b-Continuity Properties of the Cartesian Product of Tadpole Graphs and Paths

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Abstract. A b-coloring by k colors of a graph G is a proper vertex coloring of G using k colors such that in each color class, there exists a vertex adjacent to at least one vertex in every other color class and the b-chromatic number \( \chi_b(G) \) of G is the largest integer k such that there is a b-coloring by k colors. A graph G is b-continuous if G has a b-coloring by k colors for every integer k satisfying \( \chi(G) \leq k \leq \chi_b(G) \). The b-spectrum \( S_b(G) \) of G is the set of all integers k for which G has a b-coloring by k colors. The graph \( T(m, n) \) is the graph obtained by joining a vertex of the cycle \( C_m \) to a pendant vertex of the path \( P_n \) by an edge. In this paper, we find the b-chromatic number of the Cartesian product of the Tadpole graph \( T(m, n) \) and path \( P_r \) for any \( r \geq 1 \). Also, the b-continuity properties of these graphs are discussed.

Keywords: b-coloring, b-chromatic number, b-continuity, Tadpole graph, b-spectrum, Cartesian product.

AMS Mathematics Subject Classification (2010): 05C15

1. Introduction

All graphs considered in this paper are finite, simple, and undirected. For those terminologies not defined in this paper, the reader may refer to [3]. A proper k-coloring of a graph G is an assignment of k-colors to the vertices of G such that no two adjacent vertices are assigned the same color. Equivalently a proper k-coloring of G is a partition of the vertex set \( V(G) \) into \( k \) independent sets \( V_1, V_2, \ldots, V_k \). The sets \( V_i \) (\( 1 \leq i \leq k \)) are called color classes with color \( i \). The chromatic number \( \chi(G) \) is the minimum k for which G admits a proper k-coloring. Later, new types of vertex coloring were introduced and one such coloring is b-coloring. The concept of b-coloring was introduced by Irving and Manlove in 1991 [4]. A b-coloring by k-colors of G is a proper k-coloring such that in each color class, there exists a vertex adjacent to at least one vertex in every other color class. Such a vertex is called a color dominating vertex. Hence, if G has a b-coloring by k colors, then it has at least k color dominating vertices. Consequently, G has at least k vertices of degree at least \( k - 1 \). The b-chromatic number of G, denoted by \( \chi_b(G) \), is the largest integer k such that G has a b-coloring by k colors. To determine the upper bound
of \(\chi_b(G)\), the term t-degree of G, denoted by \(t(G)\) was defined as \(t(G) = \max\{i : 1 \leq i \leq |V(G)|, G \text{ has at least } i \text{ vertices of degree at least } i - 1\}\). Hence, the inequality \(\chi_b(G) \leq t(G)\) follows. In 2003, Faik [2] introduced the concept of b-continuity. It was defined as if for each integer \(k\) satisfying \(\chi(G) \leq k \leq \chi_b(G)\), G has a b-coloring by \(k\)-colors, then G is said to be b-continuous. Later the b-spectrum \(S_b(G)\) of G was defined as the set of all integers \(k\) for which G has a b-coloring by \(k\) colors. i.e. \(S_b(G) = \{k: G \text{ has a b-coloring by } k \text{ colors}\}\). If \(S_b(G)\) contains all the integers from \(\chi(G)\) to \(\chi_b(G)\), then G is b-continuous.

A Tadpole graph \(T(m, n)\) [8] is the graph obtained by joining a cycle \(C_m\), \(m \geq 3\) to a path \(P_n\), \(n \geq 1\) with a bridge.

Graphs \(T(5, 1)\) and \(T(3, 4)\) are shown in figure 1.

\[\text{Figure 1:}\]

**Definition 1.1** The Cartesian product \(G_1 \times G_2\) of two graphs \(G_1\) and \(G_2\) is the graph with vertex set \(V_1 \times V_2\), and any two distinct vertices \((u_1, v_1)\) and \((u_2, v_2)\) are adjacent in \(G_1 \times G_2\) whenever (i) \(u_1 = u_2\) and \(v_1, v_2 \in E_2\) or (ii) \(u_1 u_2 \in E_1\) and \(v_1 = v_2\).

Cartesian product \(K_2 \times P_3\) is shown in figure 2.

\[\text{Figure 2:}\]

**Structural properties of Cartesian product 1.2**

i. If \(u \in V(G_1)\) and \(v \in V(G_2)\), then \(|u| \times V(G_2) \equiv G_2\) and \(V(G_1) \times \{v\} \equiv G_1\).

ii. In \(G_1 \times G_2\), there are \(|V(G_1)|\) copies of \(G_2\) and \(|V(G_2)|\) copies of \(G_1\).

iii. \(G_1 \times K_1 \equiv G_1\) and \(K_1 \times G_2 \equiv G_2\).

