Annals of Pure and Applied Mathematics
Vol. 15, No. 2, 2017, 225-234
ISSN: 2279-087X (P), 2279-0888(online)
Published on 11 December 2017
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v15n2a9

Annals of
Pure and Applied Mathematics

b-Continuity Properties of the Cartesian Product of Tadpole Graphs and Paths

M. Poobalaranjani **and** ***S. Saraswathi***

PG & Research Department of Mathematics
Seethalakshmi Ramaswami College, Tiruchirappalli, Tamil Nadu, India.

Received 1 November 2017; accepted 9 December 2017

Abstract. A b-coloring by k colors of a graph G is a proper vertex coloring of G using k colors such that in each color class, there exists a vertex adjacent to at least one vertex in every other color class and the b-chromatic number $\chi_b(G)$ of G is the largest integer k such that there is a b-coloring by k colors. A graph G is b-continuous if G has a b-coloring by k colors for every integer k satisfying $\chi(G) \leq k \leq \chi_b(G)$. The b-spectrum $S_b(G)$ of G is the set of all integers k for which G has a b-coloring by k colors. The graph T(m, n) is the graph obtained by joining a vertex of the cycle C_m to a pendant vertex of the path P_n by an edge. In this paper, we find the b-chromatic number of the Cartesian product of the Tadpole graph T(m, n) and path P_r for any $r \geq 1$. Also, the b-continuity properties of these graphs are discussed.

Keywords: b-coloring, b-chromatic number, b-continuity, Tadpole graph, b-spectrum, Cartesian product.

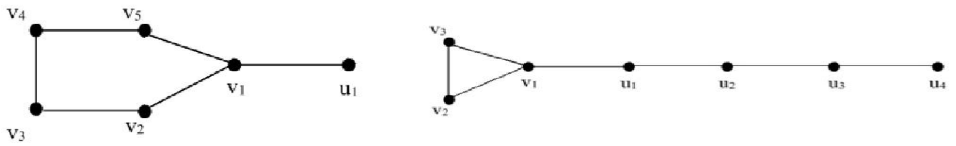
AMS Mathematics Subject Classification (2010): 05C15

1. Introduction

All graphs considered in this paper are finite, simple, and undirected. For those terminologies not defined in this paper, the reader may refer to [3]. A proper k-coloring of a graph G is an assignment of k-colors to the vertices of G such that no two adjacent vertices are assigned the same color. Equivalently a proper k-coloring of G is a partition of the vertex set V(G) into k independent sets $V_1, V_2, \ldots, V_k$. The sets V_i $(1 \leq i \leq k)$ are called color classes with color i. The chromatic number $\chi(G)$ is the minimum k for which G admits a proper k-coloring. Later, new types of vertex coloring were introduced and one such coloring is b-coloring. The concept of b-coloring was introduced by Irving and Manlove in 1991 [4]. A b-coloring by k-colors of G is a proper k-coloring such that in each color class, there exists a vertex adjacent to at least one vertex in every other color class. Such a vertex is called a color dominating vertex. Hence, if G has a b-coloring by k colors, then it has at least k color dominating vertices. Consequently, G has at least k vertices of degree at least k – 1. The b-chromatic number of G, denoted by $\chi_b(G)$, is the largest integer k such that G has a b-coloring by k colors. To determine the upper bound of $\chi_b(G)$, the term t-degree of G, denoted by t(G) was defined as $t(G) = \max\{i : 1 \le i \le |V(G)|$, G has at least i vertices of degree at least $i - 1\}$. Hence, the inequality $\chi_b(G) \le t(G)$ follows. In 2003, Faik [2] introduced the concept of b-continuity. It was defined as if for each integer k satisfying $\chi(G) \le k \le \chi_b(G)$, G has a b-coloring by k-colors, then G is said to be b-continuous. Later the b-spectrum $S_b(G)$ of G was defined as the set of all integers k for which G has a b-coloring by k colors. i.e. $S_b(G) = \{k$: G has a b-coloring by k colors$\}$. If $S_b(G)$ contains all the integers from $\chi(G)$ to $\chi_b(G)$, then G is b-continuous.

A Tadpole graph T(m, n) [8] is the graph obtained by joining a cycle C_m, $m \ge 3$ to a path P_n, $n \ge 1$ with a bridge.

Graphs T(5, 1) and T(3, 4) are shown in figure 1.

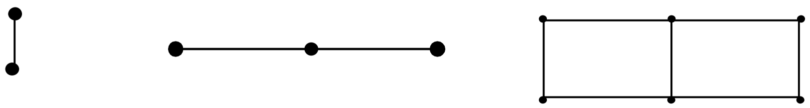


(a) T(5, 1) (b) T(3, 4)

Figure 1:

Definition 1.1 The Cartesian product $G_1 \times G_2$ of two graphs G_1 and G_2 is the graph with vertex set $V_1 \times V_2$, and any two distinct vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G_1 \times G_2$ whenever (i) $u_1 = u_2$ and $v_1v_2 \in E_2$ or (ii) $u_1u_2 \in E_1$ and $v_1 = v_2$.

