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Multi-Fuzzy BG-ideals in BG-algebra

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Abstract. Multi-fuzzy set theory is an extension of fuzzy set theory, which deals with the
multi-dimensional fuzziness. In this paper, we apply the concept of multi-fuzzy sets to
ideals in BG-algebra and introduce the notion of multi-fuzzy BG-ideals, the multi-level
subset of BG-ideals. And also we discuss some related properties of multi-fuzzy BG-
ideals based on level subset of it. Also we define the inverse homomorphic images of
multi-fuzzy BG-ideals and present some of its properties.

Keywords: BG-algebra, BG-ideal, Fuzzy BG-ideal, Multi-fuzzy BG-ideal, Multi-level
subset of multi-fuzzy BG-ideal, Homomorphism.

AMS Mathematics Subject Classification (2010): 06F35, 03G25, 08A72, 03E72, 47S40

1. Introduction

The notion of a fuzzy subset was initially introduced by Zadeh [8] in 1965, for
representing uncertainty. In 2000, Sabu and Ramakrishnan [9,10] proposed the theory of
multi-fuzzy sets in terms of multi-dimensional membership functions and investigated
some properties of multi-level fuzziness. Theory of multi-fuzzy set is an extension of
theory of fuzzy sets. Complete characterization of many real life problems can be done
by multi-fuzzy membership functions of the objects involved in the problem.

Imai and Iseki introduced two classes of abstract algebras: BCK algebras and
BCI-algebras [1,2,3]. It is shown that the class of BCK-algebras is a proper subclass of
the class of BCI-algebras. Neggers and Kim [4] introduced a new notion, called a B-
algebra. In 2005, Kim and Kim [6] introduced the notion of a BG-algebra which is a
generalization of B-algebras. With these ideas, fuzzy subalgebras of BG-algebra were
developed by Ahn and Lee [7]. Muthuraj et al. [11] presented fuzzy ideals in BG-algebra
in 2010. Muthuraj and Devi [12] introduced the concept of multi-fuzzy subalgebra of
BG-algebra in 2016. In this paper, we define a new algebraic structure of multi-fuzzy
ideals in BG-algebra and discuss some of their related properties based on level subsets.
Also, we investigate the properties of multi-fuzzy BG-ideals of BG-algebra under
homomorphism.
2. Preliminaries

In this section, the basic definitions of a BG-algebra, BG-ideal, multi-fuzzy sets are recalled. We start with

**Definition 2.1.** A non-empty set $X$ with a constant $0$ and a binary operation $\ast$ is called a BG-algebra if it satisfies the following axioms:

1. $x \ast x = 0$
2. $x \ast 0 = x$
3. $(x \ast y) \ast (0 \ast y) = x \quad \forall \ x, y \in X.$

**Example 2.2.** Let $X = \{0, 1, 2\}$ be a set with the following table:

\[
\begin{array}{ccc}
0 & 1 & 2 \\
0 & 0 & 1 \\
1 & 1 & 0 \\
2 & 2 & 0 \\
\end{array}
\]

Table 1

Then $(X; \ast, 0)$ is a BG-algebra.

**Definition 2.3.** Let $S$ be a non-empty subset of a BG-algebra $X$, then $S$ is called a subalgebra of $X$ if $x \ast y \in S$ for all $x, y \in S$.

**Definition 2.4.** Let $X$ be a BG-algebra and $I$ be a subset of $X$. Then $I$ is called a BG-ideal of $X$ if it satisfies the following conditions:

(i). $0 \in I$
(ii). $x \ast y \in I$ and $y \in I \Rightarrow x \in I$
(iii). $x \in I$ and $y \in X \Rightarrow x \ast y \in I$

**Definition 2.5.** Let $\mu$ be a fuzzy set in a BG-algebra $X$. Then $\mu$ is called a fuzzy subalgebra of $X$ if $\mu(x \ast y) \geq \min\{\mu(x), \mu(y)\}, \forall x, y \in X.$

**Definition 2.6.** Let $\mu$ be a fuzzy set in a BG-algebra $X$. Then $\mu$ is called a fuzzy BG-ideal of $X$ if it satisfies the following inequalities:

