Annals of Pure and Applied Mathematics
Vol. 16, No. 1, 2018, 1-6
ISSN: 2279-087X (P), 2279-0888(online)
Published on 1 January 2018
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v16n1a1

Annals of
Pure and Applied Mathematics

Results of Symmetric Reverse bi-derivations on Prime Rings

***C.Jaya Subba Reddy*[1], *A.Sivakameshwara Kumar*[2] and *B.Ramoorthy Reddy*[3]**

[1]Department of Mathematics, Sri Venkateswara University
Tirupati-517502, India. E-mail: cjsreddysvu@gmail.com
[2]Department of Mathematics, Rayalaseema University
Kurnool, Andhrapradesh, India. E-mail: kamesh1069@gmail.com
[3]Department of Mathematics, Sri Venkateswara University
Tirupati-517502, India. E-mail: ramoorthymaths@gmail.com

Received 3 November 17; accepted 2 December 2017

Abstract. Let R be a prime ring with char$R \neq 2,3$ and let d be trace of a nonzero symmetric reverse bi-derivation $D(.,.)$. For a fixed element a of R with $d(a) \neq 0$, if the identity $d(x)ad(x) = 0$ holds for all $x \in R$, then $a \in Z$.

Keywords: Derivation, reverse derivation, symmetric, symmetric bi-derivation, symmetric reverse bi-derivation, prime rings, trace.

AMS Mathematics Subject Classification (2010): 13D03

1. Introduction

The concept of a symmetric bi-derivation has been introduced by Maksa in [6]. In [9], Vukman has proved some results concerning symmetric bi-derivation on prime and semiprime rings. Yenigul and Argac [10] studied ideals and symmetric bi-derivations of prime and semiprime rings. Reddy et al. [5] studied symmetric reverse bi-derivations on prime rings. Sapanci et al. [8] studied few results of symmetric bi-derivation on prime rings. In this paper, we extended some results of symmetric reverse bi-derivations on prime rings.

2. Preliminaries

Throughout this paper R, will be denoted an associative ring with the center $Z(R)$. Recall that a ring R is called prime if for any $a, b \in R$, $aRb = (0)$ implies that either $a = 0$ or $b = 0$. For any $x, y \in R$, the symbol $[x, y]$ stands for commutator $xy - yx$. An additive mapping $d: R \to R$ is said to be a derivation if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. An additive mapping $d: R \to R$ is said to be a reverse derivation if $\mathrm{d}(xy) = d(y)x + yd(x)$, for all $x, y \in R$. A mapping $D(.,.): R \times R \to R$ is said to be symmetric if $D(x, y) = D(y, x)$, for all $x, y \in R$. A mapping $d: R \to R$ is called the trace of $D(.,.)$ if $d(x) = D(x, x)$, for all $x \in R$. It is obvious that if $D(.,.)$ is bi-additive (i.e., additive in both arguments), then the trace d of $D(.,.)$ satisfies the identity $d(x + y) = d(x) +$

$d(y) + 2D(x,y)$, for all $x, y \in R$. If $D(.,.)$ is bi-additive and satisfies the identities $D(xy,z) = D(x,z)y + xD(y,z)$ and

$$D(x,yz) = D(x,y)z + yD(x,z), \text{ for all } x,y,z \in R.$$

Then $D(.,.)$ is called a symmetric bi-derivation. If $D(.,.)$ is reverse bi-additive and satisfies the identity $D(xy,z) = D(y,z)x + yD(x,z)$ and $D(x,yz) = D(x,z)y + zD(x,y)$, for all $x,y,z \in R$. Then $D(.,.)$ is called a symmetric reverse bi-derivation. We shall make use of commutator identities; $[x,yz] = [x,y]z + y[x,z]$ and $[xy,z] = [x,z]y + x[y,z]$, for all $x,y,z \in R$.

Lemma 1. Let R be a prime ring with char$R \neq 2$, $D(.,.)$ a symmetric reverse bi derivation and d the trace of $D(.,.)$. If U is a nonzero ideal of R such that $ad(U) = 0$ (or, $d(U)a = 0$), then $a = 0$ or $d = 0$.