In this paper, we find the b-chromatic number of \(T(m, n) \times P_n\), the Cartesian product of a Tadpole graph and a path for all \(m \geq 3\) and \(n, r \geq 1\). Also we prove that these graphs are b-continuous.

Graph \(T(4, 3) \times P_2\) is shown in figure 3.
2. Preliminaries
In this section, some properties of the Tadpole graph $T(m, n)$ and some basic results on $T(m, n)$ are given.

Observation 2.1. [4, 5]
i) If $G$ admits a b-coloring with $k$-colors, then $G$ must have at least $k$ vertices of degree at least $k - 1$.

ii) Any proper coloring with $\chi$ colors is a b-coloring.

iii) If $G$ contains an induced path or cycle on at least 5 vertices, then

iv) $\chi_b(G)$ is at least 3.

v) If $G$ contains an induced $K_n$, then $\chi_b(G) \geq n$.

vi) $\chi(G) \leq \chi_b(G) \leq t(G)$.

vii) $\chi(G), \chi_b(G) \in S_b(G)$ and from the definition of $S_b(G)$, the minimum value of $S_b(G)$ is the chromatic number of $G$ and maximum value of $S_b(G)$ is the b-chromatic number of $G$.

Observation 2.2. For $m \geq 3$ and $n \geq 1$,
i) $T(m, n)$ has $m + n$ vertices and $m + n$ edges.

ii) $T(m, n)$ has exactly one vertex of degree 3, one vertex of degree 1 and $m + n - 2$ vertices of degree 2.

iii) $\chi(T(m, n)) = \begin{cases} 2, & \text{if } m \text{ is even} \\ 3, & \text{if } m \text{ is odd} \end{cases}$

Theorem 2.3. [8] For $m \geq 3$ and $n \geq 1$,
i) $t(T(m, n)) = 3$

ii) $2 \leq c_b(T(m, n)) \leq 3$.

iii) Tadpole graph $T(m, n)$ is a b-continuous graph.

3. Main results
In this section we prove that the Cartesian product of Tadpole graph and a path is b-continuous. To prove the theorem we use few notations and terminologies.

Notations and Terminologies 3.1
Throughout this paper, the following notations and terminologies are observed.
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i. c is a function which assigns colors to the vertices of a graph in discussion. Hence, if u is any vertex of a graph, then c(u) denotes its color.

ii. In figures, the color dominating vertices are circled.

iii. We refer to a color dominating vertex as cdv. In particular, if u is a color dominating vertex of color i, then it is referred to as i-cdv.

iv. In T(m, n) × P_r, {v_1, v_2, ..., v_m} represents the vertex set V(C_m) and {u_1, u_2, ..., u_n} represents vertex set V(P_n) of T(m, n) and {w_1, w_2, ..., w_r} represents the vertex set V(P_r). Further P_n is joined to C_m at v_1 by the edge u_1v_1.

With the above notations we observe the following.

**Observation 3.2.**

i. V(T(m, n) × P_r) = {(v_i, w_k) : i = 1 to m, k = 1 to r} ∪ {(u_j, w_k) : j = 1 to n, k = 1 to r}

ii. V(T(m, n)) × {w_k} ≅ T(m, n) for each k = 1 to r.

iii. \{v_i\} × P_r ≅ P_r for each i = 1 to m and \{u_j\} × P_r ≅ P_r for each j = 1 to n.

**Observation 3.3.** For m ≥ 3, n ≥ 1 and r ≥ 2

i. |V(T(m, n) × P_r)| = (m + n)r

ii. |E(T(m, n) × P_r)| = (2r - 1)(m + n)

iii. χ(T(m, n) × P_r) = \[
\begin{cases} 
2, & \text{if } m \text{ is even} \\
3, & \text{if } m \text{ is odd}
\end{cases}
\]

**Observation 3.4.** In T(m, n) × P_r, for m ≥ 3, n ≥ 1 and r ≥ 2

i. there are exactly 2 vertices of degree 2,

ii. there are exactly 2(m – 1) + 2(n – 1) + (r – 2) vertices of degree 3,

iii. there are exactly 2 + (m – 1)(r – 2) + (n – 1)(r – 2) vertices of degree 4

iv. there are exactly (r – 2) vertices of degree 5.