(i). $\mu(0) \geq \mu(x)$
(ii). $\mu(x) \geq \min\{\mu(x \ast y), \mu(y)\}$
(iii). $\mu(x \ast y) \geq \min\{\mu(x), \mu(y)\}, \forall x, y \in X.$

**Definition 2.7.** A mapping $f : X \rightarrow Y$ of a BG-algebra is called a homomorphism if $f(x \ast y) = f(x) \ast f(y) \quad \forall x, y \in X.$

**Remark 2.1.** If $f : X \rightarrow Y$ is a homomorphism of BG-algebra then $f(0) = 0.$

**Definition 2.8.** Let $X$ be a non-empty set. A multi-fuzzy set $A$ in $X$ is defined as a set of ordered sequences:

$A = \{(x, \mu_1(x), \mu_2(x), \ldots, \mu_i(x), \ldots) : x \in X\},$ where $\mu_i : X \rightarrow [0,1]$ for all $i$.

**Remark 2.2.**

(i). If the sequences of the membership functions have only k-terms (finite number of
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terms), k is called the dimension of A.

(ii). The set of all multi-fuzzy sets in X of dimension k is denoted by \( M^k FS(X) \).

(iii). The multi-fuzzy membership function \( \mu_A \) is a function from X to \([0,1]^k\) such that for all \( x \) in X, \( \mu_A(x) = (\mu_1(x), \mu_2(x), \ldots, \mu_k(x)) \)

(iv). For the sake of simplicity, we denote the multi-fuzzy set

\[ A = \{(x, \mu_1(x), \mu_2(x), \ldots, \mu_k(x)): x \in X\} \]

as \( A = (\mu_1, \mu_2, \ldots, \mu_k) \).

Definition 2.9. Let k be a positive integer and let A and B in \( M^k FS(X) \), where \( A = (\mu_1, \mu_2, \ldots, \mu_k) \) and \( B = (\nu_1, \nu_2, \ldots, \nu_k) \), then we have the following relations and operations:

i). \( A \subseteq B \) if and only if \( \mu_i \leq \nu_i \), for all \( i = 1, 2, \ldots, k \);

ii). \( A = B \) if and only if \( \mu_i = \nu_i \), for all \( i = 1, 2, \ldots, k \);

iii). \( A \cup B = (\mu_1 \cup \nu_1, \ldots, \mu_k \cup \nu_k) = \{ (x, \max(\mu_1(x), \nu_1(x)), \ldots, \max(\mu_k(x), \nu_k(x))) : x \in X \} \)

iv). \( A \cap B = (\mu_1 \cap \nu_1, \ldots, \mu_k \cap \nu_k) = \{ (x, \min(\mu_1(x), \nu_1(x)), \ldots, \min(\mu_k(x), \nu_k(x))) : x \in X \} \)

Definition 2.10. Let A be a multi-fuzzy set of a BG-algebra X. For any \( t = (t_1, t_2, \ldots, t_k) \) where \( t_i \in [0,1] \) for all \( i \), the set \( U(A; t) = \{ x \in X / A(x) \geq t \} \) is called the multi-level subset of A.

Definition 2.11. Let A be a multi-fuzzy set in a BG-algebra X. Then A is called a multi-fuzzy subalgebra of X if \( A(x \ast y) \geq \min\{ A(x), A(y) \} \) \( \forall x, y \in X \).

3. Multi-fuzzy BG-ideal

In this section, the notion of multi-fuzzy BG-ideal is introduced and some of its properties are discussed.

Definition 3.1. Let A be a multi-fuzzy set in X. Then A is called a multi-fuzzy BG-ideal in X if it satisfies the following conditions:

(i). \( A(0) \geq A(x) \)

(ii). \( A(0) \geq \min\{ A(x \ast y), A(y) \} \)

(iii). \( A(x \ast y) \geq \min\{ A(x), A(y) \} \) \( \forall x, y \in X \).

Example 3.2. Consider a BG-algebra \( X = \{0, 1, 2\} \) with the table 1 in Example 2.2.

Define a multi-fuzzy set \( A : X \rightarrow [0,1] \) by \( A(0) = A(1) = (r_1, r_2) \) and \( A(2) = (s_1, s_2) \) where \( r_1, r_2, s_1, s_2 \in [0,1] \) with \( r_1 < s_1 \) and \( r_2 < s_2 \). Then A is a multi-fuzzy BG-ideal in X.

