Abstract. In this article we introduce a new separation axioms to define Gem-regular space, Gem-normal space, Gem-completely normal space, Gem-perfectly normal space and $G^*-T_i$-spaces for $i = 3, 4, 5$ and $6$ under the idea of “Gem-set” and study some of its basic properties and relations among them.

Keywords: Gem-set, Gem-regular space, Gem-normal space Gem-completely normal space and Gem-perfectly normal space, and $G^*-T_i$-spaces.

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1. Introduction
The concept of ideals in topological spaces are treated in the standard text by Kuratowski [8] and Vaidyanathaswamy [16]. In ‘general topology’ Hamlett and Jankovic [2, 3, 4, 17, 18] introduced the application of topological ideal as defined below : An ideal $J$ on a topological space $(X, \tau)$ is a non empty collection of subsets of $X$ having the following properties : (i) $A \in J$ and $B \subseteq A$ implies $B \in J$. (ii) $A \in J$ and $B \in J$ implies $A \cup B \in J$. An ideal topological space is a topological space $(X, \tau)$ with an ideal $J$ on $X$ and is denoted by $(X, \tau, J)$. In addition K. Kuratowski[8] defined the local function for $A \subseteq X$ with respect to $J$ and $\tau$ as below : $A^*(J, \tau)$ or $A^*(J) = \{ x \in X : A \cap U \notin J \ \text{for any} \ U \in \tau(x) \}$ where $\tau(x) = \{ U \in \tau : x \in U \}$. We simply write $A^*$ instead of $A^*(J)$. Arenas, Dontchev and Puertas [5] introduced some weak separation axioms under the concept of ideal. Swidi and Sada[10] introduced a new type of ideal for a single point $x$ denoted as $J_x$ and is defined as below : $J_x = \{ U \subseteq X : x \in U^c \}$, where $U$ is a non-empty subset of $X$. Swidi and Nafee [9] introduced a new set in topological space namely “Gem-set” depending on the $J_x$ and defined a new separation axioms by using the idea of the “Gem-set” namely $I^*-T_i$-spaces and $I^{***-T_i}$-spaces for $i = 0, 1$ and $2$. They also defined two mappings namely “$I^*$-map” and “$I^{**}$-map” to carry properties of the “Gem-set” from one space to another space and give more properties for new separation axioms. Swidi and Ethary [12] introduced a new class of maps namely “A-map”, “AO-map” and “Am-map” under the idea of the Gem-set and studied some of its basic properties and relations as well as the properties of the separation axioms of $I^*-T_i$-spaces and $I^{***-T_i}$-spaces for $i = 0, 1$ and $2$ with the functions and their effect upon them are also established.
Aim of this article is to introduce the separation axioms to define Gem-regular space (G-T_3), Gem-normal space(G-T_4), Gem-completely normal space(G-T_5), Gem-perfectly normal space(G-T_6) and G^*T_i-spaces for i = 3, 4, 5 and 6 and study some of its basic properties. Also we study the relations as well as the properties of G-T_i-spaces and G^*T_i-spaces for i = 3, 4, 5 and 6 in connection with the functions “I^*’-map”, “I^**-map” “A-map” and “AO-map” and the effect upon them.

Throughout this paper, spaces means topological spaces on which no separation axioms are assumed unless otherwise mentioned.

2. Preliminaries

Definition 2.1. Let (X, τ) be a topological space, for A ⊆ X and x ∈ X we define A^x with respect to (X, τ) as follows:
A^x = { y ∈ X : G ∩ A /∈ τ, for every G ∈ τ(y) }, where τ(y) = { G ∈ τ : y ∈ G }. The set A^x is called “Gem-set”.

Definition 2.2. Consider the mapping f : (X, τ) → (Y, σ), then f is called
• I^*’-map if and only if, for every subset A of X, x ∈ X, f(A^x) = (f(A))^f(x).
• I^**-map if and only if, for every subset A of Y, y ∈ Y, f^{-1}(A^{*y}) = (f^{-1}(A))^{f^{-1}(y)}.

Definition 2.3. Consider the mapping f : (X, τ) → (Y, σ), then f is an
• A-map at x ∈ X, if and only if ∀ B ⊆ Y, ∃ A ⊆ X : f(A^x) ⊆ B^{f(x)}.
• A-map on X if and only if it is an A-map at each point on X.
• AO-map if and only if ∀ A ⊆ X, ∃ B ⊆ Y : B^{*y} ⊆ f(A^{f^{-1}(y)}).

3. Gem-separation axioms

In this section we define Gem-regular space, Gem-normal space, Gem-completely normal space, Gem-perfectly normal space and G^*T_i-spaces for i = 3, 4, 5 and 6 and derive some of its basic properties.