**Observation 3.5.** For m ≥ 3 and n ≥ 1

i. t(T(m, n) × P_r) = 4, r = 2

ii. t(T(m, n) × P_r) = 5, 3 ≤ r ≤ 7

iii. t(T(m, n) × P_r) = 6, r ≥ 8

From observation 2.1(v), 3.3(iii) and 3.5, the b-chromatic number of χ_b(T(m, n) × P_r) lies between 2 and 6. Also from observation 2.1(ii), to prove T(m, n) × P_r is b-continuous it is enough to prove that T(m, n) × P_r has a b-coloring by k colors for each k satisfying χ(T(m, n) × P_r) ≤ k ≤ χ_b(T(m, n) × P_r). From 1.2(iii), T(m, n) × P_r ≅ T(m, n) and from theorem 2.3(iii), T(m, n) is a b-continuous graph. Thus, T(m, n) × P_r is b-continuous for r = 1 and hence we prove theorems to find S_b(T(m, n) × P_r) for various values of m, n and r ≥ 2.
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**Theorem 3.6.** If \( m \) is even, \( m \geq 4 \) and \( n \geq 1 \), then

\[
S_b(T(m, n) \times P_r) = \begin{cases} 
\{2, 3, 4\}, & \text{if } r = 2 \\
\{2, 3, 4, 5\}, & \text{if } 3 \leq r \leq 7 \\
\{2, 3, 4, 5, 6\}, & \text{if } r \geq 8 
\end{cases}
\]

\[
\chi_b(T(m, n) \times P_r) = \begin{cases} 
4, & \text{if } r = 2 \\
5, & \text{if } 3 \leq r \leq 7 \\
6, & \text{if } r \geq 8 
\end{cases}
\]

and \( T(m, n) \times P_r \) is a b-continuous graph.

**Proof:** Since \( m \) is even, from observation 3.3(iii), \( \chi(T(m, n) \times P_r) = 2 \). Hence, \( T(m, n) \times P_r \) has a b-coloring with 2 colors.

**Case (i)** \( r = 2 \)

By observations 2.1(v) and 3.5(i),

\[ \chi_b(T(m, n) \times P_2) \leq 4 \]

We prove that \( T(m, n) \times P_r \) has a b-coloring by 3-colors and 4-colors. Let \( c(v_i, w_k) = k \), \( k = 1, 2 \). Assign colors 2, 3 to the each pair of vertices \((v_i, w_1)\) and \((v_i, w_2)\) for even \( i \) (\( 2 \leq i \leq m \)), and to \((u_j, w_1)\) and \((u_j, w_2)\) for even \( j \) (\( 2 \leq j \leq n \)). If we assign colors 3 and 1 to the each pair of vertices \((v_i, w_1)\) and \((v_i, w_2)\), for odd \( i \) (\( 3 \leq i \leq m - 1 \)), and to \((u_j, w_1)\) and \((u_j, w_2)\), for odd \( j \) (\( 1 \leq j \leq n \)). Then \((v_1, w_1)\) is 1-cdv, \((v_1, w_2)\) is a 2-cdv and \((v_2, w_2)\) is a 3-cdv. Then we get a b-coloring by 3-colors.

Next we prove that \( T(m, n) \times P_r \) has a b-coloring by 4-colors. Since there are exactly 2 vertices of degree 4, assign any two colors, namely 1, 2 to those vertices. Let \( c(v_1, w_k) = k \), \( k = 1, 2 \) and \( c(v_2, w_k) = k + 2 \), \( k = 1, 2 \). Let \( c(v_m, w_1) = 4 \) and \( c(v_m, w_2) = 3 \). Assign colors 1, 2 to the each pair of vertices \((v_i, w_1)\) and \((v_i, w_2)\) for odd \( i \) (\( 3 \leq i \leq m - 1 \)) and to \((u_j, w_1)\) and \((u_j, w_2)\) for odd \( j \) (\( 1 \leq j \leq n \)). If we assign colors 1 and 2 to the each pair of vertices \((v_i, w_1)\) and \((v_i, w_2)\) for odd \( i \) (\( 3 \leq i \leq m - 3 \)), and to \((u_j, w_1)\) and \((u_j, w_2)\) for even \( j \) (\( 1 \leq j \leq n \)), then \((v_1, w_1)\) is k-cdv and \((v_2, w_2)\) is a \((k+2)\)-cdv, \( k = 1, 2 \). Then we get a b-coloring by 4-colors.

From the above results, \( T(m, n) \times P_r \) has a b-coloring by 2-colors, 3-colors and 4-colors. Hence \( \chi_b(T(m, n) \times P_r) = 4 \) and \( S_b = \{2, 3, 4\} \).

