and \( q = |E| \). For terminology and notations not specifically defined here, we refer to Harary [3].

We recall the following definition from Harary [3].

**Definition 2.1.** Let \( G = (V, E) \) be a graph. A mapping \( f : V \to \{0, 1\} \) is called the binary vertex labeling of \( G \) and \( f(v) \) is called the label of the vertex \( v \in V \) of \( G \) under \( f \). The induced edge labeling \( f^e : E \to \{0, 1\} \) is given by \( f^e(e) = |f(u) - f(v)| \), for all \( e = uv \in E \).

We denote \( v_f(i) \) is the number of vertices of \( G \) having label \( i \) under \( f \) and \( e_f(i) \) is the number of edges of \( G \) having label \( i \) under \( f \), where \( i = 0, 1 \). Now we define cordial labeling of a graph.

**Definition 2.2.** [1] Let \( G = (V, E) \) be a graph and \( f : V \to \{0, 1\} \) be a binary vertex labeling of \( G \). The map \( f \) is called a cordial labeling if \( |v_f(1) - v_f(0)| \leq 1 \) and \( |e_f(1) - e_f(0)| \leq 1 \).

A graph \( G \) is called cordial graph if it admits cordial labeling.

**Definition 2.3.** [10] A divisor cordial labeling of a graph \( G = (V, E) \) is a bijection \( f : V \to \{1, 2, 3, \ldots , |V| \} \) such that if each edge \( uv \) is assigned the label 1 if \( f(u)f(v) \) or \( f(v)f(u) \) and the label 0 if \( f(u)f(v) \), then \( |e_f(0) - e_f(1)| \leq 1 \).

**Definition 2.4.** [4] A vertex odd divisor cordial labeling of a graph \( G = (V, E) \) is a bijection \( f : V \to \{1, 2, 3, \ldots , 2n - 1\} \) such that if each edge \( uv \) is assigned the label 1 if \( f(u)f(v) \) or \( f(v)f(u) \) and the label 0 if \( f(u)f(v) \), then \( |e_f(0) - e_f(1)| \leq 1 \).

A graph which admits odd divisor cordial labeling is called a vertex odd divisor cordial graph.

**Definition 2.5.** [9] Consider two copies of graph \( G \) namely \( G_1 \) and \( G_2 \). Then the graph \( G' = \langle G_1 \cup G_2 \rangle \) is the graph obtained by joining the apex vertices of \( G_1 \) and \( G_2 \) by an edge as well as to a new vertex \( v' \).

Note that \( \langle G_1 \cup G_2 \cup G_1 \rangle = \langle (G_1 \cup G_2) \cup G_1 \rangle \) and \( \langle G_1 \cup G_2 \cup G_3 \cup G_4 \rangle = \langle (G_1 \cup G_2 \cup G_3) \cup G_4 \rangle \).

**Definition 2.6.** The friendship graph \( F_n \) is a planar undirected graph with \( 2n+1 \) vertices and \( 3n \) edges. The friendship graph \( F_n \) can be constructed by joining \( n \) copies of the cycle.
Further Results On Vertex Odd Divisor Cordial Labeling of Some Graphs

graph $C_3$ with a common vertex.

**Definition 2.7.** A vertex switching $G_v$ of a graph $G$ is the graph obtained by taking a vertex $v$ of $G$, removing all the edges incident to $v$ and adding edges joining $v$ to every other vertex which are not adjacent to $v$ in $G$.

3. Main results

**Theorem 3.1.** The graph $G = \triangleleft F_n \triangle F_n \triangleright$ is a vertex odd divisor cordial graph.

**Proof:** Let $G_1 = G_2 = F_n$. Let $G$ be the graph $\triangleleft G_1 \triangle G_2 \triangleright$. Let $u$ and $v$ be the apex vertices of $G_1$ and $G_2$. Let $u_1, u_2, u_3, ..., u_{2n}$ be the vertices of $G_1$ and $v_1, v_2, v_3, ..., v_{2n}$ be the vertices of $G_2$ respectively. Let $v'$ be the new vertex joining with the apex vertices $u$ and $v$. Then $|V(G)| = 4n + 3$ and $|E(G)| = 6n + 3$. We define $f : V(G) \to \{1, 3, 5, ..., 8n + 5\}$, as follows $f(u) = 1$, $f(v) = 3f(v') = p$ where $p$ is the largest prime number such that $0 < p \leq 8n + 5$.

$$f(v_i) = \begin{cases} 9 + 6(i - 1) & \text{if } 1 \leq i \leq n \\ 11 + 6(i - n - 1) & \text{if } n + 1 \leq i \leq 2n \end{cases}$$

The remaining labels are assigned to the vertices $u_1, u_2, u_3, ..., u_{2n}$ in ascending order. From the above labeling pattern, we have $e_f(0) = 3n + 1$ and $e_f(1) = 3n + 2$. Therefore $|e_f(0) - e_f(1)| \leq 1$.