Definition 3.1. A topological space (X, τ) is a
• Gem-regular space or G-T_3-space if and only if for each disjoint pair consisting a point x and a set C in X, there exists subsets A, B of X such that x /∈ B^y and C /∈ A^x.
• Gem-normal space or G-T_4-space if and only if for each pair C and D of disjoint sets in X, there exists subsets A, B of X such that C /∈ B^y and D /∈ A^x.
• Gem-completely normal space or G-T_5-space if and only if for each pair of separated sets C and D in X, there exists subsets A, B of X such that C /∈ B^y and D /∈ A^x.
• Gem-perfectly normal space or G-T_6-space if and only if for each pair of C and D of disjoint sets in X, there exists continuous map f : X → [0, 1] such that C^x ≠ f^{-1}([1]) and D^{*y} ≠ f^{-1}([0]).
• G^*T_3-space if and only if for each disjoint pair consisting a point x and a set C in X, there exists subset A of X such that x /∈ A^y and C /∈ A^x.
• G^*T_4-space if and only if for each pair C and D of disjoint sets in X, there exists subset A of X such that C /∈ A^y and D /∈ A^x.
Gem-Separation Axioms in Topological Space

- $G^*-T_2$-space if and only if for each pair of separated sets $C$ and $D$ in $X$, there exists subset $A$ of $X$ such that $C \not\subseteq A^x$ and $D \not\subseteq A^x$.
- $G^*-T_6$-space if and only if for each pair $C$ and $D$ of disjoint sets in $X$, there exists a continuous map $f : X \to [0, 1]$ such that $C^{x^*} = f^{-1}((1])$ and $D^{y^*} = f^{-1}((1])$ or $C^{x^*} = f^{-1}((0))$ and $D^{y^*} \neq f^{-1}((0))$

Theorem 3.2. For a topological space $(X, \tau)$ the following properties hold good:

1. Every $T_3$-space is a $G-T_3$-space.
2. Every $T_4$-space is a $G-T_4$-space.
3. Every $T_5$-space is a $G-T_5$-space.
4. Every $T_6$-space is a $G-T_6$-space.
5. Every $T_3^*$-space is a $G^*-T_3$-space.
6. Every $T_4^*$-space is a $G^*-T_4$-space.
7. Every $T_5^*$-space is a $G^*-T_5$-space.
8. Every $T_6^*$-space is a $G^*-T_6$-space.

Proof: 1. Let $x \in X$ and $C$ be a closed set in $X$ with $x \notin C$. Since $(X, \tau)$ is a $T_3$-space. Then there exists disjoint open sets $U, V$ such that $x \in U$ and $C \subseteq V$. Then $U^{x^*} \cap V^{y^*} = \phi$. Let $A = U, B = V$. It follows that there exists subsets $A, B$ of $X$ such that $x \notin B^{y^*}$ and $C \not\subseteq A^x$. Hence $(X, \tau)$ is a $G-T_3$-space.

2. Let $C$ and $D$ be the disjoint closed sets in $X$ and $(X, \tau)$ is a $T_4$-space. Then there exists disjoint open sets $U, V$ such that $C \subseteq U$ and $D \subseteq V$. Then $U^{x^*} \cap V^{y^*} = \phi$. Let $A = U, B = V$. It follows that there exists subsets $A, B$ of $X$ such that $C \subseteq B^{y^*}$ and $D \not\subseteq A^x$. Hence $(X, \tau)$ is a $G-T_4$-space.

3. Let $C$ and $D$ be the separated sets in $X$ (i.e. $C \cap D = C \cap \bar{D} = \phi$) and $(X, \tau)$ is a $T_5$-space. Then there exists disjoint open sets $U, V$ such that $C \subseteq U$ and $D \subseteq V$. Then $U^{x^*} \cap V^{y^*} = \emptyset$. Let $A = U, B = V$. It follows that there exists subsets $A, B$ of $X$ such that $C \subseteq B^{y^*}$ and $D \not\subseteq A^x$. Hence $(X, \tau)$ is a $G-T_5$-space.

4. Let $C$ and $D$ be the disjoint closed sets in $X$ and $(X, \tau)$ is a $T_6$-space. Then there exists a continuous map $f : X \to [0, 1]$ such that $C = f^{-1}((0))$ and $D = f^{-1}((1])$. Then $C^{x^*} \cap D^{y^*} = \phi$. It follows that there exists a continuous map $f : X \to [0, 1]$ such that $C^{x^*} = f^{-1}((1])$ and $D^{y^*} \neq f^{-1}((0))$. Hence $(X, \tau)$ is a $G-T_6$-space.

5. Let $x \in X$ and $C$ be a closed set in $X$ with $x \notin C$. Since $(X, \tau)$ is a $T_3$-space. Then there exists disjoint open sets $U, V$ such that $x \in U$ and $C \subseteq V$. Then $U^{x^*} \cap V^{y^*} = \phi$. Let $U = V = A$. It follows that there exists a subset $A$ of $X$ such that $x \notin A^{y^*}$ and $C \not\subseteq A^x$. Hence $(X, \tau)$ is a $G^*-T_3$-space.

6. Let $C$ and $D$ be the disjoint closed sets in $X$ and $(X, \tau)$ is a $T_4$-space. Then there exists disjoint open sets $U, V$ such that $C \subseteq U$ and $D \subseteq V$. Then $U^{x^*} \cap V^{y^*} = \phi$. Let $U = V = A$. It follows that there exists subsets $A, B$ of $X$ such that $C \not\subseteq A^{y^*}$ and $D \not\subseteq A^x$. Hence $(X, \tau)$ is a $G^*-T_4$-space.

7. Let $C$ and $D$ be the disjoint sets in $X$ and $(X, \tau)$ is a $T_5$-space. Then there exists disjoint open sets $U, V$ such that $C \subseteq U$ and $D \subseteq V$. Then $U^{x^*} \cap V^{y^*} = \phi$. Let $U = A = V$. It follows that there exists subset $A$ of $X$ such that $C \not\subseteq A^x$. Hence $(X, \tau)$ is a $G^*-T_5$-space.

