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Monotone Iterative Technique for Caputo Fractional Differential Equations with Deviating Arguments

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Abstract. In this paper, we obtain the existence and uniqueness of solution for fractional differential equation involving the Caputo fractional derivative with deviating argument. The uniqueness of solution is obtained by using a Banach fixed point theorem also the existence of extremal solutions are obtained by a monotone iterative technique and the method of lower and upper solutions. Finally, some examples illustrate the results.

Keywords: Fractional differential equation with deviating argument, Caputo fractional derivative, existence and uniqueness, monotone iterative technique, extremal solutions.

AMS Mathematics Subject Classification (2010): 34A08, 34A12, 26A33, 34L15

1. Introduction

In this paper, we study the following problem for nonlinear initial value problem (IVP) involving Caputo fractional differential equation with deviating argument:

$$\begin{cases} {}^{C}D_{0+}^{\alpha}x(t) = f\left(t, x(t), x(\theta(t))\right), \ t \in J = [0,T], \\ \qquad x(0) = x_0, \ x'(0) = 0, \end{cases} \tag{1}$$

where $f\left(t, x(t), x(\theta(t))\right) \in C\left([0,T] \times \mathbb{R}^2, \mathbb{R}\right)$, $\theta \in C(J, J)$, $\theta(t) \le t$, $t \in J$, ${}^{C}D_{0+}^{\alpha}$ is called the Caputo fractional derivative of order α $(1 < \alpha \le 2)$.

Since $f\left(t, x(t), x(\theta(t))\right)$ is continuous, IVP (1) is equivalent to the following Volterra fractional integral equation

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, x(s), x(\theta(s))\right) ds. \tag{2}$$

Where Γ denotes the gamma function, recently, the theory of fractional differential equation has a lot of importance recently because of its applications (see [8, 20]). In the recent investigations, many researchers studied the existence and uniqueness of solution of nonlinear fractional differential equations (see [2, 3, 4, 5, 12, 15, 17] and references therein). Also, there was an important development in fractional differential equation see [7, 9, 14, 19, 22] and the papers of [6, 12, 16, 18]. Very recently, was discussed some basic theory for nonlinear IVP of fractional differential equation involving the Riemann-Liouville differential operator of order α $(0<\alpha<1)$ (see [10, 11] and the references therein). However, discussion on initial value problems of fractional differential equation with deviating argument is rare. We know that it is important build a comparison result relative to lower and upper solutions of Caputo which we using in the monotone iterative technique for initial value problem of order α $(1<\alpha\le 2)$.

The paper is organized as follows: In Section 2, we present some definitions and fundamental facts of fractional calculus theory. In Section 3, we will prove the uniqueness of solution for nonlinear IVP (1) by using Banach fixed point theorem. In Section 4, by the utility of the monotone iterative technique and the method of lower and upper solutions, we prove that nonlinear IVP (1) has extremal solutions. Lastly, we illustrate our results with suitable examples.

2. Preliminaries

Let $C(J,\mathbb{R})=\{x\ :\ x(t)$ is continuous on $C(J,\mathbb{R})\}$ with the norm $\|x\|_C=\max_{t\in J}|x(t)|$. Obviously, $C(J,\mathbb{R})$ is Banach space. For the convenience of the readers, we first present some useful definitions and fundamental facts of fractional calculus theory, which can be found in [7, 19].

Definition 2.1. Caputo's derivative for a function $f(t)\in C^n(J,\mathbb{R})$ can be written as

$$ {}_0^C D_t^{\alpha} f(t)=\left(I_{0+}^{n-\alpha}D^n f\right)(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(s)ds}{(t-s)^{\alpha+1-n}}, $$

where $D=\dfrac{d}{dt}$ and $n=[\alpha]+1,\ [\alpha]$ denotes the integer part of real number $\alpha>0$.

Definition 2.2. For $\alpha>0$, the integral

$$ I_{0+}^{\alpha} f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds, $$

is called the Riemann-Liouville fractional integral operator of order α .