Cartesian product $K_2 \times P_3$ is shown in figure 2.



(a) K_2 (b) P_3 (c) $K_2 \times P_3$

Figure 2:

Structural properties of Cartesian product 1.2

i. If $u \in V(G_1)$ and $v \in V(G_2)$, then $\{u\} \times V(G_2) \cong G_2$ and $V(G_1) \times \{v\} \cong G_1$.
ii. In $G_1 \times G_2$, there are $|V(G_1)|$ copies of G_2 and $|V(G_2)|$ copies of G_1.
iii. $G_1 \times K_1 \cong G_1$ and $K_1 \times G_2 \cong G_2$.

In this paper, we find the b-chromatic number of $T(m, n) \times P_r$, the Cartesian product of a Tadpole graph and a path for all $m \ge 3$ and $n, r \ge 1$. Also we prove that these graphs are b-continuous.

Graph $T(4, 3) \times P_2$ is shown in figure 3.

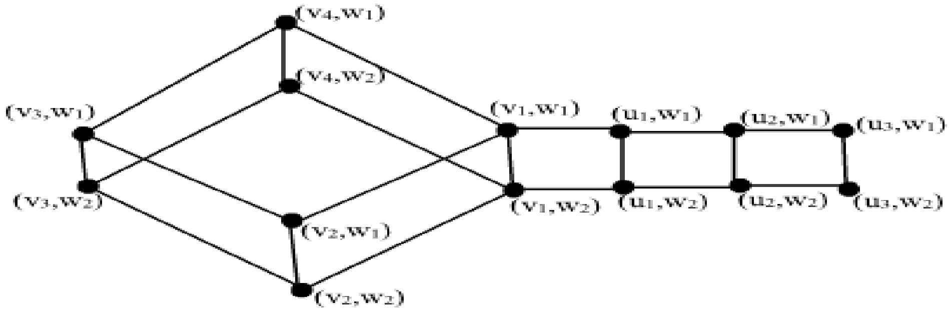


Figure 3:

2. Preliminaries

In this section, some properties of the Tadpole graph T(m, n) and some basic results on T(m, n) are given.

Observation 2.1. [4, 5]

i) If G admits a b-coloring with k-colors, then G must have at least k vertices of degree at least k – 1.
ii) Any proper coloring with χ colors is a b-coloring.
iii) If G contains an induced path or cycle on at least 5 vertices, then
iv) $\chi_b(G)$ is at least 3.
v) If G contains an induced K_n, then $\chi_b(G) \geq n$.
vi) For a graph G, $\chi(G) \leq \chi_b(G) \leq t(G)$.
vii) $\chi(G), \chi_b(G) \in S_b(G)$ and from the definition of $S_b(G)$, the minimum
viii) value of $S_b(G)$ is the chromatic number of G and maximum value of
ix) $S_b(G)$ is the b-chromatic number of G.

Observation 2.2. For $m \geq 3$ and $n \geq 1$,

i. T(m, n) has m + n vertices and m + n edges.
ii. T(m, n) has exactly one vertex of degree 3, one vertex of degree 1 and m + n – 2 vertices of degree 2.
iii. $\chi(T(m,n)) = \begin{cases} 2, & if \quad m \quad is \quad even \\ 3, & if \quad m \quad is \quad odd \end{cases}$

Theorem 2.3. [8] For $m \geq 3$ and $n \geq 1$,

i. t(T(m, n)) = 3
ii. $2 \leq \chi_b(T(m, n)) \leq 3$.
iii. Tadpole graph T(m, n) is a b-continuous graph.

3. Main results

In this section we prove that the Cartesian product of Tadpole graph and a path is b-continuous. To prove the theorem we use few notations and terminologies.

Notations and Terminologies 3.1

Throughout this paper, the following notations and terminologies are observed.

i. c is a function which assigns colors to the vertices of a graph in discussion. Hence, if u is any vertex of a graph, then c(u) denotes its color.
ii. In figures, the color dominating vertices are circled.
iii. We refer to a color dominating vertex as ***cdv***. In particular, if u is a color dominating vertex of color i, then it is referred to as ***i-cdv***.
iv. In $T(m, n) \times P_r$, $\{v_1, v_2, \dots, v_m\}$ represents the vertex set $V(C_m)$ and $\{u_1, u_2, \dots, u_n\}$ represents vertex set $V(P_n)$ of $T(m, n)$ and $\{w_1, w_2, \dots, w_r\}$ represents the vertex set $V(P_r)$. Further P_n is joined to C_m at v_1 by the edge u_1v_1.

With the above notations we observe the following.