Theorem 3.3. Let X be a BG-algebra. Then A is a multi-fuzzy BG-ideal of X if and only if A is a multi-fuzzy subalgebra of X.

Proof: Every multi-fuzzy BG-ideal of a BG-algebra X is a multi-fuzzy subalgebra of X. Conversely, let A be a multi-fuzzy subalgebra in X.

Let \( x, y \in X \).

\[ \begin{align*}
  i) \quad A(0) & \geq \min\{ A(x), A(x) \} = A(x) \quad \forall x \in X \\
  ii) \quad A(x) & \geq A(x \ast y \ast (0 \ast y)) \geq \min\{ A(x \ast y), A(0 \ast y) \}
\end{align*} \]
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\[ \geq \min \{ A(x \ast y) \} \]

\[ \geq \min \{ A(x \ast y) , A(y) \} \]

Hence \( A \) is a multi-fuzzy BG-ideal in \( X \).

**Theorem 3.4.** Let \( A_1 \) and \( A_2 \) be two multi-fuzzy BG-ideals of a BG-algebra \( X \). Then \( A_1 \cap A_2 \) is also a multi-fuzzy BG-ideal in \( X \).

**Proof:** Let \( x , y \in A_1 \cap A_2 \)

Then \( x , y \in A_1 \) and \( x , y \in A_2 \)

i) \( A_1 \cap A_2(0) = A_1 \cap A_2(x \ast x) \)

\[ = \min \{ A_1(x \ast x) , A_2(x \ast x) \} \]

\[ \geq \min \{ \min \{ A_1(x) , A_2(x) \} , \min \{ A_2(x) , A_2(x) \} \} \]

\[ = \min \{ A_1(x) , A_2(x) \} \]

\[ = A_1 \cap A_2(x) \]

ii) \( A_1 \cap A_2(x) = \min \{ A_1(x) , A_2(x) \} \)

\[ \geq \min \{ A_1(x \ast y) , A_1(y) \} , \min \{ A_2(x \ast y) , A_2(y) \} \]

\[ = \min \{ A_1(x \ast y) , A_2(x \ast y) \} , \min \{ A_1(y) , A_2(y) \} \]

\[ = \min \{ A_1 \cap A_2(x \ast y) , A_1 \cap A_2(y) \} \]

iii) \( A_1 \cap A_2(x \ast y) = \min \{ A_1(x \ast y) , A_2(x \ast y) \} \)

\[ \geq \min \{ \min \{ A_1(x) , A_2(x) \} , \min \{ A_2(x) , A_2(y) \} \} \]

\[ = \min \{ \min \{ A_1(x) , A_2(x) \} , \min \{ A_1(y) , A_2(y) \} \} \]

\[ = \min \{ A_1 \cap A_2(x) , A_1 \cap A_2(y) \} \]

Hence \( A_1 \cap A_2 \) is a multi-fuzzy BG-ideal in \( X \).

**Theorem 3.5.** Let \( A \) be a multi-fuzzy BG-ideal of a BG-algebra \( X \). If \( x \leq y \) then \( A(x) \geq A(y) \) i.e., order reversing.

**Proof:** Let \( x , y \in X \) such that \( x \leq y \).

Then \( x \ast y = 0 \)

Since \( A \) is a multi-fuzzy BG-ideal in \( X \), \( A(x) \geq \min \{ A(x \ast y) , A(y) \} \)

\[ = \min \{ A(0) , A(y) \} \]

\[ = A(y) \]

Hence it completes the proof.

**Theorem 3.6.** Let \( A \) be a multi-fuzzy BG-ideal of \( X \). If the inequality \( x \ast y \leq z \) holds in \( X \), then \( A(x) \geq \min \{ A(y) , A(z) \} \) for all \( x , y , z \in X \).