Hence, $G$ is a vertex odd divisor cordial graph.

**Example 3.2.** Vertex odd divisor cordial labeling of the graph $\triangleleft F_7 \triangle F_7 \triangleright$ is shown in Figure 1.

![Figure 1: Vertex odd divisor cordial labeling of $\triangleleft F_7 \triangle F_7 \triangleright$](image-url)
Theorem 3.3. Theta graph is a vertex odd divisor cordial graph.

Proof: Let $v_1, v_2, v_3, v_4, v_5, v_6$ be the vertices of Theta graph. Let $G$ be the Theta graph. Then $|V(G)| = 6$ and $|E(G)| = 7$.

We define $f: V(G) \rightarrow \{1,3,5,7,9,11\}$ as follows

$$f(v_1) = 1, f(v_{2i}) = 3 + 6(i-1), 1 \leq i \leq 2.$$  

Except the labels assigned from the set $\{1,3,5,...,11\}$ the remaining labels are assigned to $v_4, v_5, v_6$ in any order. We observe that, from the above labeling pattern, we have $e_f(1) = 4$ and $e_f(0) = 3$. Therefore $|e_f(0) - e_f(1)| \leq 1$.

Hence $G$ is a vertex odd divisor cordial graph.

Theorem 3.4. Switching of a vertex in a Petersen graph admits vertex odd divisor cordial graph.

Proof: Let $G$ be the Petersen graph and let $V(G) = \{v, v_i : 1 \leq i \leq 9\}$ be the vertex set. Let $G_v$ be the graph obtained from $G$ by switching the vertex $v$.

Then $|V(G_v)| = 10$ and $|E(G_v)| = 18$.

We define $f: V(G) \rightarrow \{1,3,5,...,19\}$, as follows

$$f(v) = 1, f(v_i) = 3 + 6(i-1), 1 \leq i \leq 2 f(v_3) = 7, f(v_{2i+2}) = 5 + 10(i-1), 1 \leq i \leq 2.$$  

Figure 2: Vertex odd divisor cordial labeling of theta graph

Figure 3: Petersen graph
Further Results On Vertex Odd Divisor Cordial Labeling of Some Graphs

Except the labels assigned from the set \( \{1, 3, 5, \ldots, 19\} \) the remaining labels are assigned to \( \{v_6, v_7, v_8, v_9, v_{10}\} \) in any order.

In view of above defined labeling pattern, we have \( e_f(0) = e_f(1) = 9 \).

Therefore \( |e_f(0) - e_f(1)| \leq 1 \).

Hence \( G \) is a vertex odd divisor cordial graph.

**Theorem 3.5.** The graph \( G = \langle K_{1,n} \triangle K_{1,n} \rangle \) is a vertex odd divisor cordial graph.

**Proof:** Let \( G_1 = G_2 = G_3 = K_{1,n} \). Let \( G \) be the graph \( G = \langle K_{1,n} \triangle K_{1,n} \rangle \). Let \( u, v \) and \( w \) be the apex vertices of \( G_1, G_2 \) and \( G_3 \). Let \( \{u_i, v_i, w_i : 1 \leq i \leq n\} \) be the pendant vertices, where \( u_i, v_i, w_i (1 \leq i \leq n) \) are attached with \( u, v \) and \( w \) respectively. Let \( v' \) and \( v'' \) be the new vertices joining with apex vertices \( u \) and \( v, v \) and \( w \) respectively. Then \( |V(G)| = 3n + 5 \) and \( |E(G)| = 3n + 6 \).

We define \( f : V(G) \rightarrow \{1, 3, 5, \ldots, 6n + 9\} \), as follows

\[
f(u) = p, f(v) = 1, f(w) = 3, f(v') = 7, f(v'') = 5 \]

where \( p \) is the largest prime number such that \( 0 < p \leq 6n + 9 \).

**Case 1:** For \( n = 2 \).

![Figure 4: Switching of a vertex \( v \) in Petersen graph](image)

From the above figure we observe that \( e_f(0) = 6 = e_f(1) \).

Hence \( \langle K_{1,2} \triangle K_{1,2} \rangle \) is a vertex odd divisor cordial graph.

**Case 2:** when \( n \) is even \((n > 2)\)
A. Sugumaran and K. Suresh

\[
f(v_i) = \begin{cases} 
9 + 6(i-1) & \text{if } 1 \leq i \leq \frac{n-2}{2} \\
11 + 6(i - \frac{n}{2}) & \text{if } \frac{n}{2} \leq i \leq n 
\end{cases}
\]

Case 3: when \( n \) is odd

\[
f(v_i) = \begin{cases} 
9 + 6(i-1) & \text{if } 1 \leq i \leq \frac{n-1}{2} \\
11 + 6(i - \frac{n-1}{2}) & \text{if } \frac{n+1}{2} \leq i \leq n 
\end{cases}
\]

Except the labels assigned from the set \( \{1, 3, 5, \ldots, 6n + 9\} \) the remaining labels are assigned to \( u_i, v_i (1 \leq i \leq n) \) in any order.