Observation 3.2.
i. $V(T(m, n) \times P_r) = \{(v_i , w_k) : i = 1 \text{ to } m, k= 1 \text{ to } r\} \cup \{(u_j , w_k) : j = 1 \text{ to } n, k = 1 \text{ to } r\}$
ii. $V(T(m, n)) \times \{w_k\} \cong T(m, n)$ for each k = 1 to r.
iii. $\{v_i\} \times P_r \cong P_r$ for each i = 1 to m and $\{u_j\} \times P_r \cong P_r$ for each j = 1 to n.

Observation 3.3. For $m \geq 3$, $n \geq 1$ and $r \geq 2$
i. $|V(T(m,n) \times P_r| = (m + n) r$
ii. $|E(T(m,n) \times P_r| = (2r - 1)(m + n)$
iii. $\chi(T(m,n) \times P_r) = \begin{cases} 2, & if \quad m \quad is \quad even \\ 3, & if \quad m \quad is \quad odd \end{cases}$

Observation 3.4. In $T(m, n) \times P_r$, for $m \geq 3$, $n \geq 1$ and $r \geq 2$
i. there are exactly 2 vertices of degree 2,
ii. there are exactly $2(m – 1) + 2(n – 1) + (r – 2)$ vertices of degree 3,
iii. there are exactly $2 + (m – 1)(r – 2) + (n – 1)(r – 2)$ vertices of degree 4
iv. there are exactly $(r – 2)$ vertices of degree 5.

Observation 3.5. Form ≥ 3 and $n \geq 1$
i. $t(T(m, n) \times P_r) = 4$, $r = 2$
ii. $t(T(m, n) \times P_r) = 5$, $3 \leq r \leq 7$
iii. $t(T(m, n) \times P_r) = 6$, $r \geq 8$

From observation 2.1(v), 3.3(iii) and 3.5, the b-chromatic number of $\chi_b(T(m, n)\times P_r$ lies between 2 and 6. Also from observation 2.1(ii), to prove $T(m, n) \times P_r$ is b-continuous it is enough to prove that $T(m, n) \times P_r$ has a b-coloring by k colors for each k satisfying $\chi(T(m, n) \times P_r)<k\leq t(T(m, n)\times P_r)$. From 1.2(iii), $T(m, n)\times P_1 \cong T(m, n)$ and from theorem 2.3(iii), $T(m, n)$ is a b-continuous graph. Thus, $T(m, n) \times P_r$ is b-continuous for r = 1 and hence we prove theorems to find $S_b(T(m, n) \times P_r)$ for various values of m, n and $r \geq 2$.

Theorem 3.6. If m is even, $m \geq 4$ and $n \geq 1$, then

$$S_b(T(m, n) \times P_r) = \begin{cases} \{2, 3, 4\}, & if \quad r = 2 \\ \{2, 3, 4, 5\}, & if \quad 3 \leq r \leq 7 \\ \{2, 3, 4, 5, 6\}, & if \quad r \geq 8 \end{cases},$$

$$\chi_b(T(m, n) \times P_r) = \begin{cases} 4, & if \quad r = 2 \\ 5, & if \quad 3 \leq r \leq 7 \\ 6, & if \quad r \geq 8 \end{cases}$$

and $T(m, n) \times P_r$ is a b-continuous graph.

Proof: Since m is even, from observation 3.3(iii), $\chi(T(m, n) \times P_r) = 2$. Hence, $T(m, n) \times P_r)$ has a b-coloring with 2 colors.

Case (i) r = 2

By observations 2.1(v) and 3.5(i),

$$2 \leq \chi_b(T(m, n) \times P_r) \leq 4$$

We prove that $T(m, n) \times P_r$ has a b-coloring by 3-colors and 4-colors. Let $c(v_1, w_k) = k$, k = 1, 2. Assign colors 2, 3 to the each pair of vertices (v_i, w_1) and (v_i, w_2) for even i $(2 \leq i \leq m)$, and to (u_j, w_1) and (u_j, w_2) for even j $(2 \leq j \leq n)$. If we assign colors 3 and 1 to the each pair of vertices (v_i, w_1) and $(v_i\ w_2)$, for odd i $(3 \leq i \leq m\text{-}1)$, and to (u_j, w_1) and (u_j,w_2), for odd j $(1 \leq j \leq n)$. Then (v_1, w_1) is 1-cdv, (v_1, w_2) is a 2-cdv and (v_2, w_2) is a 3-cdv. Then we get a b-coloring by 3-colors.