**Case (ii)** \( 3 \leq r \leq 7 \)

By observations 2.1(v) and 3.5(ii),

\[ \chi_b(T(m, n) \times P_r) \leq 5 \]

We prove that \( T(m, n) \times P_r \) has a b-coloring by 3-colors, 4-colors and 5-colors. Since \( T(m, n) \times P_2 \) is an induced sub graph of \( T(m, n) \times P_r \), we apply the same color scheme as given in case (i) to \( T(m, n) \times P_r \). In addition, for each odd \( k \), \( 3 \leq k \leq r \), \( c(v_i, w_k) = c(v_i, w_1) \), for all \( i = 1 \) to \( m \), and \( c(u_j, w_k) = c(u_j, w_1) \), for all \( j = 1 \) to \( n \). Similarly, for each even \( k \), \( 3 \leq k \leq r \), \( c(v_i, w_k) = c(v_i, w_2) \), for all \( i = 1 \) to \( m \), and \( c(u_j, w_k) = c(u_j, w_2) \), for all \( j = 1 \) to \( n \). Then we get a b-coloring by 3-colors and 4-colors.

Next we prove that \( T(m, n) \times P_r \) has a b-coloring by 5-colors. For \( k \), \( 1 \leq k \leq r \), assign colors 5, 1, 3 to the vertices \((v_i, w_k)\), colors 4, 2, 5 to the vertices \((v_2, w_k)\) and colors 2, 4,
5 to the vertices \((v_m, w_k)\), in cyclic order. For each odd \(i\), \(3 \leq i \leq m - 1\), assign colors 5, 3, 1 to the vertices \((v_i, w_k)\), and for each even \(i\), \(3 \leq i \leq m - 1\) colors 2, 4, 5 to the vertices \((v_i, w_k)\) in cyclic order.

Similarly, for each odd \(j\), \(1 \leq j \leq n\), assign colors 3, 2, 1 to the vertices \((u_j, w_k)\), and for each even \(j\), \(1 \leq j \leq n\) colors 2, 4, 5 to the vertices \((u_j, w_k)\) in cyclic order.

Therefore \((v_1, w_2)\), \((v_2, w_2)\), \((v_3, w_2)\), \((v_m, w_2)\) and \((v_1, w_1)\) are 1, 2, 3, 4 and 5 color dominating vertices respectively. Then we get a b-coloring by 5-colors.

From the above results, \(T(m, n) \times P_r\) has a b-coloring by 2-colors, 3-colors, 4-colors and 5-colors. Hence \(\chi_b(T(m, n) \times P_r) = 5\) and \(S_b = \{2, 3, 4, 5\}\).

**Case (iii) \(r \geq 8\)**

By observations 2.1(v) and 3.5(iii),

\[2 \leq \chi_b(T(m, n) \times P_r) \leq 6\]

Let us show that \(T(m, n) \times P_r\) has a b-coloring by 3-colors, 4-colors, 5-colors and 6-colors. Since \(T(m, n) \times P_r\), \(3 \leq r \leq 7\) is an induced sub graph of \(T(m, n) \times P_r\), \(r \geq 8\), we apply the same color scheme as given in case (ii) to \(T(m, n) \times P_r\), \(r \geq 8\). Then we get a b-coloring by 3-colors, 4-colors and 5-colors.

Next we prove that \(T(m, n) \times P_r\) has a b-coloring by 6-colors. For each \(k = 1\) to \(r\), assign colors 6, 1, 2, 3, 4, 5 to the vertices \((v_1, w_k)\), colors 3, 4, 5, 6, 1, 2 to the vertices \((v_m, w_k)\) and 4, 5, 6, 1, 2, 3 to the vertices \((v_2, w_k)\), colors 2, 3, 4, 5, 6, 1 to the vertices \((u_1, w_k)\), \(1 \leq k \leq r\) in cyclic order. For each odd \(i\), \(2 \leq i \leq m - 1\), \(c(v_i, w_k) = c(v_1, w_k)\), for each even \(i\), \(4 \leq i \leq m - 2\), \(c(v_i, w_k) = c(v_2, w_k)\) and for each odd \(j\), \(3 \leq j \leq n\), \(c(u_j, w_k) = c(u_1, w_k)\) and for each even \(j\), \(2 \leq j \leq n\), \(c(u_j, w_k) = c(u_2, w_k)\), for all \(k = 1\) to \(r\). Therefore \((v_i, w_{k+1})\) is \(k\)-cdv for \(k = 1\) to 6. Then we get a b-coloring by 6-colors.

From the above results, \(T(m, n) \times P_r\) has a b-coloring by 2-colors, 3-colors, 4-colors, 5-colors and 6-colors. Hence \(\chi_b(T(m, n) \times P_r) = 6\) and \(S_b = \{2, 3, 4, 5, 6\}\).