Lemma 2.1. [7] Let $x(t)\in C^n[0,1]$ and $\alpha\in(n-1,n],\ n\in\mathbb{N}$. Then for $t\in[0,1]$,

$$ I_{0+}^{\alpha}\,{}^{C}D_{0+}^{\alpha}x(t)=x(t)-\sum_{k=0}^{n-1}\frac{t^k}{k!}x^k(0). $$

3. Uniqueness of solution

In this section, we discuss uniqueness of solution for nonlinear IVP (1) for Caputo fractional differential equation with deviating argument under the following condition:

(H_1) There exist nonnegative constants L_1, L_2 such that

$$ \left|f(t,v_1,v_2)-f(t,u_1,u_2)\right|\le L_1|v_1-u_1|+L_2|v_2-u_2|,\ \forall t\in J,\ v_i,u_i\in\mathbb{R},\ i=1,2. $$

Lemma 3.1. Let $f \in C\left(J \times \mathbb{R}^2, \mathbb{R}\right)$, $x \in C(J, \mathbb{R})$ is the solution of the nonlinear IVP (1), if and only if $x(t)$ is a solution of integral equation

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, x(s), x(\theta(s))\right) ds. \tag{3}$$

Proof: Assume that $x(t)$ satisfies IVP (1). From the first equation of IVP (1) and Lemma 2.1, we have

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, x(s), x(\theta(s))\right) ds.$$

Conversely, assume that $x(t)$ satisfies (3). Applying the operator ${}_0^C D_t^\alpha$ to both sides of (3), we have

$${}_0^C D_t^\alpha x(t) = f\left(t, x(t), x(\theta(t))\right).$$

In addition, we have $x(0) = x_0$, $x'(0) = 0$. The proof is complete.

Theorem 3.1. Assume that (H_1) hold, $f \in C\left(J \times \mathbb{R}^2, \mathbb{R}\right)$. Then the nonlinear IVP (1) has a unique solution.

Proof: Define the operator $A : C(J, \mathbb{R}) \to C(J, \mathbb{R})$ by

$$(Ax)(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, x(s), x(\theta(s))\right) ds.$$

It is easy to check that the operator $(Ax)(t) \in C(J, \mathbb{R})$. Next we show that A is a contraction operator $C(J, \mathbb{R})$. For convenience, let

$$\frac{\left(L_1 + L_2\right) T^\alpha}{\Gamma(1+\alpha)} < 1. \tag{4}$$

For any $x, y \in C(J, \mathbb{R})$, we have

$$\begin{aligned}
\|Ax - Ay\|_C &= \max_{t \in J} \left|(Ax)(t) - (Ay)(t)\right| \\
&\le \max_{t \in J} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| f\left(s, x(s), x(\theta(s))\right) - f\left(s, y(s), y(\theta(s))\right) \right| ds \\
&\le \frac{1}{\Gamma(\alpha)} \max_{t \in J} \int_0^t (t-s)^{\alpha-1} \left[L_1 |x(s) - y(s)| + L_2 |x(\theta(s)) - y(\theta(s))| \right] ds \\
&\le \frac{1}{\Gamma(\alpha)} \max_{t \in J} \int_0^t (t-s)^{\alpha-1} \left[L_1 \|x - y\|_C + L_2 \|x - y\|_C \right] ds \\
&\le \frac{\left(L_1 + L_2\right)}{\Gamma(\alpha)} \|x - y\|_C \max_{t \in J} \int_0^t (t-s)^{\alpha-1} ds \\
&\le \frac{\left(L_1 + L_2\right)}{\Gamma(\alpha)} \|x - y\|_C \max_{t \in J} t^\alpha \int_0^1 (1-\eta)^{\alpha-1} d\eta
\end{aligned}$$

$$\leq \left\{\left(\frac{L_1+L_2}{\Gamma(\alpha)}\right)\left(\frac{\Gamma(\alpha)T^{\alpha}}{\Gamma(1+\alpha)}\right)\right\}\|x-y\|_C$$
$$\leq \frac{(L_1+L_2)T^{\alpha}}{\Gamma(1+\alpha)}\|x-y\|_C.$$

According to (4) and using Banach fixed point theorem, the nonlinear IVP (1) has a unique solution. The proof is complete.

Lemma 3.2. Let $M, N \geq 0$ are constants, $h \in C(J, \mathbb{R})$. The linear initial value problem:

$$\begin{cases} {}^{C}D_{0+}^{\alpha}x(t) - Mx(t) - Nx(\theta(t)) = h(t),\ 1<\alpha\leq 2,\ t\in[0,T], \\ \quad x(0)=x_0,\ \ x'(0)=0, \end{cases} \tag{5}$$

has a unique solution

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big[Mx(s)+Nx(\theta(s))+h(s)\big]ds. \tag{6}$$

Proof: By Theorem 3.1, the linear IVP (5) is equivalent to solving a fixed point problem with operator A_h defined by

$$A_h x(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big[M(s)x(s)+N(s)x(\theta(s))+h(s)\big]ds.$$

For any $h \in C_{1-\alpha}(J)$. Then the operator A_h has a unique fixed point.