Next we prove that $T(m, n) \times P_r$ has a b-coloring by 4-colors. Since there are exactly 2 vertices of degree 4, assign any two colors, namely 1, 2 to those vertices. Let $c(v_1, w_k) = k$, k = 1, 2 and $c(v_2, w_k) = k + 2$, k = 1, 2. Let $c(v_m, w_1) = 4$ and $c(v_m, w_2) = 3$. Assign colors 2, 1 to the each pair of vertices (v_i, w_1) and (v_i, w_2) for odd i $(3 \leq i \leq m - 1)$ and to (u_j, w_1) and (u_j, w_2) for odd j $(1 \leq j \leq n)$. If we assign colors 1 and 2 to the each pair of vertices (v_{i+1}, w_1) and $(v_{i+1}\ w_2)$ for odd i $(3 \leq i \leq m - 3)$, and to $(u_j, w_1)$and (u_j, w_2) for even j, $(1 \leq j \leq n)$, then (v_1, w_k) is k-cdv and (v_2, w_k) is a (k+ 2)-cdv, k = 1, 2. Then we get a b-coloring by 4-colors.

From the above results, $T(m, n) \times P_r$ has a b-coloring by 2-colors, 3-colors and 4-colors. Hence $\chi_b(T(m, n) \times P_r) = 4$ and $S_b = \{2, 3, 4\}$.

Case (ii) $3 \leq r \leq 7$

By observations 2.1(v) and 3.5(ii),

$$2 \leq \chi_b(T(m, n) \times P_r) \leq 5$$

We prove that $T(m, n) \times P_r$ has a b-coloring by 3-colors, 4-colors and 5-colors. Since $T(m, n) \times P_2$ is an induced sub graph of $T(m, n) \times P_{r,}$ we apply the same color scheme as given in case (i)to $T(m, n) \times P_r$. In addition, for each odd k, $3 \leq k \leq r$, $c(v_i, w_k) = c(v_i, w_1)$, for all i = 1 to m, and $c(u_j, w_k) = c(u_j, w_1)$, for all j = 1 to n.

Similarly, for each even k, $3 \leq k \leq r$, $c(v_i, w_k) = c(v_i, w_2)$, for all i = 1 to m, and $c(u_j, w_k) = c(u_j, w_1)$, for all j = 1 to n. Then we get a b-coloring by 3-colors and 4-colors.

Next we prove that $T(m, n) \times P_r$ has a b-coloring by 5-colors. For k, $1 \leq k \leq r$, assign colors 5, 1, 3 to the vertices (v_1, w_k), colors 4, 2, 5 to the vertices (v_2, w_k) and colors 2, 4, 5 to the vertices (v_m, w_k), in cyclic order. For each odd i, $3 \le i \le m - 1$, assign colors 5, 3, 1 to the vertices (v_i, w_k), and for each even i, $3 \le i \le m -1$ colors 2, 4, 5 to the vertices (v_i, w_k) in cyclic order.

Similarly, for each odd j, $1 \le j \le n$, assign colors 3, 2, 1 to the vertices (u_j, w_k), and for each even j, $1 \le j \le n$, $c(u_j, w_k) = c(v_1, w_k)$, for all k = 1 to r in cyclic order. Therefore (v_1, w_2), (v_2, w_2), (v_3, w_2), (v_m, w_2) and (v_1, w_1) are 1, 2, 3, 4 and 5 color dominating vertices respectively. Then we get a b-coloring by 5-colors.

From the above results, $T(m, n) \times P_r$ has a b-coloring by 2-colors, 3-colors, 4-colors and 5-colors. Hence $\chi_b(T(m, n) \times P_r) = 5$ and $S_b = \{2, 3, 4, 5\}$.

Case (iii) $r \ge 8$

By observations 2.1(v) and 3.5(iii),

$$2 \le \chi_b(T(m, n) \times P_r) \le 6$$

Let us show that $T(m, n) \times P_r$ has a b-coloring by 3-colors, 4-colors, 5-colors and 6-colors. Since $T(m, n) \times P_r$ $(3 \le r \le 7)$ is an induced sub graph of $T(m, n) \times P_r$ $(r \ge 8)$, we apply the same color scheme as given in case (ii) to $T(m, n) \times P_r$, $r \ge 8$. Then we get a b-coloring by 3-colors, 4-colors and 5-colors

Next we prove that $T(m, n) \times P_r$ has a b-coloring by 6-colors.For each k = 1 to r, assign colors 6, 1, 2, 3, 4, 5 to the vertices (v_1, w_k), colors 3, 4, 5, 6, 1, 2 to the vertices (v_m, w_k) and 4, 5, 6, 1, 2, 3 to the vertices (v_2, w_k), colors 2, 3, 4, 5, 6, 1 to the vertices (u_1, w_k), $1 \le k \le r$ in cyclic order. For each odd i, $2 \le i \le m - 1$, $c(v_i, w_k) = c(v_1, w_k)$, for each even i, $4 \le i \le m - 2$, $c(v_i, w_k) = c(v_2, w_k)$ and for each odd j, $3 \le j \le n$, $c(u_j, w_k) = c(u_1, w_k)$ and for each even j, $2 \le j \le n$, $c(u_j, w_k) = c(v_1, w_k)$, for all k = 1 to r,. Therefore (v_1, w_{k+1}) is k-cdv for k = 1 to 6. Then we get a b-coloring by 6-colors.