Proof: Given that $ad(U) = 0$, for some nonzero ideal U.

By linearizing above equation , we get

$ad(u+v) = 0$, for all $u, v \in U$.

$ad(u) + ad(v) + a2D(u,v) = 0$

Since $d(u) = d(v) = 0$ and char$R \neq 2$, then

$aD(u,v) = 0$, for all $u, v \in U$.

Replacing v by uv in above equation, we get

$aD(u,uv) = 0$

$a(D(u,v)u + vD(u,u)) = 0$

$aD(u,v)u + avD(u,u) = 0$

$avd(u) = 0$

$aRd(u) = 0$

Since R is a prime ring, then $a = 0$ or $d = 0$.

Lemma 2. [1, Theorem 3.1.3] Let R be a prime ring with char $R \neq 2$, $D(.,.)$ a symmetric bi-derivation and d the trace of $D(.,.)$. For a fixed element $a \in R$, we have

(i) If $[a, d(x)] = 0$, for all $x \in R$, then $a \in Z$ or $d = 0$.

(ii) If $[a, d(x)] \in Z$, for all $x \in R$ and for nonzero trace d with $d(a) \neq 0$, then $a \in Z$.

Lemma 3. Let R be a prime ring and let $a, b, c \in R$. If $axb = cxa$ for all $x \in R$, then $a = 0$ or $b = c$.

Proof: Given that $axb = cxa$, for all $x \in R$.

Replacing x by xay in above equation, we get

$axayb = cxaya$.

But $ayb = cya$ and $cxa = axb$ then, we get

$axcya = axbya$

$axcya - axbya = 0$

$ax(c-b)ya = 0$

Since R is a prime ring, then $a = 0$ or $b = c$.

Lemma 4. Let R be a prime ring with char$R \neq 2$ and let d_1 and d_2 be traces of symmetric reverse bi-derivations $D_1(.,.)$ and $D_2(.,.)$ respectively. If the identity

$d_1(x)d_2(y) = d_2(x)d_1(y)$ holds and $d_1 \neq 0$, then there exists $\lambda \in C$ such that $d_2(x) = \lambda d_1(x)$.
Proof: Given $d_1(x)d_2(y) = d_2(x)d_1(y)$, for all $x, y, z \in R$. (1)
Replacing y by $z + y$ in equation (1), we get
$$d_1(x)d_2(z+y) = d_2(x)d_1(z+y)$$
$$d_1(x)d_2(z) + d_1(x)d_2(y) + d_1(x)2D_2(z,y)$$
$$= d_2(x)d_1(z) + d_2(x)d_1(y) + d_2(x)2D_1(z,y)$$
$$d_1(x)D_2(z,y) = d_2(x)D_1(z,y) \quad (2)$$
Replacing z by yz in equation (2) leads to the identity
$$d_1(x)D_2(yz,y) = d_2(x)D_1(yz,y)$$
$$d_1(x)zD_2(y,y) + d_1(x)D_2(z,y)y = d_2(x)zD_1(y,y) + d_2(x)D_1(z,y)y$$
$$d_1(x)zd_2(y) = d_2(x)zd_1(y) \quad (3)$$
Again replacing y by x in equation (3), we get
$$d_1(x)zd_2(x) = d_2(x)zd_1(x) \quad (4)$$
Thus if $d_1(x) \neq 0$, then by (4) and [4, Corollary to Lemma 1.3.2] we have $d_2(x) = \lambda(x)d_1(x)$ for some $\lambda(x) \in C$. Hence if $d_1(x) \neq 0$ and $d_1(y) \neq 0$, then $\big(\lambda(y) - \lambda(x)\big)d_1(x)zd_1(y) = 0$ by equation (3). Since R is prime, it follows from Lemma 1 that $\lambda(x) = \lambda(y)$.This shows that there exists $\lambda \in C$ such that $d_2(x) = \lambda d_1(x)$ under the condition $d_1(x) \neq 0$. On the other hand, assume that $d_1(x) = 0$. Since $d_1 \neq 0$ and R is a prime, it follows from equation (3) that $d_2(x) = 0$ as well. Thus $d_2(x) = \lambda d_1(x)$. This completes the proof.