**Proof:** Assume the inequality \( x \ast y \leq z \) holds in \( X \).

Then \( (x \ast y) \ast z = 0 \).

\[ A(x) \geq \min \{ A((x \ast y) \ast z) , A(z) \} , A(y) \}

\[ = \min \{ \min \{ A(0) , A(z) \} , A(y) \} \}

\[ = \min \{ A(y) , A(z) \} \}

**Definition 3.7.** Let \( A \) be a multi-fuzzy set in a BG-algebra \( X \). Then \( A \) is called multi-fuzzy closed ideal in \( X \) if it satisfies the following conditions:

i) \( A(x) \geq \min \{ A(x \ast y) , A(y) \} \)
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ii) \( A(0 \ast x) \geq A(x) \)

**Example 3.8.** Consider a BG-algebra \( X = \{ 0, 1, 2, 3 \} \) with the following Cayley table

<table>
<thead>
<tr>
<th>( \ast )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

**Table 2:**

Let \( A : X \rightarrow I \) be a multi-fuzzy set defined by
\( A(0) = A(1) = (0.6, 0.8) \) and \( A(2) = A(3) = (0.3, 0.4) \)

Then \( A \) is a multi-fuzzy closed ideal in \( X \).

**Theorem 3.9.** Every multi-fuzzy closed ideal is a multi-fuzzy ideal in \( X \).

**Proof:** Let \( A \) be a multi-fuzzy closed ideal of \( X \). It is enough to prove that \( A(0) \geq A(x) \)

Now, \( A(0) \geq \min \{ A(0 \ast x) \ast A(x) \} \)

\( \geq \min \{ A(x), A(x) \} \)

\( = A(x) \)

**Remark 3.3.** The converse of the above theorem is not true in general.

**Theorem 3.10.** Every multi-fuzzy closed ideal of a BG-algebra is a multi-fuzzy BG-subalgebra of \( X \).

**Proof:** Let \( A \) be a multi-fuzzy closed ideal of \( X \).

Now, \( A(x \ast y) \geq \min \{ A((x \ast y) \ast (0 \ast y)) \ast A(0 \ast y) \} \)

\( = \min \{ A(x), A(0 \ast y) \} \)

\( \geq \min \{ A(x), A(y) \} \)

Hence the proof.

**Theorem 3.11.** If \( A \) is a multi-fuzzy BG-ideal in \( X \), then the set \( U(A ; t) \) is a BG-ideal in \( X \) for \( t = (t_1, t_2, \ldots, t_k) \) where \( t_i \in [0,1] \), for all \( i \)

**Proof:** Let \( A \) be a multi-fuzzy BG-ideal in \( X \).

i) Since \( A(0) \geq A(x) \geq t \), \( 0 \in U(A ; t) \)

ii) Let \( x \ast y \in U(A ; t) \) and \( y \in U(A ; t) \)

Then \( A(x \ast y) \geq t \) and \( A(y) \geq t \)

Now, \( A(x) \geq \min \{ A(x \ast y) \ast A(y) \} \)

\( \geq \min \{ t, t \} \)

\( = t \)

This implies that \( x \in U(A ; t) \).

iii) Let \( x \in U(A ; t) \) and \( y \in X \)

Choose \( y \) in \( X \) such that \( A(y) \geq t \)

\( A(x \ast y) \geq \min \{ A(x), A(y) \} \)
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\[ \geq \min \{ t, t \} \]

This implies that \( x \ast y \in U(A ; t) \)
Hence \( U(A ; t) \) is a BG-ideal in \( X \).

**Theorem 3.12.** If \( X \) be a BG-algebra and \( U(A ; t) \) for \( t = (t_1, t_2, \ldots, t_k) \) where \( t_i \in [0,1] \), for all \( i \) is a BG-ideal in \( X \), then \( A \) is a multi-fuzzy BG-ideal in \( X \).

**Proof:** Let \( U(A ; t) \) be a BG-ideal in \( X \).
Let \( x, y \in U(A ; t) \)
Then \( A(x) \geq t \) and \( A(y) \geq t \)
i). Let \( A(x) = r \) and \( A(y) = s \) and such that \( r \leq s \) where \( r = (r_1, r_2, \ldots, r_k) \) and \( s = (s_1, s_2, \ldots, s_k) \) for \( r_i \) and \( s_i \in [0,1] \) for all \( i \).