From the above results, $T(m, n) \times P_r$ has a b-coloring by 2-colors, 3-colors, 4-colors, 5-colors and 6-colors. Hence $\chi_b(T(m, n) \times P_r) = 6$ and $S_b = \{2, 3, 4, 5, 6\}$.

From case (i), (ii) and (iii), $T(m, n) \times P_r$ is a b-continuous graph for m is even, $m \ge 4$ and n, $r \ge 1$.

Theorem 3.7. For m = 3,

$$S_b(T(m, n) \times P_r) = \begin{cases} \{3, 4\} & , \ if \quad r = 2, \quad n \ge 1 \\ \{3, 4\} & , \ if \quad r = 3, \quad n = 1 \\ \{3, 4, 5\} & , \ if \quad r = 3, \quad n \ge 2 \\ \{3, 4, 5\} & , \ if \quad 4 \le r \le 7, \quad n \ge 1 \\ \{3, 4, 5, 6\} & , \ if \quad r \ge 8, \quad n \ge 1 \end{cases},$$

$$\chi_b(T(m, n) \times P_r) = \begin{cases} 4 & , \ if \quad r = 2, \quad n \ge 1 \\ 4 & , \ if \quad r = 3, \quad n = 1 \\ 5 & , \ if \quad r = 3, \quad n \ge 2 \\ 5 & , \ if \quad 4 \le r \le 7, \quad n \ge 1 \\ 6 & , \ if \quad r \ge 8, \quad n \ge 1 \end{cases}$$

and $T(m, n) \times P_r$ is a b-continuous graph.

Proof: Since m = 3, from observation 3.3(iii), $\chi(T(m, n) \times P_r) = 3$. Hence, $T(m, n) \times P_r$ has a b-coloring with 3 colors.

Case (i) r = 2 and $n \geq 1$

By observations 2.1(v) and 3.5(i),

$$3 \leq \chi_b(T(m, n) \times P_r) \leq 4$$

Since $T(m, n) \times P_2$ contains K_3 as an induced sub graph, assign distinct colors to the vertices of K_3. Let $c(v_1, w_k) = k$, $c(v_2, w_k) = k+2$, k = 1, 2, $c(v_3, w_1) = 2$, $c(v_3, w_2) = 1$, $c(u_1, w_1) = 4$ and $(u_1, w_2) = 3$. Then each (v_1, w_k) is a k-color dominating vertex, and (v_2, w_k) is a (k+2)-color dominating vertex, k = 1, 2. For each even j, $c(u_j, w_k) = c(v_1, w_k)$ and for each odd j, $c(u_j, w_k) = c(u_1, w_k)$, $(2 \leq j \leq n)$, for all k = 1 to r. Then we get a b-coloring by 4 colors. Hence $\chi_b(T(m, n)\times P_r) = 4$ and $S_b = \{3, 4\}$.

Case (ii) r = 3 and n = 1

By observations 2.1(v) and 3.5(ii),

$$3 \leq \chi_b(T(m, n) \times P_r) \leq 5$$

Since $T(m, n) \times P_2$ is an induced sub graph of $T(m, n) \times P_r$, we apply the same color scheme as given in case (i) to $T(m, n) \times P_r$. We can get the color dominating vertices. In addition, let each i, $1 \leq i \leq 3$, $c(v_i, w_3) = c(v_i, w_1)$ and $c(u_1, w_3) = c(u_1, w_1)$. Then we get a b-coloring by 4-colors which is shown in figure 4.

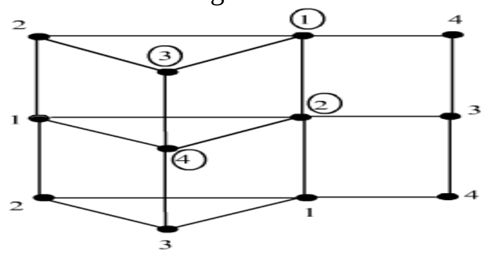


Figure 4:

Next we prove that $T(m, n) \times P_r$ has no b-coloring by 5-colors.

By observation 3.4, there is exactly one vertex of degree 5 and 5 vertices of degree 4. From the five vertices of degree at least 4, we must get five color dominating vertices. Assign distinct colors namely 1, 2, 3, 4, 5 to these vertices. Let $c(v_i, w_2) = i$, $1 \leq i \leq 3$; $c(v_1, w_1) = 4$, and $c(v_1, w_3) = 5$. Then (v_1, w_2) is a 1-cdv. To get 3-cdv, let $c(v_3, w_3) = 4$ and $c(v_3, w_1) = 5$. Then (v_3, w_2) is a 3-cdv. To get 2-cdv, assign colors 4 and 5 to the vertices (v_2, w_3) and $c(v_2, w_1)$. But this is impossible. From the above discussion, we cannot get a b-coloring by 5-colors. Hence $\chi_b(T(m, n) \times P_r) = 4$ and $S_b = \{3, 4\}$.