8. Let $C$ and $D$ be the disjoint closed sets in $X$ and $(X, \tau)$ is a $T_6$-space. Then there exists a continuous map $f : X \to [0, 1]$ such that $C = f^{-1}((0))$ and $D = f^{-1}((1])$. Then
Let \( A \) and \( B \) of \( Y \) such that \( \text{disjoint pairs} \) of \( Y \) is a \( G^* \)-space.

**Remark:** The converse of the above theorem need not be true.

### 3.1. \( G-T_3 \)-space

In this section we proved some theorems in connection with \( I^* \)-map, \( I^{**} \)-map, A-map and AO-map for \( G-T_3 \)-space.

**Theorem 3.1.1.** If \( f : (X, \tau) \to (Y, \sigma) \) is one-one \( I^* \)-map of a \( G-T_3 \)-space \( X \) onto a space \( Y \), then \( Y \) is a \( G-T_3 \)-space.

**Proof:** Let \( x_1 \) and \( C_1 \) of \( X \) such that \( f(x_1) = y_1 \) and \( f(C_1) = C_2 \). Since \( (X, \tau) \) is \( G-T_3 \)-space, there exists subsets \( A, B \) of \( X \) such that \( x_1 \notin B^{x_2} \) and \( C_1 \notin A^{x_1} \), so that \( f(x_1) \notin f(B^{x_2}) = f(B)^{f(x_2)} \) and \( f(C_1) \notin f(A^{x_1}) = (f(A))^{f(x_1)} \). Thus \( y_1 \notin (f(B))^{f(x_2)=y_2} \). Thus \( Y \) is a \( G-T_3 \)-space.

**Theorem 3.1.2.** If \( f : (X, \tau) \to (Y, \sigma) \) is one-one \( I^{**} \)-map of a space \( X \) onto \( G-T_3 \)-space \( Y \), then \( X \) is a \( G-T_3 \)-space.

**Proof:** Let \( x_1 \) and \( C_1 \) be a disjoint pairs of \( X \). Since \( f \) is one-one and onto, there exists disjoint pairs \( y_1 \) and \( C_2 \) of \( Y \) such that \( f(x_1) = y_1 \) and \( f(C_1) = C_2 \). Since \( (Y, \sigma) \) is \( G-T_3 \)-space, there exists subsets \( A, B \) of \( Y \) such that \( y_1 \notin B^{y_2} \) and \( C_2 \notin A^{y_1} \), so that \( f^{-1}(y_1) \notin f^{-1}(B^{y_2}) = (f^{-1}(B))^f((y_2)) \) and \( f^{-1}(C_2) \notin f^{-1}(A^{y_1}) = (f^{-1}(A))^{f^{-1}(y_1)} \). This implies \( x_1 \notin (f^{-1}(B))^{x_2} \) and \( C_1 \notin (f^{-1}(A))^{x_1} \). Thus \( X \) is a \( G-T_3 \)-space.

**Theorem 3.1.3.** If \( f : (X, \tau) \to (Y, \sigma) \) is one-one \( A \)-map of a \( G-T_3 \)-space \( X \) onto a space \( Y \), then \( Y \) is a \( G-T_3 \)-space.

**Proof:** Let \( x_1 \) and \( C_1 \) be a disjoint pair of \( X \). Since \( f \) is one-one and onto, there exists a disjoint pair \( y_1 \) and \( C_2 \) of \( X \) such that \( f(x_1) = y_1 \) and \( f(C_1) = C_2 \). Since \( (X, \tau) \) is \( G-T_3 \)-space, there exists subsets \( A_1, A_2 \) of \( X \) such that \( x_1 \notin A_2^{x_2} \) and \( C_1 \notin A_1^{x_1} \), so that \( f(x_1) \notin f(A_2^{x_2}) \subseteq B_2^{f(x_2)} \) and \( f(C_1) \notin f(A_1^{x_1}) \subseteq B_1^{f(x_1)} \). This implies \( y_1 \notin B_2^{y_2} \) and \( C_2 \subseteq B_1^{y_1} \). Thus \( Y \) is a \( G-T_3 \)-space.

**Theorem 3.1.4.** If \( f : (X, \tau) \to (Y, \sigma) \) is one-one \( AO \)-map of a space \( X \) onto \( G-T_3 \)-space \( Y \), then \( X \) is a \( G-T_3 \)-space.

**Proof:** Let \( x_1 \) and \( C_1 \) be a disjoint pair of \( X \). Since \( f \) is one-one and onto, there exists a disjoint pair \( y_1 \) and \( C_2 \) of \( Y \) such that \( f(x_1) = y_1 \) and \( f(C_1) = C_2 \). Since \( (Y, \sigma) \) is \( G-T_3 \)-space, there exists subsets \( B_1, B_2 \) of \( Y \) such that \( y_1 \notin B_2^{y_2} \subseteq f(A_2^{f^{-1}(y_2)}) \) and \( C_2 \subseteq B_2^{y_1} \subseteq f(A_1^{f^{-1}(y_1)}) \), so that \( f^{-1}(y_1) \notin f^{-1}(f(A_1^{f^{-1}(y_1)})) \) and \( f^{-1}(C_2) \notin f^{-1}(f(A_2^{f^{-1}(y_2)})) \). This implies \( x_1 \notin A_2^{x_2} \) and \( C_1 \subseteq A_1^{x_1} \). Thus \( X \) is a \( G-T_3 \)-space.
Gem-Separation Axioms in Topological Space

3.2. $G_{T_4}$-space

In this section we proved some theorems in connection with $I^*$-map, $I^{**}$-map, A-map and AO-map for $G_{T_4}$-space.