Remark 1.

i. Putting $N=0$, in the above linear IVP (5), we have the result obtained by Devi [1], Mcrae [13], Sambandham et al., [21];
ii. Putting $N=0$ and $h(t)\equiv 0$, in the above linear IVP (5), we get the solution of the corresponding homogenous IVP (5) by Yakar [23];
iii. When $\alpha=1$, the problem (5) reduces to the following
$$\begin{cases} x'(t) = Mx(t)+Nx(\theta(t))+h(t),\ t\in J, \\ \quad x(0)=x_0,\ \ x'(0)=0, \end{cases}$$
has a unique solution which satisfies Eq.(6);
iv. Putting $M=0$ and $N=0$, in the above linear IVP (5), the initial value problem (5) has a unique solution
$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds.$$

4. The monotone iterative technique

In this section, we mainly prove the existence of extremal solutions of the nonlinear IVP (1) by monotone iterative technique combined with the method of lower and upper solutions. We need the following Lemma and definition.

Lemma 4.1. Suppose that M, $N \geq 0$ are constants and the inequality

$$\frac{(M+N)T^{\alpha}}{\Gamma(1+\alpha)} < 1, \tag{7}$$

holds, and $p \in C(J,\mathbb{R})$ satisfies

$$\begin{aligned} &p(t) \leq p(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[Mp(s) + Np(\theta(s))\right]ds,\ t \in J, \\ &p(0) \leq 0, \end{aligned} \tag{8}$$

then $p(t) \leq 0$ for all $t \in J$.

Proof: Suppose that $p(t) \leq 0,\ \forall t \in J$ is not true. So there exists at least one $t_* \in J$ such that $p(t_*) > 0$. Without loss of generality, we assume

$$p(t_*) = \max_{t\in J}\{p(t)\} = \rho_1 > 0.$$

We obtain that

$$\begin{aligned} p(t) &\leq p(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[Mp(s) + Np(\theta(s))\right]ds \\ &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[Mp(s) + Np(\theta(s))\right]ds \\ &\leq \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} p(s)ds + \frac{N}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} p(\theta(s))ds. \end{aligned}$$

Let $t = t_*$, we have

$$\rho_1 \leq \left(\frac{(M+N)}{\Gamma(1+\alpha)}T^{\alpha}\right)\rho_1$$

So

$$\frac{(M+N)}{\Gamma(1+\alpha)}T^{\alpha} \geq 1.$$

This is a contradiction. Hence $p(t) \leq 0$ for all $t \in J$. The proof is complete.

Definition 4.1. We say that $\alpha_0 \in C(J,\mathbb{R})$ is called a lower solution of IVP (1) if

$$\begin{cases} \alpha_0(t) \leq \alpha_0(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\left(s,\alpha_0(s),\alpha_0(\theta(s))\right)ds,\ t \in J, \\ \alpha_0(0) \leq \alpha_0,\ \alpha_0'(0) \leq 0. \end{cases}$$

We say that $\beta_0 \in C(J,\mathbb{R})$ is called a upper solution of IVP (1) if

$$\begin{cases} \beta_0(t) \geq \beta_0(0) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{\alpha-1} f\left(s,\beta_0(s),\beta_0(\theta(s))\right)ds,\ t \in J, \\ \beta_0(0) \geq \beta_0,\ \ \beta_0'(0) \geq 0. \end{cases}$$

In the following discussion, we need the following assumptions:

(H_2) Functions α_0 and β_0 are ordered lower and upper solutions of nonlinear IVP (1) such that

$$\alpha_0(t) \le \beta_0(t),\ t \in J.$$

(H_3) There exist constants $M, N \ge 0$ such that

$$f(t,v_1,v_2) - f(t,u_1,u_2) \ge M(v_1 - u_1) + N(v_2 - u_2),$$

where $\alpha_0(t) \le u_1 \le v_1 \le \beta_0(t),\ \ \alpha_0(\theta(t)) \le u_2 \le v_2 \le \beta_0(\theta(t)),\ \ t \in J.$

Let $[\alpha_0, \beta_0] = \{z \in C(J,\mathbb{R}):\ \alpha_0(t) \le z(t) \le \beta_0(t),\ t \in J,\ \alpha_0(0) \le z(0) \le \beta_0(0)\}.$

Theorem 4.1. Let (H_2)-(H_3) and inequality (7) hold. Then there exist two monotone sequences $\{\alpha_n\},\{\beta_n\} \subset [\alpha_0, \beta_0]$ both converge uniformly to the extremal solutions of nonlinear IVP (1) in $[\alpha_0, \beta_0]$.