Case (iii) r = 3 and $n \geq 2$

By observations 2.1(v) and 3.5(ii),

$$3 \leq \chi_b(T(m, n) \times P_r) \leq 5.$$

We prove that $T(m, n) \times P_r$ has a b-coloring by 4-colors and 5-colors. Since $T(m, 1) \times P_r$ is an induced sub graph of $T(m, n) \times P_r$, we apply the same color scheme as in case (ii) to $T(m, n) \times P_r$. In addition, for even j, $c(u_j, w_k) = c(v_1, w_k)$ and for odd j, $c(u_j, w_k) = c(u_1, w_k)$, $(2 \leq j \leq n)$ for all k = 1 to 3. Hence we get a b-coloring by 4 colors.

Next we prove that $T(m, n) \times P_r$ has a b-coloring by 5-colors. Since there are 6 vertices of degree at least 4, we assign colors 1, 2, 3, 4 and 5 to any five of these vertices. Let $c(v_i, w_2) = i$, $i = 1, 2, 3$; $c(u_1, w_2) = 4$ and $c(v_1, w_3) = 5$, then (v_1, w_3) is 1-cdv. Since $c(v_2, w_2) = 2$ and (v_2, w_2) is adjacent to the vertices of colors 1, 3, assign colors 4 and 5 properly to the adjacent vertices (which are not yet colored) of (v_2, w_2). Therefore, $c(v_2, w_3) = 4$, $c(v_2, w_1) = 5$. Then (v_2, w_2) is 2-cdv. To get (v_2, w_3) is 3-cdv, we must assign color 4 and 5 to (v_3, w_3) and (v_3, w_1). Since (v_3, w_3) is adjacent to the vertices of colors 4 and 5, $c(v_3, w_3) \neq 4$ and 5, Therefore (v_3, w_2) cannot be 3-cdv. Hence assign color $c(v_3, w_2)$ to (v_1, w_1). Since $c(v_1, w_1) = 3$ and (v_1, w_1) is adjacent to the vertices of colors 1 and 5, assign colors 2 and 4 properly to the adjacent vertices (which are not yet colored) of (v_1, w_1). Let $c(v_3, w_1) = 4$ and $c(u_1, w_1) = 2$, then (v_1, w_1) is 3-cdv. Since $c(v_1, w_3) = 5$ and (v_1, w_3) is adjacent to the vertices of colors 1 and 4, assign colors 2 and 3 properly to the adjacent vertices (which are not yet colored) of (v_1, w_3). Therefore $c(v_3, w_3) = 2$, $c(u_1, w_3) = 3$. Hence (v_1, w_3) is 5-cdv. By observation 3.4, $T(m, n) \times P_r$ has one more vertex of degree 4, namely (u_1, w_2). Therefore, we use the vertex (u_1, w_2), to get 4-cdv. Since $c(u_1, w_2) = 4$ and (u_1, w_2) is adjacent to the vertices of colors 1, 2 and 3, assign color 5 to (u_2, w_2). Hence (u_1, w_2) is 4-cdv.Let $c(u_2, w_1) = 3$ and $c(u_2, w_3) = 2$. In addition for each odd j, $c(u_j, w_k) = c(u_1, w_k)$ and for each even j, $c(u_j, w_k) = c(u_2, w_k)$, $(3 \leq j \leq n)$ for all k = 1 to 3. Then we get a b-coloring by 5-colors. Hence $\chi_b(T(m, n) \times P_r) = 5$ and $S_b = \{3, 4, 5\}$.

Case (iv) $4 \leq r \leq 7$ and $n \geq 1$

By observations 2.1(v) and 3.5(iii),

$$3 \leq \chi_b(T(m, n) \times P_r) \leq 5$$

We prove that $T(m, n) \times P_r$ has a b-coloring by 4-colors and 5-colors. Assign colors 1, 2, 3, 4 to the vertices (v_1, w_k), colors 2, 3, 4, 1 to the vertices (v_2, w_k), colors 3, 4, 1, 2 to the vertices (v_3, w_k) and colors 4, 3, 2, 1 to the vertices (u_1, w_k) for all k = 1 to r in cyclic order. For each odd j, $c(u_j, w_k) = c(u_1, w_k)$, and for each even j, $c(u_j, w_k) = c(v_1, w_k)$, $(2 \leq j \leq n)$ for all k = 1 to r. Then (v_1, w_k) is k-cdv, k = 1 to 4 and also we get a b-coloring by 4-colors.