From case (i), (ii) and (iii), \(T(m, n) \times P_r\) is a b-continuous graph for \(m\) is even, \(m \geq 4\) and \(n, r \geq 1\).

**Theorem 3.7.** For \(m = 3\),

\[
S_b(T(m, n) \times P_r) = \begin{cases} 
\{3, 4\}, & \text{if } r = 2, \quad n \geq 1 \\
\{3, 4\}, & \text{if } r = 3, \quad n = 1 \\
\{3, 4, 5\}, & \text{if } 4 \leq r \leq 7, \quad n \geq 1 \\
\{3, 4, 5\}, & \text{if } r \geq 8, \quad n \geq 1
\end{cases}
\]

\[
\chi_b(T(m, n) \times P_r) = \begin{cases} 
4, & \text{if } r = 2, \quad n \geq 1 \\
4, & \text{if } r = 3, \quad n = 1 \\
5, & \text{if } r = 3, \quad n \geq 2 \\
5, & \text{if } 4 \leq r \leq 7, \quad n \geq 1 \\
6, & \text{if } r \geq 8, \quad n \geq 1
\end{cases}
\]
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and \( T(m, n) \times P_r \) is a b-continuous graph.

**Proof:** Since \( m = 3 \), from observation 3.3(iii), \( \chi(T(m, n) \times P_l) = 3 \). Hence, \( T(m, n) \times P_l \) has a b-coloring with 3 colors.

**Case (i)** \( r = 2 \) and \( n \geq 1 \)

By observations 2.1(v) and 3.5(i),

\[ 3 \leq \gamma_b(T(m, n) \times P_r) \leq 4 \]

Since \( T(m, n) \times P_2 \) contains \( K_3 \) as an induced sub graph assign distinct colors to the vertices of \( K_3 \). Let \( c(v_1, w_k) = k \), \( c(v_2, w_k) = k+2 \), \( k = 1, 2 \), \( c(v_3, w_1) = 2 \), \( c(v_3, w_2) = 1 \), \( c(u_i, w_i) = 4 \) and \( (u_i, w_2) = 3 \). Then each \( (v_i, w_k) \) is a \( k \)-color dominating vertex, and \( (v_2, w_k) \) is a \((k+2)\)-color dominating vertex, \( k = 1, 2 \). For each even \( j \), \( c(u_j, w_k) = c(v_1, w_k) \) and for each odd \( j \), \( c(u_j, w_k) = c(u_1, w_k) \), \( (2 \leq j \leq n) \), for all \( k = 1 \) to \( r \). Then we get a b-coloring by 4 colors. Hence \( \gamma_b(T(m, n) \times P_r) = 4 \) and \( S_b = \{ 3, 4 \} \).

**Case (ii)** \( r = 3 \) and \( n = 1 \)

By observations 2.1(v) and 3.5(ii),

\[ 3 \leq \gamma_b(T(m, n) \times P_r) \leq 5 \]

Since \( T(m, n) \times P_3 \) is an induced sub graph of \( T(m, n) \times P_r \), we apply the same color scheme as given in case (i) to \( T(m, n) \times P_r \). We can get the color dominating vertices. In addition, let each \( i \), \( 1 \leq i \leq 3 \), \( c(v_i, w_3) = c(v_i, w_1) \) and \( c(u_1, w_3) = c(u_1, w_1) \). Then we get a b-coloring by 4-colors which is shown in figure 4.

![Figure 4](image)

Next we prove that \( T(m, n) \times P_r \) has no b-coloring by 5-colors.

By observation 3.4, there is exactly one vertex of degree 5 and 5 vertices of degree 4. From the five vertices of degree at least 4, we must get five color dominating vertices. Assign distinct colors namely 1, 2, 3, 4, 5 to these vertices. Let \( c(v_i, w_2) = i \), \( 1 \leq i \leq 3 \); \( c(v_1, w_1) = 4 \), and \( c(v_1, w_3) = 5 \). Then \( (v_1, w_2) \) is a 1-cdv. To get 3-cdv, let \( c(v_3, w_3) = 4 \) and \( c(v_3, w_1) = 5 \). Then \( (v_3, w_2) \) is a 3-cdv. To get 2-cdv, assign colors 4 and 5 to the vertices \( (v_2, w_3) \) and \( c(v_2, w_1) \). But this is impossible. From the above discussion, we cannot get a b-coloring by 5-colors. Hence \( \gamma_b(T(m, n) \times P_r) = 4 \) and \( S_b = \{ 3, 4 \} \).