In view of above defined labeling pattern, we have

\[
e_f(1) = \begin{cases} 
3n & \text{if } \ n \ \text{is even} \\
3n - 1 & \text{if } \ n \ \text{is odd and}
\end{cases}
\]

\[
e_f(0) = \begin{cases} 
3n & \text{if } \ n \ \text{is even} \\
3n - 2 & \text{if } \ n \ \text{is odd}
\end{cases}
\]

Therefore \( |e_f(0) - e_f(1)| \leq 1 \).

Hence \( G \) is vertex odd divisor cordial labeling graph.

**Example 3.6.** Vertex odd divisor cordial labeling for \( \prec K_{1,7} \triangle K_{1,7} \triangle K_{1,7} \succ \) and \( \prec K_{1,8} \triangle K_{1,8} \triangle K_{1,8} \succ \) are shown in Figures 6 and 7.

![Figure 6: Vertex odd divisor cordial labeling of \( \prec K_{1,7} \triangle K_{1,7} \triangle K_{1,7} \succ \)](image)
Further Results On Vertex Odd Divisor Cordial Labeling of Some Graphs

![Graph Image]

**Figure 7:** Vertex odd divisor cordial labeling of $\prec K_{1,8} \sqcup K_{1,8} \sqcup K_{1,8} \succ$

**Theorem 3.7.** The graph $G = \prec K_{1,n} \sqcup K_{1,n} \sqcup K_{1,n} \sqcup K_{1,n} \succ$ is vertex odd divisor cordial graph.

**Proof:** Let $G = G_1 = G_2 = G_3 = G_4 = K_{1,n}$. Let $G$ be the graph

$G = \prec K_{1,n} \sqcup K_{1,n} \sqcup K_{1,n} \sqcup K_{1,n} \succ$. Let $u_1, u_2, u_3$ and $u_4$ be the apex vertices of $G_1$, $G_2$, $G_3$ and $G_4$. Let $\{u'_i, u''_i, u'''_i : 1 \leq i \leq n\}$ be the pendant vertices attached with $u_1, u_2, u_3$ and $u_4$ respectively. Let $v', v''$ and $v'''$ be the new vertices joining with apex vertices $u_1$ and $u_2$, $u_2$ and $u_3$, $u_3$ and $u_4$ respectively.

Then $|V(G)| = 4n + 7$ and $|E(G)| = 4n + 9$.

We define $f : V(G) \to \{1, 3, 5, \ldots, 8n + 13\}$ as follows

- $f(u_1) = p_1$,
- $f(u_2) = 1$,
- $f(u_3) = 3$,
- $f(u_4) = p_2$,
- $f(v') = 11$,
- $f(v'') = 7$, $f(v''') = 5$

where $p_1$ and $p_2$ ($p_1 \neq p_2$) are the largest prime number such that $0 < p_2 < p_1 \leq 8n + 13$. $f(u'_i) = 9 + 6(i - 1); 1 \leq i \leq n$ Except the labels assigned from the set $\{1, 3, 5, \ldots, 8n + 13\}$ the remaining labels are assigned to $u'_1, u''_1, u'''_1 (1 \leq i \leq n)$ in any order.

In view of above defined labeling pattern, we have $e_f(0) = 2n + 5$ and $e_f(1) = 2n + 4$.

Therefore $|e_f(0) - e_f(1)| \leq 1$. Hence $G$ is vertex odd divisor cordial labeling graph.

**Example 3.8.** Vertex odd divisor cordial labeling for $\prec K_{1,8} \sqcup K_{1,8} \sqcup K_{1,8} \sqcup K_{1,8} \succ$ is shown in Figure 8.
The vertex odd divisor cordial labeling is a variation of cordial labeling. It is very interesting to study graph or families of graph which are vertex odd divisor cordial as all the graphs do not admit vertex odd divisor cordial labeling. In this paper we proved that the graphs $\langle K_{1,8} \downarrow K_{1,8} \uparrow K_{1,8} \uparrow K_{1,8} \rangle$, Theta graph, Switching of a vertex in a Petersen graph, $\langle K_{1,\alpha} \downarrow K_{1,\alpha} \downarrow K_{1,\alpha} \rangle$ and $\langle K_{1,\alpha} \downarrow K_{1,\alpha} \downarrow K_{1,\alpha} \downarrow K_{1,\alpha} \rangle$ are vertex odd divisor cordial graphs.

REFERENCES