Next we prove that $T(m, n) \times P_r$ has a b-coloring by 5-colors. Assign colors 1, 2, 3, 4 to the vertices (v_1, w_k), colors 3, 4, 5, 1 to the vertices (v_2, w_k), colors 5, 1, 2, 3 to (v_3, w_k) and colors 4, 5, 1, 2 to the vertices (u_1, w_k) for all k = 1 to r in cyclic order. For each even j, $c(u_j, w_k) = c(v_1, w_k)$, and for each odd j, $c(u_j, w_k) = c(u_1, w_k)$, $(2 \leq j \leq n)$ for all k = 1 to r. Then (v_1, w_k) is k-cdv, k = 1 to 3, (v_2, w_2) is 4-cdv and (v_2, w_3) is 5-cdv Thus we get a b-coloring by 5-colors. Hence $\chi_b(T(m, n) \times P_r) = 5$ and $S_b = \{3, 4, 5\}$.

Case (v) $r \geq 8$ and $n \geq 1$

By observations 2.1(v) and 3.5(iii),

$$3 \leq \chi_b(T(m, n) \times P_r) \leq 6$$

In this case we prove that $T(m, n) \times P_r$ has a b-coloring by 4-colors, 5-colors and 6-colors. Since $T(m, n) \times P_r$, $(4 \leq r \leq 7)$ is an induced sub graph of $T(m, n) \times P_r$, $(r \geq 8)$, we apply the same color scheme as in case(iv) to $T(m, n) \times P_r$ $(r \geq 8)$. Then we get a b-coloring by 4-colors and 5-colors.

Next we prove that $T(m, n) \times P_r$ has a b-coloring by 6-colors. If we assign colors 6, 1, 2, 3, 4, 5 to the vertices (v_1, w_k), colors 2, 3, 4, 5, 6, 1 to the vertices (v_2, w_k), colors

3, 4, 5, 6, 1, 2 to (v_3, w_k) and colors 4, 5, 6, 1, 2, 3 to the vertices (u_1, w_k) for all k = 1 to r in cyclic order, then (v_1, w_k) is (k – 1)-cdv for k = 2 to 7. For each even j, $c(u_j, w_k) = c(v_1, w_k)$ and for each odd j, $c(u_j, w_k) = c(u_1, w_k)$, $(2 \le j \le n)$ for all k = 1 to r. Then we get a b-coloring by 6-colors. Hence $\chi_b(T(m, n) \times P_r) = 6$ and $S_b = \{3, 4, 5, 6\}$.

From case (i), (ii), (iii), (iv) and (v), $T(m, n) \times P_r$ is a b-continuous graph for m = 3 and n, $r \ge 1$.

Theorem 3.8. If m is odd, $m \ge 5$ and $n \ge 1$, then

$$S_b(T(m, n) \times P_r) = \begin{cases} \{3, \ 4\}, & if \quad r = 2 \\ \{3, \ 4 \ \ 5\}, & if \quad 3 \le r \le 7 \\ \{3, \ 4, \ 5, \ 6\}, & if \quad r \ge 8 \end{cases},$$

$$\chi_b(T(m, n) \times P_r) = \begin{cases} 4, & if \quad r = 2 \\ 5, & if \quad 3 \le r \le 7 \\ 6, & if \quad r \ge 8 \end{cases}$$

and $T(m, n) \times P_r$ is a b-continuous graph.

Proof: Since m is odd, from observation 3.3(iii), $\chi(T(m, n) \times P_r) = 3$. Hence, $T(m, n) \times P_r$ has a b-coloring with 3 colors.

Case (i) r = 2

By observations 2.1(v) and 3.5(i),

$$3 \le \chi_b(T(m, n) x P_r) \le 4$$

Now we prove that $T(m, n) \times P_r$ has a b-coloring by 4-colors. Assign colors 1, 3 to (v_1, w_k), colors 2, 4 to (v_2, w_k), colors 3, 1 to (v_3, w_k) for all k = 1, 2 in order. Let $c(u_1, w_1) = 4$, $c(u_1, w_2) = 2$. Then (v_i, w_1) is i-cdv and (v_i, w_2) is a (i + 2)-cdv, for all i = 1, 2. For each even i, $c(v_i, w_1) = 2$, $c(v_i, w_2) = 4$ and for each odd i, $c(v_i, w_1) = 5$, $c(v_i, w_2) = 1$ $(4 \le i \le m)$. Also, for each even j, $c(u_j, w_k) = c(v_1, w_k)$ and for each odd j, $c(u_j, w_k) = c(u_1, w_k)$, $(2 \le j \le n)$ for k = 1, 2. Then we get a b-coloring by 4 colors. Hence $\chi_b(T(m, n) \times P_r) = 4$ and $S_b = \{3, 4\}$.