**Case (iii)** \( r = 3 \) and \( n \geq 2 \)

By observations 2.1(v) and 3.5(ii),

\[ 3 \leq \gamma_b(T(m, n) \times P_r) \leq 5. \]

We prove that \( T(m, n) \times P_r \) has a b-coloring by 4-colors and 5-colors. Since \( T(m, 1) \times P_r \) is an induced sub graph of \( T(m, n) \times P_r \), we apply the same color scheme as in case (ii) to \( T(m, n) \times P_r \). In addition, for even \( j \), \( c(u_j, w_k) = c(v_1, w_k) \) and for odd \( j \), \( c(u_j, w_k) = c(u_1, w_k) \), \( (2 \leq j \leq n) \) for all \( k = 1 \) to \( 3 \). Hence we get a b-coloring by 4 colors.

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Next we prove that $T(m, n) \times P_r$ has a b-coloring by 5-colors. Since there are 6 vertices of degree at least 4, we assign colors 1, 2, 3, 4 and 5 to any five of these vertices. Let $c(v_i, w_j) = i$, $i = 1, 2, 3$; $c(u_i, w_j) = 4$ and $c(v_i, w_j) = 5$, then $(v_i, w_j)$ is 1-cdv. Since $c(v_2, w_2) = 2$ and $(v_2, w_2)$ is adjacent to the vertices of colors 1, 3, assign colors 4 and 5 properly to the adjacent vertices (which are not yet colored) of $(v_2, w_2)$. Therefore, $c(v_2, w_2) = 4$, $c(v_2, w_1) = 5$. Then $(v_2, w_2)$ is 2-cdv. To get $(v_2, w_1)$ is 3-cdv, we must assign color 4 and 5 to $(v_2, w_1)$ and $(v_3, w_1)$. Since $(v_3, w_1)$ is adjacent to the vertices of colors 4 and 5, $c(v_3, w_1) \neq 4$ and 5, Therefore $(v_3, w_1)$ cannot be 3-cdv. Hence assign color $c(v_3, w_2)$ to $(v_1, w_1)$. Since $c(v_1, w_1) = 3$ and $(v_1, w_1)$ is adjacent to the vertices of colors 1 and 5, assign colors 2 and 4 properly to the adjacent vertices (which are not yet colored) of $(v_1, w_1)$. Let $c(v_3, w_1) = 4$ and $c(u_1, w_1) = 2$, then $(v_1, w_1)$ is 3-cdv. Since $c(v_1, w_3) = 5$ and $(v_1, w_3)$ is adjacent to the vertices of colors 1 and 4, assign colors 2 and 3 properly to the adjacent vertices (which are not yet colored) of $(v_1, w_3)$. Therefore $c(v_3, w_3) = 2$, $c(u_1, w_3) = 3$. Hence $(v_1, w_3)$ is 5-cdv. By observation 3.4, $T(m, n) \times P_r$ has one more vertex of degree 4, namely $(u_1, w_2)$. Therefore, we use the vertex $(u_1, w_2)$, to get 4-cdv. Since $c(u_1, w_2) = 4$ and $(u_1, w_2)$ is adjacent to the vertices of colors 1, 2, 3, assign color 5 to $(u_2, w_2)$. Hence $(u_1, w_2)$ is 4-cdv. Let $c(u_1, w_1) = 3$ and $c(u_2, w_2) = 2$. In addition for each odd $j$, $c(u_j, w_k) = c(u_j, w_k)$ and for each even $j$, $c(u_j, w_k) = c(u_j, w_k)$, $(3 \leq j \leq n)$ for all $k = 1$ to 3. Then we get a b-coloring by 5-colors. Hence $\chi_b(T(m, n) \times P_r) = 5$ and $S_6 = \{3, 4, 5\}$.

**Case (iv)** $4 \leq r \leq 7$ and $n \geq 1$

By observations 2.1(v) and 3.5(iii),

$$3 \leq \chi_b(T(m, n) \times P_r) \leq 5$$

We prove that $T(m, n) \times P_r$ has a b-coloring by 4-colors and 5-colors. Assign colors 1, 2, 3, 4 to the vertices $(v_i, w_k)$, colors 2, 3, 4, 1 to the vertices $(v_2, w_k)$, colors 3, 4, 1, 2 to the vertices $(v_3, w_k)$ and colors 4, 3, 2, 1 to the vertices $(u_1, w_k)$ for all $k = 1$ to $r$ in cyclic order. For each odd $j$, $c(u_j, w_k) = c(u_j, w_k)$, and for each even $j$, $c(u_j, w_k) = c(v_j, w_k)$, $(2 \leq j \leq n)$ for all $k = 1$ to $r$. Then $(v_i, w_k)$ is k-cdv, $k = 1$ to 4 and also we get a b-coloring by 4-colors.