Proof: This proof consists of the following three steps.

Step 1: Construct two the sequences $\{\alpha_n\}$ and $\{\beta_n\}$. For any $\eta \in [\alpha_0, \beta_0]$ such that $\eta \in C(J,\mathbb{R})$, we consider the following linear initial value problem:

$$\begin{cases} {}^{C}D^{\alpha}_{0+}x(t) - Mx(t) - Nx(\theta(t)) = f\big(t, \eta(t), \eta(\theta(t))\big) - M\eta(t) - N\eta(\theta(t)),\ t \in J, \\ \quad x(0) = x_0,\ \ x'(0) = 0. \end{cases} \tag{9}$$

In the present context $h(t) = f\big(t, \eta(t), \eta(\theta(t))\big) - M\eta(t) - N\eta(\theta(t))$ and is to be replaced in (5) by this new value. By Lemma 3.2, the linear IVP (9) has a unique solution

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big[M(s)x(s) + N(s)x(\theta(s)) + h(s)\big]ds, \tag{10}$$

Suppose that $x_1(t)$ and $x_2(t)$ be two solutions of linear IVP (9). Let $p(t) = x_1(t) - x_2(t)$. Applying Lemma 4.1 again one can prove that $p(t) \le 0$, and thus $x_1(t) \le x_2(t)$. As the same argument is valid for $x_2(t) - x_1(t)$, we conclude that $x_1(t) = x_2(t)$. This proves uniqueness.

Now, we define a mapping $A : [\alpha_0, \beta_0] \to [\alpha_0, \beta_0]$ by $x = A\eta$, where x is the unique solution of linear IVP (9) It is easy to check that the operator A is monotone nondecreasing on $[\alpha_0, \beta_0]$, let $\eta, \mu \in [\alpha_0, \beta_0]$ such that $\eta \le \mu$. Suppose that $z_1 = A\eta$ and $z_2 = A\mu$. Setting $p(t) = z_1(t) - z_2(t)$, we obtain

$$\begin{aligned} p(t) &= p(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\begin{pmatrix} f\big(s, \eta(s), \eta(\theta(s))\big) - f\big(s, \mu(s), \mu(\theta(s))\big) \\ -M\big(\eta(s) - z_1(s)\big) - N\big(\eta(\theta(s)) - z_1(\theta(s))\big) \\ +M\big(\mu(s) - z_2(s)\big) + N\big(\mu(\theta(s)) - z_2(\theta(s))\big) \end{pmatrix} ds \\ &\le p(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big(Mp(s) + Np(\theta(s))\big)ds. \end{aligned}$$

Besides,

$$p(0) = z_1(0) - z_2(0) = z_0 - z_0 \le 0.$$

By Lemma 4.1, we get $p(t) \le 0$, implies that $A\eta \le A\mu$ for all $t \in J$. It means that A is monotone nondecreasing on $[\alpha_0, \beta_0]$. Obviously, we can easily get that A is a continuous map. It is now easy to define sequences $\{\alpha_n\}$ and $\{\beta_n\}$ such that $\alpha_n = A\alpha_{n-1},\ \beta_n = A\beta_{n-1},\ n = 1, 2, \ldots.$

Step 2: The sequences $\{\alpha_n\}$ and $\{\beta_n\}$ converge uniformly to $\alpha^*,\ \beta^*$ respectively. In fact $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following relation:

$$\alpha_0 \le \alpha_1 \le \cdots \le \alpha_n \le \cdots \alpha^* \le \beta^* \cdots \le \beta_n \le \cdots \le \beta_1 \le \beta_0. \tag{11}$$

Setting $p(t) = \alpha_0(t) - \alpha_1(t)$ and $\alpha_0(t)$ is the lower solution of IVP (1), we obtain

$$\begin{aligned} p(t) &\le p(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \begin{bmatrix} f(s, \alpha_0(s), \alpha_0(\theta(s))) - f(s, \alpha_0(s), \alpha_0(\theta(s))) \\ + M(\alpha_0(s) - \alpha_1(s)) + N(\alpha_0(\theta(s)) - \alpha_1(\theta(s))) \end{bmatrix} ds \\ &\le p(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \big(Mp(s) + Np(\theta(s)) \big) ds. \end{aligned}$$