Theorem 1. Let R be a prime ring with char$R \neq 2$ and let $d_1(\neq 0)$, d_2, d_3, $d_4(\neq 0)$ be traces of symmetric reverse bi-derivations $D_1(.,.)$, $D_2(.,.)$, $D_3(.,.)$ and $D_4(.,.)$ respectively. If the identity $d_1(x)d_2(y) = d_3(x)d_4(y)$, holds for all $x, y \in R$, then there exists $\lambda \in C$ such that $d_2(x) = \lambda d_4(x)$ and $d_3(x) = \lambda d_1(x)$.
Proof: Given $d_1(x)d_2(y) = d_3(x)d_4(y)$, for all $x, y \in R$. (5)
Replacing y by $z + y$ in equation (5), we get
$$d_1(x)d_2(z+y) = d_3(x)d_4(z+y)$$
$$d_1(x)d_2(z) + d_1(x)d_2(y) + d_1(x)2D_2(z,y)$$
$$= d_3(x)d_4(z) + d_3(x)d_4(y) + d_3(x)2D_4(z,y)$$
$$d_1(x)D_2(z,y) = d_3(x)D_4(z,y) \quad (6)$$
Replacing z by yz in equation (6) and using equation (6) leads to the identity
$$d_1(x)D_2(yz,y) = d_3(x)D_4(yz,y)$$
$$d_1(x)zD_2(y,y) + d_1(x)D_2(z,y)y = d_3(x)zD_4(y,y) + d_3(x)D_4(z,y)y$$
$$d_1(x)zd_2(y) = d_3(x)zd_4(y) \quad (7)$$
It follows from replacing z by $zd_4(w)$ in equation (7), and using equation (7), we get
$$d_1(x)zd_4(w)d_2(y) = d_3(x)zd_4(w)d_4(y) = d_1(x)zd_2(w)d_4(y)$$
So that $d_1(x)z(d_4(w)d_2(y) - d_2(w)d_4(y)) = 0$, for all $x, y, z, w \in R$.
Since $d_1 \neq 0$ and R is a prime, it follows that $d_4(w)d_2(y) = d_2(w)d_4(y)$. Applying Lemma 4, there exists $\lambda \in C$ such that $d_2(y) = \lambda d_4(y)$, Which implies from equation (7) that $\big(\lambda d_1(x) - d_3(x)\big)zd_4(y) = 0$ so that $d_3(x) = \lambda d_1(x)$. This completes the proof.

Theorem 2. Let R be a prime ring with char$R \neq 2,3$ and let d be trace of a nonzero symmetric reverse bi-derivations $D(.,.)$. For a fixed element a of R with $d(a) \neq 0$, if the identity $d(x)ad(x) = 0$ holds for all $x \in R$, then $a \in Z$.

Proof: Given $d(x)ad(x) = 0$, for all $x \in R$, $a \in Z$. (8)

By linearizing x by $x + y$ in equation (8) and using equation (8), we get

$d(x + y)ad(x + y) = 0$

$\big(d(x) + d(y) + 2D(x, y)\big)a\big(d(x) + d(y) + 2D(x, y)\big) = 0$

$d(x)ad(x) + d(y)ad(x) + 2D(x, y)ad(x) + d(x)ad(y) + d(y)ad(y) + 2D(x, y)ad(y) + d(x)a2D(x, y) + d(y)a2D(x, y) + 2D(x, y)a2D(x, y) = 0$

$d(x)ad(y) + d(y)ad(x) + 2d(x)aD(x, y) + 2d(y)aD(x, y) + 2D(x, y)ad(x) + 2D(x, y)ad(y) + 4D(x, y)aD(x, y) = 0$, for all $x, y \in R$. (9)

Substituting x for $-x$ in equation (9), we get

$d(-x)ad(y) + d(y)ad(-x) + 2d(-x)aD(-x, y) + 2d(y)aD(-x, y) + 2D(-x, y)ad(-x) + 2D(-x, y)ad(y) + 4D(-x, y)aD(-x, y) = 0$