**Theorem 3.2.1.** If $f : (X, \tau) \rightarrow (Y, \sigma)$ is one-one $I^*$-map of a $G_{T_4}$-space $X$ onto a space $Y$, then $Y$ is a $G_{T_4}$-space.

**Proof:** Let $C_2$ and $D_2$ be two disjoint sets in $Y$. Since $f$ is one-one and onto, there exists disjoint sets $C_1$ and $D_1$ of $X$ such that $f(C_1) = C_2$ and $f(D_1) = D_2$. Since $(X, \tau)$ is $G_{T_4}$-space, there exists subsets $A$ and $B$ of $X$ such that $f^{-1}(C_2) \not\subseteq B^{x_2}$ and $f^{-1}(D_2) \not\subseteq A^{x_1}$, so that $f^{-1}(f(C_2)) = f^{-1}(f(D_2))$. This implies $C_1 \not\subseteq (f^{-1}(B))^{x_2}$ and $D_1 \not\subseteq (f^{-1}(A))^{x_1}$. Thus $Y$ is a $G_{T_4}$-space.

**Theorem 3.2.2.** If $f : (X, \tau) \rightarrow (Y, \sigma)$ is one-one $I^{**}$-map of a space $X$ onto $G_{T_4}$-space $Y$, then $X$ is a $G_{T_4}$-space.

**Proof:** Let $C_1$ and $D_1$ be two disjoint sets in $X$. Since $f$ is one-one and onto, there exists disjoint sets $C_2$ and $D_2$ of $Y$ such that $f(C_1) = C_2$ and $f(D_1) = D_2$. Since $(X, \sigma)$ is $G_{T_4}$-space, there exists subsets $A$, $B$ of $Y$ such that $C_2 \not\subseteq B^{y_2}$ and $D_2 \not\subseteq A^{y_1}$, so that $f^{-1}(f^{-1}(B^{y_2})) = f^{-1}(f^{-1}(A^{y_1}))$. This implies $C_1 \not\subseteq (f^{-1}(B))^{x_2}$ and $D_1 \not\subseteq (f^{-1}(A))^{x_1}$. Thus $X$ is a $G_{T_4}$-space.

**Theorem 3.2.3.** If $f : (X, \tau) \rightarrow (Y, \sigma)$ is one-one A-map of a $I^*$-space $X$ onto a space $Y$, then $Y$ is a $G_{T_4}$-space.

**Proof:** Let $C_2$ and $D_2$ be two disjoint sets in $Y$. Since $f$ is one-one and onto, there exists a disjoint sets $C_1$ and $D_1$ of $X$ such that $f(C_1) = C_2$ and $f(D_1) = D_2$. Since $(X, \tau)$ is $G_{T_4}$-space, there exists subsets $A_1$, $A_2$ of $X$ such that $C_1 \not\subseteq A_2^{x_2}$ and $D_1 \not\subseteq A_1^{x_1}$, so that $f^{-1}(f^{-1}(B^{y_2})) = f^{-1}(f^{-1}(A^{y_1}))$. This implies $C_2 \not\subseteq B_2^{y_2}$ and $D_2 \not\subseteq B_1^{y_1}$. Thus $Y$ is a $G_{T_4}$-space.

**Theorem 3.2.4.** If $f : (X, \tau) \rightarrow (Y, \sigma)$ is one-one AO-map of a space $X$ onto $G_{T_4}$-space $Y$, then $X$ is a $G_{T_4}$-space.

**Proof:** Let $C_1$ and $D_1$ be two disjoint sets in $X$. Since $f$ is one-one and onto, there exists a disjoint sets $C_2$ and $D_2$ of $Y$ such that $f(C_1) = C_2$ and $f(D_1) = D_2$. Since $(X, \sigma)$ is $G_{T_4}$-space, there exists subsets $B_1$, $B_2$ of $Y$ such that $C_2 \not\subseteq B_2^{y_2}$ and $D_2 \not\subseteq B_1^{y_1}$, so that $f^{-1}(f^{-1}(B^{y_2})) = f^{-1}(f^{-1}(A^{y_1}))$. This implies $C_1 \not\subseteq A_2^{x_2}$ and $D_1 \not\subseteq A_1^{x_1}$. Thus $X$ is a $G_{T_4}$-space.

3.3. $G_{T_5}$-space

In this section we proved some theorems in connection with $I^*$-map, $I^{**}$-map, A-map and AO-map for $G_{T_5}$-space.
Theorem 3.3.1. If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is one-one \( I^* \)-map of a \( G-T_5 \)-space \( X \) onto a space \( Y \), then \( Y \) is a \( G-T_5 \)-space.