Since \( A(x) = r \), \( x \in U(A ; r) \)
\( x \in U(A ; r) \) and \( y \in X \) implies \( x \ast y \in U(A ; r) \)
That is \( A(x) \geq r \)
\[ = \min \{ r, s \} \]
\[ = \min \{ A(x) , A(y) \} \]

ii). \( A(0) \geq A(x \ast x) \)
\[ = \min \{ A(x) , A(x) \} \]
by (i)
\[ = A(x) \]

iii). \( A(x) = A((x \ast y) \ast (0 \ast y)) \)
\[ \geq \min \{ A(x \ast y) , A(0 \ast y) \} , \]
by (i)
\[ \geq \min \{ A(x \ast y) , \min \{ A(0) , A(y) \} \} \]
\[ = \min \{ A(x \ast y) , A(y) \} \]

Hence \( A \) is a multi-fuzzy BG-ideal in \( X \).

4. Homomorphism of multi-fuzzy BG-ideals
In this section, the properties of multi-fuzzy BG-ideals are discussed under homomorphism.

**Definition 4.1.** Let \( f : X \to Y \) be a mapping of BG-algebra and \( A \) be a multi-fuzzy set of \( Y \) then \( f^{-1}(A) \) is the pre-image of \( A \) under \( f \) if \( f^{-1}(A) = A(f(x)) \) \( \forall x \in X \).

**Theorem 4.2.** Let \( f : X \to Y \) be a homomorphism of BG-algebra. If \( A \) is a multi-fuzzy BG-ideal of \( Y \), then \( f^{-1}(A) \) is a multi-fuzzy BG-ideal of \( X \).

**Proof:** For any \( x \in X \),
i) \[ f^{-1}(A)(x) = A(f(x)) \]
\[ \leq A(0) \]
\[ = A(f(0)) \]
\[ = f^{-1}(A)(0) \]

ii) \[ f^{-1}(A)(x) = A(f(x)) \]
\[ \geq \min \{ A(f(x)) \ast A(f(y)) , A(f(y)) \} \]
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\[ A(f(x \ast y)) = \min \{ A(f(x)) , A(f(y)) \} \]
\[ \geq \min \{ f^{-1}(A)(x) , f^{-1}(A)(y) \} \]

iii) \[ f^{-1}(A)(x \ast y) = A(f(x \ast y)) = A(f(x) \ast f(y)) \]
\[ \geq \min \{ A(f(x) , A(f(y)) \} \]
\[ = \min \{ f^{-1}(A)(x) , f^{-1}(A)(y) \} \]

Hence the proof.

**Theorem 4.3.** Let \( f : X \rightarrow Y \) be an epimorphism of a BG-algebra. If \( f^{-1}(A) \) is a multi-fuzzy ideal in \( X \) then \( A \) is a multi-fuzzy ideal in \( Y \).

**Proof:**

i) Let \( y \in Y \) there exists \( x \in X \) such that \( f(x) = y \)

\[ A(y) = A(f(x)) = f^{-1}(A)(x) \leq f^{-1}(A)(0) = A(f(0)) = A(0) \]

That is \( A(0) \geq A(y) \)

ii) Let \( x, y \in Y \) there exists \( a, b \in X \) such that \( f(a) = x, f(b) = y \)

\[ A(x) = A(f(a)) = f^{-1}(A)(a) \geq \min \{ f^{-1}(A)(a \ast b) , f^{-1}(A)(b) \} \]
\[ = \min \{ A(f(a \ast b)) , A(f(b)) \} \]
\[ = \min \{ A(f(a) \ast f(b)) , A(f(b)) \} \]
\[ = \min \{ A(x \ast y) , A(y) \} \]

iii). \[ A(x \ast y) = A(f(a) \ast f(b)) = A(f(a \ast b)) = f^{-1}(A)(a \ast b) \geq \min \{ f^{-1}(A)(a) , f^{-1}(A)(b) \} \]
\[ = \min \{ A(f(a)) , A(f(b)) \} \]
\[ = \min \{ A(x) , A(y) \} \]

Hence \( A \) is a multi-fuzzy BG-ideal in \( Y \).

5. Conclusion

In this paper, we introduced the concept of multi-fuzzy BG-ideals in BG-algebra and discussed some of its properties based on level sets and also presented some results under homomorphism.

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