$d(x)ad(y) - 2d(x)aD(x, y) + d(y)ad(x) - 2d(y)aD(x, y) - 2D(x, y)ad(x) - 2D(x, y)ad(y) + 4D(x, y)aD(x, y) = 0$ (10)

By adding equations (9) and (10), we get

$2d(x)ad(y) + 2d(y)ad(x) + 8D(x, y)aD(x, y) = 0$

$2(d(x)ad(y) + d(y)ad(x) + 4D(x, y)aD(x, y)) = 0$

Since R is a char$R \neq 2$ then, we get

$d(x)ad(y) + d(y)ad(x) + 4D(x, y)aD(x, y) = 0$ (11)

Now we replacing x by $x + y$ in equation (11) and expand it, and then we use equations (8), (11) and the fact that char$R \neq 2$ then, we get

$d(x + y)ad(y) + d(y)ad(x + y) + 4D(x + y, y)aD(x + y, y) = 0$

$\big(d(x) + d(y) + 2D(x, y)\big)ad(y) + d(y)a(d(x) + d(y) + 2D(x, y)) + 4\big(D(x, y) + D(y, y)\big)a\big(D(x, y) + D(y, y)\big) = 0$

$d(x)ad(y) + d(y)ad(y) + 2D(x, y)ad(y) + d(y)ad(x) + d(y)ad(y) + 2d(y)aD(x, y) + 4D(x, y)aD(x, y) + 4D(x, y)aD(y, y) + 4D(y, y)aD(x, y) + 4D(y, y)aD(y, y) = 0$

$d(x)ad(y) + 2D(x, y)ad(y) + d(y)ad(x) + 2d(y)aD(x, y) + 4D(x, y)aD(x, y) + 4D(x, y)ad(y) + 4d(y)aD(x, y) + 4d(y)ad(y) = 0$

$d(x)ad(y) + d(y)ad(x) + 4D(x, y)aD(x, y) + 2D(x, y)ad(y) + 2d(y)aD(x, y) + 4d(y)aD(x, y) + 4D(x, y)ad(y) = 0$

$2D(x, y)ad(y) + 2d(y)aD(x, y) + 4d(y)aD(x, y) + 4D(x, y)ad(y) = 0$

Since R is a char$R \neq 2$ then, we get

$D(x, y)ad(y) + d(y)aD(x, y) + 2d(y)aD(x, y) + 2D(x, y)ad(y) = 0$

$3D(x, y)ad(y) + 3d(y)aD(x, y) = 0$

$D(x, y)ad(y) + d(y)aD(x, y) = 0$ (12)