**Proof:** Let \( C_2 \) and \( D_2 \) be separated sets in \( Y \). Since \( f \) is one-one and onto, there exists separated sets \( C_1 \) and \( D_1 \) of \( X \) such that \( f(C_1) = C_2 \) and \( f(D_1) = D_2 \). Since \( (X, \tau) \) is \( G-T_5 \)-space, there exists subsets \( A \) and \( B \) of \( X \) such that \( C_1 \notin B^{x_2} \) and \( D_1 \notin A^{x_1} \), so that \( f(C_1) \notin f(B^{x_2}) = f(B)^{f(x_2)} \) and \( f(D_1) \notin f(A^{x_1}) = f(A)^{f(x_1)} \). Thus \( C_2 \notin (f(B))^{f(x_2) = y_2} \) and \( D_2 \notin (f(A))^{f(x_1) = y_1} \). Thus \( Y \) is a \( G-T_5 \)-space.

Theorem 3.3.2. If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is one-one \( I^{**} \)-map of a space \( X \) onto \( G-T_5 \)-space \( Y \), then \( X \) is a \( G-T_5 \)-space.

**Proof:** Let \( C_2 \) and \( D_2 \) be separated sets in \( Y \). Since \( f \) is one-one and onto, there exists separated sets \( C_1 \) and \( D_1 \) of \( X \) such that \( f(C_1) = C_2 \) and \( f(D_1) = D_2 \). Since \( (X, \tau) \) is \( G-T_5 \)-space, there exists subsets \( A \), \( B \) of \( Y \) such that \( C_2 \notin B^{y_2} \) and \( D_2 \notin A^{y_1} \), so that \( f^{-1}(C_2) \notin f^{-1}(B^{y_2}) = (f^{-1}(B))^{f^{-1}(y_2)} \) and \( f^{-1}(D_2) \notin f^{-1}(A^{y_1}) = (f^{-1}(A))^{f^{-1}(y_1)} \). This implies \( C_1 \notin (f^{-1}(B))^{x_2} \) and \( D_1 \notin (f^{-1}(A))^{x_1} \). Thus \( X \) is a \( G-T_5 \)-space.

Theorem 3.3.3. If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is one-one \( A \)-map of an \( G-T_5 \)-space \( X \) onto a space \( Y \), then \( Y \) is a \( G-T_5 \)-space.

**Proof:** Let \( C_2 \) and \( D_2 \) be separated sets in \( Y \). Since \( f \) is one-one and onto, there exists separated sets \( C_1 \) and \( D_1 \) of \( X \) such that \( f(C_1) = C_2 \) and \( f(D_1) = D_2 \). Since \( (X, \tau) \) is \( G-T_5 \)-space, there exists subsets \( A_1 \), \( A_2 \) of \( X \) such that \( C_1 \notin A_2^{x_2} \) and \( D_1 \notin A_1^{x_1} \), so that \( f(C_1) \notin f(A_2^{x_2}) \subseteq B_2^{f(x_2)} \) and \( f(D_1) \notin f(A_1^{x_1}) \subseteq B_1^{f(x_1)} \). This implies \( C_2 \notin B_2^{y_2} \) and \( D_2 \notin B_1^{y_1} \). Thus \( Y \) is a \( G-T_5 \)-space.

Theorem 3.3.4. If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is one-one \( AO \)-map of a space \( X \) onto \( G-T_5 \)-space \( Y \), then \( X \) is a \( G-T_5 \)-space.

**Proof:** Let \( C_1 \) and \( D_1 \) be separated sets in \( X \). Since \( f \) is one-one and onto, there exists separated sets \( C_2 \) and \( D_2 \) of \( Y \) such that \( f(C_1) = C_2 \) and \( f(D_1) = D_2 \). Since \( (X, \tau) \) is \( G-T_5 \)-space, there exists subsets \( B_1 \), \( B_2 \) of \( Y \) such that \( C_2 \notin B_2^{y_2} \subseteq f(A_2^{f^{-1}(y_2)}) \) and \( D_2 \notin B_1^{y_1} \subseteq f(A_1^{f^{-1}(y_1)}) \), so that \( f^{-1}(C_2) \notin f^{-1}(B_2^{y_2}) \subseteq f^{-1}(A_2^{f^{-1}(y_2)}) \) and \( f^{-1}(D_2) \notin f^{-1}(B_1^{y_1}) \subseteq f^{-1}(A_1^{f^{-1}(y_1)}) \). This implies \( C_1 \notin A_2^{x_2} \) and \( D_1 \notin A_1^{x_1} \). Thus \( X \) is a \( G-T_5 \)-space.

### 3.4. \( G-T_6 \)-space

In this section we proved some theorems in connection with \( I^* \)-map, \( I^{**} \)-map, \( A \)-map and \( AO \)-map for \( G-T_6 \)-space.

Theorem 3.4.1. If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is one-one \( I^* \)-map of a space \( X \) onto \( G-T_6 \)-space \( Y \), then \( X \) is a \( G-T_6 \)-space.