Next we prove that $T(m, n) \times P_r$ has a b-coloring by 5-colors. Assign colors 1, 2, 3, 4 to the vertices $(v_i, w_k)$, colors 3, 4, 5, 1 to the vertices $(v_2, w_k)$, colors 5, 1, 2, 3 to $(v_3, w_k)$ and colors 4, 5, 1, 2 to the vertices $(u_1, w_k)$ for all $k = 1$ to $r$ in cyclic order. For each even $j$, $c(u_j, w_k) = c(v_j, w_k)$, and for each odd $j$, $c(u_j, w_k) = c(u_j, w_k)$, $(2 \leq j \leq n)$ for all $k = 1$ to $r$. Then $(v_i, w_k)$ is k-cdv, $k = 1$ to 3, $(v_2, w_2)$ is 4-cdv and $(v_2, w_3)$ is 5-cdv. Thus we get a b-coloring by 5-colors. Hence $\chi_b(T(m, n) \times P_r) = 5$ and $S_6 = \{3, 4, 5\}$.

**Case (v)** $r \geq 8$ and $n \geq 1$

By observations 2.1(v) and 3.5(iii),

$$3 \leq \chi_b(T(m, n) \times P_r) \leq 6$$

In this case we prove that $T(m, n) \times P_r$ has a b-coloring by 4-colors, 5-colors and 6-colors. Since $T(m, n) \times P_r$ has an induced sub graph of $T(m, n) \times P_r$ $(r \geq 8)$, we apply the same color scheme as in case(iv) to $T(m, n) \times P_r$ $(r \geq 8)$. Then we get a b-coloring by 4-colors and 5-colors.

Next we prove that $T(m, n) \times P_r$ has a b-coloring by 6-colors. If we assign colors 6, 1, 2, 3, 4, 5 to the vertices $(v_i, w_k)$, colors 2, 3, 4, 5, 6, 1 to the vertices $(v_2, w_k)$, colors
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3, 4, 5, 6, 1, 2 to (v_3, w_k) and colors 4, 5, 6, 1, 2, 3 to the vertices (u_i, w_k) for all k = 1 to r in cyclic order, then (v_1, w_k) is (k – 1)-cdv for k = 2 to 7. For each even j, c(u_j, w_k) = c(v_1, w_k) and for each odd j, c(u_j, w_k) = c(u_i, w_k), (2 ≤ j ≤ n) for all k = 1 to r. Then we get a b-coloring by 6-colors. Hence \( \chi_b(T(m, n) \times P_r) = 6 \) and \( S_b = \{3, 4, 5, 6\} \).

From case (i), (ii), (iii), (iv) and (v), \( T(m, n) \times P_r \) is a b-continuous graph for \( m = 3 \) and \( n, r \geq 1 \).

**Theorem 3.8.** If \( m \) is odd, \( m \geq 5 \) and \( n \geq 1 \), then

\[
S_b(T(m, n) \times P_r) = \begin{cases} 
\{3, 4\}, & \text{if } r = 2 \\
\{3, 4, 5\}, & \text{if } 3 \leq r \leq 7 \\
\{3, 4, 5, 6\}, & \text{if } r \geq 8
\end{cases}
\]

\[
\chi_b(T(m, n) \times P_r) = \begin{cases} 
5, & \text{if } r = 2 \\
6, & \text{if } r \geq 8
\end{cases}
\]

and \( T(m, n) \times P_r \) is a b-continuous graph.

**Proof:** Since \( m \) is odd, from observation 3.3(iii), \( \chi(T(m, n) \times P_r) = 3 \). Hence, \( T(m, n) \times P_r \) has a b-coloring with 3 colors.

**Case (i) \( r = 2 \)**

By observations 2.1(v) and 3.5(i),

\[
3 \leq \chi_b(T(m, n) \times P_r) \leq 4
\]

Now we prove that \( T(m, n) \times P_r \) has a b-coloring by 4-colors. Assign colors 1, 3 to \((v_1, w_k)\), colors 2, 4 to \((v_2, w_k)\), colors 3, 1 to \((v_3, w_k)\) for all \( k = 1, 2 \) in order. Let \( c(u_1, w_2) = 4 \), \( c(u_1, w_2) = 2 \). Then \((v_i, w_1)\) is i-cdv and \((v_i, w_2)\) is a \((i + 2)\)-cdv, for all \( i = 1, 2 \). For each even \( i, c(v_i, w_1) = 2 \), \( c(v_i, w_2) = 4 \) and for each odd \( i, c(v_i, w_1) = 5 \), \( c(v_i, w_2) = 1 \) \((4 \leq i \leq m)\). Also, for each even \( i, c(u_j, w_1) = c(v_i, w_1) \) and for each odd \( j, c(u_j, w_1) = c(u_i, w_1) \), \((2 \leq j \leq n)\) for \( k = 1, 2 \). Then we get a b-coloring by 4 colors. Hence \( \chi_b(T(m, n) \times P_r) = 4 \) and \( S_b = \{3, 4\} \).