Case (ii) $3 \le r \le 7$

By observations 2.1(v) and 3.5(ii),

$$3 \le \chi_b(T(m, n) \times P_r) \le 5$$

Since $T(m, n) \times P_2$ is an induced sub graph of $T(m, n) \times P_r$, , $3 \le r \le 7$, we get four color dominating vertices. In addition, for each odd k, $c(v_i, w_k) = c(v_i, w_1)$ and $c(u_j, w_k) = c(u_j, w_1)$, for each even k, $c(v_i, w_k) = c(v_i, w_2)$ and $c(u_j, w_k) = c(u_j, w_2)$, $(3 \le k \le r)$ for all i = 1 to m and for all j = 1 to n. Then we get a b-coloring by 4-colors.

Let $c(v_1, w_k) = k$, k = 1 to 3. Assign colors 3, 4, 5 to (v_m, w_k), colors 4, 5, 1 to (v_2, w_k). In addition, for each even i, assign colors 5, 1, 4 to (v_i, w_k) and for each odd i, assign colors 2, 3, 5 to (v_i, w_k) $(3 \le i \le m - 1)$ for all k = 1 to 3 in order. Also assign colors 5, 1, 4 to (u_1, w_k) for all k = 1 to 3 in order.

For each even j, $c(u_j, w_k) = c(v_1, w_k)$ and for each odd j, $c(u_j, w_k) = c(u_1, w_k)$, $(2 \le j \le n)$ for all k = 1 to 3. Each $c(v_1, w_k) = k$, k = 1 to 3, is k-cdv. Also (v_m, w_2) is 4-cdv and (v_2, w_2) is 5-cdv. In addition, for each odd k, $c(v_i, w_k) = c(v_i, w_3)$, and for each even k,

$c(v_i, w_k) = c(v_i, w_2)$, $(4 \le k \le r)$ for all i = 1 to m. Then we get a b-coloring by 5-colors. Hence $\chi_b(T(m, n) \times P_r) = 5$ and $S_b = \{3, 4, 5\}$.

Case (iii) $r \ge 8$

By observations 2.1(v) and 3.5(iii),

$$3 \le \chi_b(T(m, n) \times P_r) \le 6$$

We show that $T(m, n) \times P_r$ has a b-coloring by 4-colors, 5-colors and 6-colors. Since $T(m, n) \times P_r$, $3 \le r \le 7$, is an induced sub graph of $T(m, n) \times P_r$, $r \ge 8$, we apply the same color scheme as given in case (ii) to $T(m, n) \times P_r$, $r \ge 8$. Then we get a b-coloring by 4-colors and 5-colors. Next we prove that $T(m, n) \times P_r$ has a b-coloring by 6-colors. Assign colors 6, 1, 2, 3, 4, 5 to the vertices (v_1, w_k), colors 2, 3, 4, 5, 6, 1 to the vertices (u_1, w_k), colors 3, 4, 5, 6, 1, 2 to (v_m, w_k) and colors 4, 5, 6, 1, 2, 3 to the vertices (v_2, w_k) for all k = 1 to r in cyclic order. For each odd i, $c(v_i, w_k) = c(v_m, w_k)$, and for each even i, $c(v_i, w_k) = c(v_2, w_k)$, $(3 \le i \le m - 1)$ for all k = 1 to r. $(v_1\ w_{k+1})$ is k-cdv, k = 1 to 6. Then we get a b-coloring by 6-colors. Hence $\chi_b(T(m, n) \times P_r) = 6$ and $S_b = \{3, 4, 5, 6\}$.

From case (i), (ii) and (iii), $T(m, n) \times P_r$ is a b-continuous graph for m is odd, $m \ge 5$ and n, $r \ge 1$.

4. Conclusion

In this paper, we found the b-chromatic number of $T(m, n) \times P_r$ and proved that it is a b-continuous graph. This paper can be further extended to the Cartesian product of Tadpole graph and cycle.

REFERENCES

1. Dominique Barth, Johanne Cohen and Taoufil Faik, On the b-continuity property of graph, *Discrete Applied Mathematics*, 155 (2007) 1761 – 1768.
2. T. Faik and J. F. Scale, Some b-continuous classes of graphs, *LRI*, 01(2003) Rapport de Recherche N 1350, 1-10
3. F.Harary, *Graph Thoery*, Narosa /Addison Wesley, Indian Student Edition, 1988.
4. R.W. Irving and D.F. Manlove, The b-chromatic number of a graph, *Discrete. Appl. Math.*, 91(1991) 127-141.
5. M. Kouider and M. Maheo, Some bounds for the b-chromatic number of a graph, *Discrete Mathematics*, 256 (2002) 267-277.
6. M. Kouider, M. Maheo, The b-chromatic number of the Cartesian product of two graphs, *Studia Sci. Math. Hungar.*, 44 (2007) 49-55.
7. M. Poobalaranjani and S. Saraswathi, b-Continuity properties of graphs related to square of Tadpole graphs, *International Journal of Engineering Science, Advanced computing and Bio-Technoloy*, 8(2) (2017) 82-91.
8. http://Mathworld.wolfram.com