**Proof:** Let \( C_1 \) and \( D_1 \) be two disjoint sets in \( X \). Since \( f \) is one-one and onto, there exists disjoint sets \( C_2 \) and \( D_2 \) of \( Y \) such that \( f(C_1) = C_2 \) and \( f(D_1) = D_2 \). Since \( f \) is an \( I^* \)-map, so that \( f(C_1) = f(C_2) = f(D_1) = f(D_2) \), \( C_2 \) and \( D_2 \) are \( G-T_6 \)-space. Since \( (Y, \sigma) \) is \( G-T_6 \)-space, there exists a continuous map \( g : Y \rightarrow [0, 1] \) such that \( g \notin g^{-1}([1]) \) and...
Gem-Separation Axioms in Topological Space

**Theorem 3.4.3.** If \( f : (X, \{1\}) \) and \( g(f(\{0\})) \). Thus

**Theorem 3.4.2.** Proof: Let

**Theorem 3.4.4.** If \( f : (X, \{0\}) \). Thus

In this section we proved some theorems in connection with

**3.5. \( G^*T_3 \)-space**

In this section we proved some theorems in connection with \( I^* \)-map, \( I^{**} \)-map, A-map and AO-map for \( G^*T_3 \)-space.
Theorem 3.5.1. If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is one-one \( I^* \)-map of a \( G^*-T_3 \)-space \( X \) onto a space \( Y \), then \( Y \) is a \( G^*-T_3 \)-space.

**Proof:** Let \( y_1 \) and \( C_2 \) be a disjoint pair of \( Y \). Since \( f \) is one-one and onto, there exists disjoint pair \( x_1 \) and \( C_1 \) of \( X \) such that \( f(x_1) = y_1 \) and \( f(C_1) = C_2 \). Since \( (X, \tau) \) is \( G^*-T_3 \)-space, there exists subset \( A \) of \( X \) such that \( x_1 \notin A^{x_2} \) and \( C_1 \notin A^{x_1} \), so that \( f(x_1) \notin f(A^{x_2}) = (f(A))^{f(x_2)} \) and \( f(C_1) \notin f(A^{x_1}) = (f(A))^{f(x_1)} \). Thus \( y_1 \notin (f(A))^{f(x_2)} = y_2 \) and \( C_2 \notin (f(A))^{f(x_1)} = y_1 \). Thus \( Y \) is a \( G^*-T_3 \)-space.

Theorem 3.5.2. If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is one-one \( I^{**} \)-map of a space \( X \) onto \( G^*-T_3 \)-space \( Y \), then \( X \) is a \( G^*-T_3 \)-space.

**Proof:** Let \( x_1 \) and \( C_2 \) be a disjoint pair of \( X \). Since \( f \) is one-one and onto, there exists disjoint pairs \( y_1 \) and \( C_2 \) of \( Y \) such that \( f(x_1) = y_1 \) and \( f(C_2) = C_2 \). Since \( (Y, \sigma) \) is \( G^*-T_3 \)-space, there exists subset \( A \) of \( Y \) such that \( y_1 \notin A^{y_2} \) and \( C_2 \notin A^{y_1} \), so that \( f^{-1}(y_1) \notin f^{-1}(A^{y_2}) = (f^{-1}(A))^{f^{-1}(y_2)} \) and \( f^{-1}(C_2) \notin f^{-1}(A^{y_1}) = (f^{-1}(A))^{f^{-1}(y_1)} \). This implies \( x_1 \notin (f^{-1}(A))^{x_2} \) and \( C_2 \notin (f^{-1}(A))^{x_1} \). Thus \( X \) is a \( G^*-T_3 \)-space.

Theorem 3.5.3. If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is one-one \( A \)-map of an \( G^*-T_3 \)-space \( X \) onto a space \( Y \), then \( Y \) is a \( G^*-T_3 \)-space.

**Proof:** Let \( y_1 \) and \( C_2 \) be a disjoint pair of \( Y \). Since \( f \) is one-one and onto, there exists a disjoint pair \( x_1 \) and \( C_1 \) of \( X \) such that \( f(x_1) = y_1 \) and \( f(C_1) = C_2 \). Since \( (X, \tau) \) is \( G^*-T_3 \)-space, there exists subsets \( A \) of \( X \) such that \( x_1 \notin A^{x_2} \) and \( C_1 \notin A^{x_1} \), so that \( f(x_1) \notin f(A^{x_2}) \subseteq B^{f(x_2)} \) and \( f(C_1) \notin f(A^{x_1}) \subseteq B^{f(x_1)} \). This implies \( y_1 \notin B^{y_2} \) and \( C_2 \notin B^{y_1} \). Thus \( Y \) is a \( G^*-T_3 \)-space.

Theorem 3.5.4. If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is one-one \( AO \)-map of a space \( X \) onto \( G^*-T_3 \)-space \( Y \), then \( X \) is a \( G^*-T_3 \)-space.

**Proof:** Let \( x_1 \) and \( C_1 \) be a disjoint pair of \( X \). Since \( f \) is one-one and onto, there exists a disjoint pair \( y_1 \) and \( C_2 \) of \( Y \) such that \( f(x_1) = y_1 \) and \( f(C_1) = C_2 \). Since \( (Y, \sigma) \) is \( G^*-T_3 \)-space, there exists subset \( B \) of \( Y \) such that \( y_1 \notin B^{y_2} \subseteq f(A^{f^{-1}(y_2)}) \) and \( C_2 \notin B^{y_1} \subseteq f(A^{f^{-1}(y_1)}) \), so that \( f^{-1}(y_1) \notin f^{-1}(f(A^{f^{-1}(y_2)})) \) and \( f^{-1}(C_2) \notin f^{-1}(f(A^{f^{-1}(y_1)})) \). This implies \( x_1 \notin A^{x_2} \) and \( C_1 \notin A^{x_1} \). Thus \( X \) is a \( G^*-T_3 \)-space.