Replacing y by $x + y$ in equation (12) then, we get

$D(x, x + y)ad(x + y) + d(x + y)aD(x, x + y) = 0$

$\big(D(x,x)+D(x,y)\big)a(d(x)+d(y)+2D(x,y))+$
$\big(d(x)+d(y)+2D(x,y)\big)a(D(x,x)+\ D(x,y))=0$
$d(x)ad(x)+d(x)ad(y)+2d(x)aD(x,y)+D(x,y)ad(x)+D(x,y)ad(y)+$
$2D(x,y)aD(x,y)+d(x)ad(x)+d(x)aD(x,y)+d(y)ad(x)+d(y)aD(x,y)+$
$2D(x,y)ad(x)+2D(x,y)aD(x,y)=0$
$d(x)ad(y)+d(y)ad(x)+4D(x,y)aD(x,y)+D(x,y)ad(y)+d(y)aD(x,y)+$
$3D(x,y)ad(x)+3d(x)aD(x,y)=0$
Using equations (8),(11),(12) and the fact that char$R\neq 3$,we get
$$D(x,y)ad(x)+d(x)aD(x,y)=0 \qquad (13)$$
Replacing y by zy in equation (13), and using equation (13), we get
$D(x,zy)ad(x)+d(x)aD(x,zy)=0$
$yD(x,z)ad(x)+D(x,y)zad(x)+d(x)ayD(x,z)+d(x)aD(x,y)z=0$
$yD(x,z)ad(x)+D(x,y)zad(x)+d(x)ayD(x,z)-D(x,y)ad(x)z=0$
$yD(x,z)ad(x)+D(x,y)[zad(x)-ad(x)z)]+d(x)ayD(x,z)=0$
$yD(x,z)ad(x)+D(x,y)[z,ad(x)]+d(x)ayD(x,z)=0$
$-yd(x)aD(x,z)+D(x,y)[z,ad(x)]+d(x)ayD(x,z)=0$
$[d(x)a,y]D(x,z)+D(x,y)[z,ad(x)]=0$
$D(x,y)[z,ad(x)]=-[d(x)a,y]D(x,z)$
$D(x,y)[z,ad(x)]=[y,d(x)a]D(x,z)$
Interchanging x to y, and y to x and applying symmetric if $D(x,y)=D(y,x)$
$D(y,x)[z,ad(y)]=[x,d(y)a]D(y,z)$
$$D(x,y)[z,ad(y)]=[x,d(y)a]D(z,y) \qquad (14)$$
Replacing x by wx in equation (14) and using equation (14) again, we get
$D(wx,y)[z,ad(y)]=[wx,d(y)a]D(z,y)$
$xD(w,y)[z,ad(y)]+D(x,y)w[z,ad(y)]=w[x,d(y)a]D(z,y)+[w,d(y)a]xD(z,y)$
$xD(w,y)[z,ad(y)]+D(x,y)w[z,ad(y)]=wD(x,y)[z,ad(y)]+[w,d(y)a]xD(z,y)$
Replacing x to w then, we get
$$wD(w,y)[z,ad(y)]+D(w,y)w[z,ad(y)]$$
$$=wD(w,y)[z,ad(y)]+[w,d(y)a]wD(z,y)$$
$$D(w,y)w[z,ad(y)]=[w,d(y)a]wD(z,y) \qquad (15)$$
Replacing z to w in equation (15) then, we get
$D(w,y)w[w,ad(y)]=[w,d(y)a]wD(w,y)$
Replacing w to x then, we get
$$D(x,y)x[x,ad(y)]=[x,d(y)a]xD(x,y) \qquad (16)$$
It follows from Lemma 3 that $D(x,y)=0$ or $[x,ad(y)]=[x,d(y)a]$. In other words, R is the union of its subsets $A=\{x\in R/D(x,y)=0$ for all $y\in R\}$ and $B=\{x\in R/[x,ad(y)-d(y)a]=0$ for all $y\in R\}$. Note that A and B are additive subgroups of R. Since R cannot be written as the union of A and B, it follows that $A=R$ or $B=R$. So from the hypothesis that $R=B$. This implies that $[a,d(y)]\in Z$ for all $y\in R$. By Lemma 2(ii), we know that $a\in Z$. This completes the proof.

REFERENCES

1. N.Argac, On prime rings with derivation, *Ph.D. Thesis*, Ege university,1992.
2. M.Bresar, A note on derivations, *Math. J. Okayama Univ.*, 32 (1990) 83-88.

3. M.Bresar, Centralizing mappings and derivations in prime rings, *J. Algebra,* 156 (1993) 385-394.
4. I.N. Herstein, *Rings with involution,* Univ. of Chicago Press, Chicago, 1969.
5. C.Jaya Subba Reddy, Symmetric reverse bi-derivations on prime rings, *Research J. Pharm. and Tech*, 9(9) (2016) 1496-1500.
6. Gy.Maksa, On the trace of symmetric bi-derivations, *C.R. Math. Rep. Sci. Canada,* 9 (1987) 303-307.
7. J.Mayne, Centralizing mappings of prime rings, *Canadian Math. Bull,* 27(1) (1984) 122-126.
8. M. Sapanci, M.A.Ozturk and Y.B.Jun, Symmetric bi-derivations on prime rings, *East Asian Math .J,* 14(1) (1999) 105-109.
9. J.Vukman, Symmetric bi-derivations on prime ring and semi-prime rings, *Aeuationes Mathematics*, 38 (1989) 245-254.
10. M.S.Yenigul and N.Argac, Ideals and symmetric bi-derivations of prime and semi-prime rings, *Math. J. Okayama Univ,* 35 (1993) 189-192.