Besides,

$$p(0) = \alpha_0(0) - \alpha_1(0) \le \alpha_0 - \alpha_0 = 0.$$

By Lemma 4.1, we get $p(t) \le 0$, implies that $\alpha_0(t) \le \alpha_1(t)$ for all $t \in J$. Similarly, we can show that $\beta_1 \le \beta_0$ for all $t \in J$. Applying the operator A to both sides of $\alpha_0 \le \alpha_1$, $\beta_1 \le \beta_0$, $\alpha_1 \le \beta_1$ and $\alpha_0 \le \beta_0$, we can easily get (11). Obviously, the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are uniformly bounded and equicontinuous on J. Then by using the Ascoli-Arzela criterion, we can conclude that the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ converge uniformly on J with $\lim_{n\to\infty} \alpha_n = \alpha^*$, $\lim_{n\to\infty} \beta_n = \beta^*$ uniformly on J.

Step 3: Prove that $\alpha^*,\ \beta^*$ are extremal solutions of nonlinear IVP (1) and $\alpha^*,\ \beta^*$ are solutions of nonlinear IVP (1) on $[\alpha_0, \beta_0]$ because of the continuity of operator A. Let $z \in [\alpha_0, \beta_0]$ be any solution of nonlinear IVP (1). That is,

$$\begin{cases} z(t) = z(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, z(s), z(\theta(s))) ds, \\ z(0) = z_0,\ z'(0) = 0. \end{cases}$$

Suppose that there exists a positive integer n such that $\alpha_n(t) \le z(t) \le \beta_n(t)$ on J. Let $p(t) = \alpha_{n+1}(t) - z(t)$, we obtain

$$p(t)= p(0)+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\begin{pmatrix} f\big(s,\alpha_n(s),\alpha_n(\theta(s))\big)-f\big(s,z(s),z(\theta(s))\big) \\ -M\big(\alpha_n(s)-\alpha_{n+1}(s)\big)-N\big[\alpha_n(\theta(s))-\alpha_{n+1}(\theta(s))\big]\end{pmatrix}ds$$

$$p(t)\le p(0)+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big(Mp(s)+Np(\theta(s))\big)ds.$$

Besides,

$$p(0)=\alpha_{n+1}(0)-z(0)=\alpha_n-z_0\le 0.$$

By Lemma 4.1, we get $p(t)\le 0$, implies $\alpha_{n+1}(t)\le z(t)$ for all $t\in J$. Similarly, we obtain that $z(t)\le \beta_{n+1}(t)$ on J.

Since $\alpha_0(t)\le z(t)\le \beta_0(t)$ on J, by induction we get that $\alpha_n(t)\le z(t)\le \beta_n(t)$ on J for all n. Therefore, $\alpha^*(t)\le z(t)\le \beta^*(t)$ on $[0,T]$ by taking $n\to\infty$. The proof is complete.

5. Examples

Example 5.1. Consider the following problem:

$$\begin{cases} {}^C D_{0+}^{\frac{3}{2}} x(t)=\frac{1}{30}x(t)+\frac{1}{15}x(t^2),\ t\in[0,1],\ 1<\alpha\le 2, \\ \qquad x(0)=1,\ x'(0)=0. \end{cases} \tag{12}$$

where $\alpha=\frac{3}{2}$, $T=1$, $\theta(t)=t^2$ and $f\big(t,x(t),x(t^2)\big)=\frac{1}{30}x(t)+\frac{1}{15}x(t^2)$. Obviously, $f\big(t,x(t),x(t^2)\big)$ satisfies Lipschitz condition and there exist constants $L_1=\frac{1}{30}$, $L_2=\frac{1}{15}$ such that

$$\left| f(t,x(t),x(t^2))-f(t,y(t),y(t^2))\right|\le \frac{1}{30}\left|x(t)-y(t)\right|+\frac{1}{15}\left|x(t^2)-y(t^2)\right| \text{ if } t\in J,$$

So that condition (H_1) of Theorem 3.1 hold, the problem (12) has a unique solution.

Consider the same equation as (12), taking $x_0(t)=0,\ \ y_0(t)=5t^{-\frac{1}{2}}+6,$ and then we have $y_0(0)=5.$ Moreover,

$$y_0(t)=5t^{-\frac{1}{2}}+6\ge 5t^{-\frac{1}{2}}+\frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-s)^{-\frac{1}{2}}\left(\frac{1}{30}(5s^{-\frac{1}{2}}+6)+\frac{1}{15}(5s^{-\frac{1}{2}}+6)^2\right)ds,$$

On the other hand, it is easy to check that $x_0\le y_0$ and (H_2) of Theorem 4.1 holds.