3.6. \( G^*-T_4 \)-space

In this section we proved some theorems in connection with \( I^* \)-map, \( I^{**} \)-map, \( A \)-map and \( AO \)-map for \( G^*-T_4 \)-space.

Theorem 3.6.1. If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is one-one \( I^* \)-map of a \( G^*-T_4 \)-space \( X \) onto a space \( Y \), then \( Y \) is a \( G^*-T_4 \)-space.

**Proof:** Let \( C_2 \) and \( D_2 \) be two disjoint sets in \( Y \). Since \( f \) is one-one and onto, there exists disjoint sets \( c_2 \) and \( D_1 \) of \( X \) such that \( f(C_2) = c_2 \) and \( f(D_2) = D_2 \). Since \( (X, \tau) \) is \( G^*-T_4 \)-space, there exists a subset \( A \) of \( X \) such that \( C_1 \notin A^{x_2} \) and \( D_1 \notin A^{x_1} \), so that \( f(C_1) \notin f(A^{x_2}) = (f(A))^{f(x_2)} \) and \( f(D_1) \notin f(A^{x_1}) = (f(A))^{f(x_1)} \). Thus
Gem-Separation Axioms in Topological Space

Let $C_2 \not\subseteq (f(A))^{f(x_2)=y_2}$ and $D_2 \not\subseteq (f(A))^{f(x_1)=y_1}$. Thus $Y$ is a $G^*-T_4$-space.

**Theorem 3.6.2.** If $f : (X, \tau) \to (Y, \sigma)$ is one-one $I''$-map of a space $X$ onto $G^*-T_4$-space $Y$, then $X$ is a $G^*-T_4$-space.

**Proof:** Let $C_1$ and $D_1$ be two disjoint sets in $X$. Since $f$ is one-one and onto, there exists disjoint sets $C_2$ and $D_2$ of $Y$ such that $f(C_1) = C_2$ and $f(D_1) = D_2$. Since $(Y, \sigma)$ is $G^*-T_4$-space, there exists a subset $A$ of $Y$ such that $C_2 \not\subseteq A^{y_2}$ and $D_2 \not\subseteq A^{y_1}$, so that $f^{-1}(C_2) \not\subseteq f^{-1}(A^{y_2}) = (f^{-1}(A))^{f^{-1}(y_2)}$ and $f^{-1}(D_2) \not\subseteq f^{-1}(A^{y_1}) = (f^{-1}(A))^{f^{-1}(y_1)}$. This implies $C_1 \not\subseteq (f^{-1}(A))^{x_2}$ and $D_1 \not\subseteq (f^{-1}(A))^{x_1}$. Thus $X$ is a $G^*-T_4$-space.

**Theorem 3.6.3.** If $f : (X, \tau) \to (Y, \sigma)$ is one-one A-map of a space $X$ onto a space $Y$, then $Y$ is a $G^*-T_4$-space.

**Proof:** Let $C_2$ and $D_2$ be two disjoint sets in $Y$. Since $f$ is one-one and onto, there exists a disjoint sets $C_1$ and $D_1$ of $X$ such that $f(C_1) = C_2$ and $f(D_1) = D_2$. Since $(Y, \sigma)$ is $G^*-T_4$-space, there exists a subset $A$ of $X$ such that $C_1 \not\subseteq A^{x_2}$ and $D_1 \not\subseteq A^{x_1}$, so that $f(C_2) \not\subseteq f(A^{x_2}) \subseteq B^{f(x_2)}$ and $f(D_2) \not\subseteq f(A^{x_1}) \subseteq B^{f(x_1)}$. This implies $C_2 \not\subseteq B^{y_2}$ and $D_2 \not\subseteq B^{y_1}$. Thus $Y$ is a $G^*-T_4$-space.

**Theorem 3.6.4.** If $f : (X, \tau) \to (Y, \sigma)$ is one-one AO-map of a space $X$ onto $G^*-T_4$-space $Y$, then $X$ is a $G^*-T_4$-space.

**Proof:** Let $C_1$ and $D_1$ be two disjoint sets in $X$. Since $f$ is one-one and onto, there exists a disjoint sets $C_2$ and $D_2$ of $Y$ such that $f(C_1) = C_2$ and $f(D_1) = D_2$. Since $(Y, \sigma)$ is $G^*-T_4$-space, there exists subsets $B$ of $Y$ such that $C_2 \not\subseteq B^{y_2} \subseteq f(A^{f^{-1}(y_2)})$ and $D_2 \not\subseteq B^{y_1} \subseteq f(A^{f^{-1}(y_1)})$, so that $f^{-1}(C_2) \not\subseteq f^{-1}(f(A^{f^{-1}(y_2)}))$ and $f^{-1}(D_2) \not\subseteq f^{-1}(f(A^{f^{-1}(y_1)}))$. This implies $C_1 \not\subseteq A^{x_2}$ and $D_1 \not\subseteq A^{x_1}$. Thus $X$ is a $G^*-T_4$-space.

### 3.7. $G^*-T_5$-space

In this section we proved some theorems in connection with $I^*$-map, $I''$-map, A-map and AO-map for $G^*-T_5$-space.