**Case (ii) \( 3 \leq r \leq 7 \)**

By observations 2.1(v) and 3.5(ii),

\[
3 \leq \chi_b(T(m, n) \times P_r) \leq 5
\]

Since \( T(m, n) \times P_r \) is an induced sub graph of \( T(m, n) \times P_r \), \( 3 \leq r \leq 7 \), we get four color dominating vertices. In addition, for each odd \( k, c(v_i, w_k) = c(u_i, w_k) \) and \( c(u_j, w_k) = c(u_i, w_k) \), \((3 \leq k \leq 7)\) for all \( k = 1 \) to \( m \) and for all \( j = 1 \) to \( n \). Then we get a b-coloring by 4-colors.

Let \( c(v_1, w_1) = k \), \( k = 1 \) to 3. Assign colors 3, 4, 5 to \((v_m, w_k)\), colors 4, 5, 1 to \((v_2, w_k)\). In addition, for each even \( i, \) assign colors 5, 1, 4 to \((v_i, w_k)\) and for each odd \( i, \) assign colors 2, 3, 5 to \((v_i, w_k)\) \((3 \leq i \leq m - 1)\) for all \( k = 1 \) to 3 in order. Also assign colors 5, 1, 4 to \((u_k, w_1)\) for all \( k = 1 \) to 3 in order.

For each even \( j, c(u_j, w_1) = c(v_i, w_1) \) and for each odd \( j, c(u_j, w_1) = c(u_i, w_1) \), \((2 \leq j \leq n)\) for all \( k = 1 \) to 3. Each \( c(v_i, w_1) = k, k = 1 \) to 3, is \( k\)-cdv. Also \((v_m, w_2)\) is \( 4\)-cdv and \((v_2, w_2)\) is \( 5\)-cdv. In addition, for each odd \( k, c(v_i, w_1) = c(v_i, w_3) \), and for each even \( k, \)
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\[ c(v_i, w_k) = c(v_i, w_2), \quad (4 \leq k \leq r) \] for all \( i = 1 \) to \( m \). Then we get a b-coloring by 5-colors. Hence \( \chi_b(T(m, n) \times P_r) = 5 \) and \( S_b = \{3, 4, 5\} \).

**Case (iii) \( r \geq 8 \)**

By observations 2.1(v) and 3.5(iii),
\[ 3 \leq \chi_b(T(m, n) \times P_r) \leq 6 \]

We show that \( T(m, n) \times P_r \) has a b-coloring by 4-colors, 5-colors and 6-colors. Since \( T(m, n) \times P_r, 3 \leq r \leq 7, \) is an induced sub graph of \( T(m, n) \times P_r, \) \( r \geq 8, \) we apply the same color scheme as given in case (ii) to \( T(m, n) \times P_r, r \geq 8. \) Then we get a b-coloring by 4-colors and 5-colors. Next we prove that \( T(m, n) \times P_r \) has a b-coloring by 6-colors. Assign colors 6, 1, 2, 3, 4, 5, 1 to the vertices \( (v_1, w_k) \), colors 2, 3, 4, 5, 6, 1 to the vertices \( (u_1, w_k) \), colors 3, 4, 5, 6, 1, 2 to \( (v_m, w_k) \) and colors 4, 5, 6, 1, 2, 3 to the vertices \( (v_2, w_k) \) for all \( k = 1 \) to \( r \) in cyclic order. For each odd \( i, c(v_i, w_k) = c(v_m, w_k), \) and for each even \( i, c(v_i, w_k) = c(v_2, w_k), (3 \leq i \leq m - 1) \) for all \( k = 1 \) to \( r. (v_1 w_{k+1}) \) is k-cdv, \( k = 1 \) to 6. Then we get a b-coloring by 6-colors. Hence \( \chi_b(T(m, n) \times P_r) = 6 \) and \( S_b = \{3, 4, 5, 6\} \).

From case (i), (ii) and (iii), \( T(m, n) \times P_r \) is a b-continuous graph for \( m \) is odd, \( m \geq 5 \) and \( n, r \geq 1. \)

**4. Conclusion**

In this paper, we found the b-chromatic number of \( T(m, n) \times P_r \) and proved that it is a b-continuous graph. This paper can be further extended to the Cartesian product of Tadpole graph and cycle.

**REFERENCES**


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