And let $M=\frac{1}{30},\ \ N=\frac{1}{15},$ we get that

$$f(t,x(t),x(t^2))-f(t,y(t),y(t^2))\ge \frac{1}{30}\big[x(t)-y(t)\big]+\frac{1}{15}\big[x(t^2)-y(t^2)\big],$$

where $x_0\le u_1\le v_1\le y_0,\ x_0(t^2)\le u_2\le v_2\le y_0(t^2)$. So (H_3) is satisfied. Furthermore, we get that

$$\frac{(M+N)T^{\alpha}}{\Gamma(1+\alpha)}=\frac{2}{15\sqrt{\pi}}<1.$$

It is easy to see that (7) holds. Thus, all conditions of Theorem 4.1 are satisfied. Therefore, problem (12) has extremal solutions.

Example 5.2. Consider the following problem:

$$\begin{cases} {}^{C}D_{0+}^{\frac{3}{2}}x(t)=\frac{t^2}{60}x(t)+\frac{t^3}{30}x(\frac{1}{2}t),\ t\in[0,1],\ 1<\alpha\le 2, \\ \qquad x(0)=1,\ x'(0)=0. \end{cases} \tag{13}$$

Where $\alpha=\frac{3}{2}$ $,T=1,\ \theta(t)=(\frac{1}{2}t)$ and $f\left(t,x(t),x(\frac{1}{2}t)\right)=\frac{t^2}{60}x(t)+\frac{t^3}{30}x(\frac{1}{2}t).$ Obviously, $f\left(t,x(t),x(\frac{1}{2}t)\right)$ satisfies Lipschitz condition and there exist constants $L_1=\frac{1}{60}$, $L_2=\frac{1}{30}$ such that

$$\left|f(t,x(t),x(\frac{1}{2}t))-f(t,y(t),y(\frac{1}{2}t))\right|\le\frac{1}{60}\left|x(t)-y(t)\right|+\frac{1}{30}\left|x(\frac{1}{2}t)-y(\frac{1}{2}t)\right| \text{ if } t\in J.$$

So that condition (H_1) of Theorem 3.1 holds, the problem (13) has a unique solution.

Example 5.3. Consider the following problem:

$$\begin{cases} {}^{C}D_{0+}^{\frac{3}{2}}x(t)=\frac{t^2+1}{30}x(t)+\frac{t^4+1}{15}x(\frac{1}{2}t^2),\ t\in[0,1],\ 1<\alpha\le 2, \\ \qquad x(0)=1,\ x'(0)=0. \end{cases} \tag{14}$$

Obviously, $\alpha=\frac{3}{2}$, $T=1,\ \theta(t)=(\frac{1}{2}t^2)$ and $f\left(t,x(t),x(\frac{1}{2}t^2)\right)=\frac{t^2+1}{30}x(t)+\frac{t^4+1}{15}x(\frac{1}{2}t^2).$

Taking $x_0(t)=0,\ \ y_0(t)=t^{-\frac{1}{2}}+6,$ then we have $x_0(0)=0$, $y_0(0)=1.$ Moreover,

$$y_0(t)=t^{-\frac{1}{2}}+6\ge t^{-\frac{1}{2}}+\frac{1}{\Gamma(\frac{1}{2})}\int_0^t(t-s)^{-\frac{1}{2}}\left(\frac{(s^2+1)}{30}(s^{-\frac{1}{2}}+6)+\frac{s^4+1}{15}\left(\frac{1}{2}(s^{-\frac{1}{2}}+6)^2\right)\right)ds,$$

So that condition (H_2) of Theorem 4.1 holds. On the other hand, it is easy to check that (H_3) of Theorem 4.1 holds, therefore

$$f(t,x(t),x(\tfrac{1}{2}t^2))-f(t,y(t),y(\tfrac{1}{2}t^2))\ge\frac{1}{30}\left[x(t)-y(t)\right]+\frac{1}{15}\left[x(\tfrac{1}{2}t^2)-y(\tfrac{1}{2}t^2)\right],$$

where $x_0\le u_1\le v_1\le y_0$, $x_0(\frac{1}{2}t^2)\le u_2\le v_2\le y_0(\frac{1}{2}t^2)$.

We see that $M=\frac{1}{30}$, $N=\frac{1}{15}$, which satisfied inequality (7). All conditions of Theorem 4.1 are satisfied. So problem (14) has extremal solutions.

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