**Theorem 3.7.1.** If $f : (X, \tau) \to (Y, \sigma)$ is one-one $I^*$-map of a $G^*-T_5$-space $X$ onto a space $Y$, then $Y$ is a $G^*-T_5$-space.

**Proof:** Let $C_2$ and $D_2$ be separated sets in $Y$. Since $f$ is one-one and onto, there exists separated sets $C_1$ and $D_1$ of $X$ such that $f(C_1) = C_2$ and $f(D_1) = D_2$. Since $(X, \tau)$ is $G^*-T_5$-space, there exists subset $A$ of $X$ such that $C_1 \not\subseteq A^{x_2}$ and $D_1 \not\subseteq A^{x_1}$, so that $f(C_1) \not\subseteq f(A^{x_2}) = (f(A))^{f(x_2)}$ and $f(D_1) \not\subseteq f(A^{x_1}) = (f(A))^{f(x_1)}$. Thus $C_2 \not\subseteq (f(A))^{f(x_2)=y_2}$ and $D_2 \not\subseteq (f(A))^{f(x_1)=y_1}$. Thus $Y$ is a $G^*-T_5$-space.

**Theorem 3.7.2.** If $f : (X, \tau) \to (Y, \sigma)$ is one-one $I''$-map of a space $X$ onto $G^*-T_5$-space $Y$, then $X$ is a $G^*-T_5$-space.

**Proof:** Let $C_1$ and $D_1$ be separated sets in $X$. Since $f$ is one-one and onto, there exists separated sets $C_2$ and $D_2$ of $Y$ such that $f(C_1) = C_2$ and $f(D_1) = D_2$. Since $(Y, \sigma)$ is $G^*-T_5$-space, there exists subset $A$ of $Y$ such that $C_2 \not\subseteq A^{y_2}$ and $D_2 \not\subseteq A^{y_1}$, so that
R. Rathinam and C. Elango

\[ f^{-1}(C_2) \neq f^{-1}(A^{y2}) = (f^{-1}(A))^{f^{-1}(y_2)} \text{ and } f^{-1}(D_2) \neq f^{-1}(A^{y_2}) = (f^{-1}(A))^{f^{-1}(y_1)} \text{. This implies } C_3 \nsubseteq (f^{-1}(A))^{x_2} \text{ and } D_3 \nsubseteq (f^{-1}(A))^{x_1} \text{. Thus } X \text{ is a } G^*\text{-}T_5 \text{-space.} \]

**Theorem 3.7.3.** If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is one-one A-map of an \( G^*\text{-}T_5 \)-space \( X \) onto a space \( Y \), then \( Y \) is a \( G^*\text{-}T_5 \)-space.

**Proof:** Let \( C_2 \) and \( D_2 \) be separated sets in \( Y \). Since \( f \) is one-one and onto, there exists separated sets \( C_1 \) and \( D_1 \) of \( X \) such that \( f(C_1) = C_2 \) and \( f(D_1) = D_2 \). Since \( (X, \tau) \) is \( G^*\text{-}T_5 \)-space, there exists subset \( A \) of \( X \) such that \( C_1 \nsubseteq A^{x_2} \text{ and } D_1 \nsubseteq A^{x_1} \), so that \( f(C_1) \nsubseteq f(A^{x_2}) \subseteq B^{f(x_2)} \) and \( f(D_1) \nsubseteq f(A^{x_1}) \subseteq B^{f(x_1)} \). This implies \( C_2 \nsubseteq B^{y_2} \text{ and } D_2 \nsubseteq B^{y_1} \). Thus \( Y \) is a \( G^*\text{-}T_5 \)-space.

**Theorem 3.7.4.** If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is one-one AO-map of a space \( X \) onto \( G^*\text{-}T_5 \)-space \( Y \), then \( X \) is a \( G^*\text{-}T_5 \)-space.

**Proof:** Let \( C_1 \) and \( D_1 \) be separated sets in \( X \). Since \( f \) is one-one and onto, there exists separated sets \( C_2 \) and \( D_2 \) of \( Y \) such that \( f(C_1) = C_2 \) and \( f(D_1) = D_2 \). Since \( (Y, \sigma) \) is \( G^*\text{-}T_5 \)-space, there exists subset \( B \) of \( Y \) such that \( C_2 \nsubseteq B^{y_2} \subseteq f(A^{y_2}) \text{ and } D_2 \nsubseteq B^{y_1} \subseteq f(A^{y_1}) \), so that \( f^{-1}(C_2) \nsubseteq f^{-1}(f(A^{y_1})) \) and \( f^{-1}(D_2) \nsubseteq f^{-1}(f(A^{y_2})) \). This implies \( C_1 \nsubseteq A^{x_2} \text{ and } D_1 \nsubseteq A^{x_1} \). Thus \( X \) is a \( G^*\text{-}T_5 \)-space.

### 3.8. \( G^*\text{-}T_6 \)-space

In this section we proved some theorems in connection with \( I^*\text{-}map \), \( I^{**}\text{-}map \), A-map and AO-map for \( G^*\text{-}T_6 \)-space.

**Theorem 3.8.1.** If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is one-one \( I^*\)-map of a space \( X \) onto \( G^*\text{-}T_6 \)-space \( Y \), then \( X \) is a \( G^*\text{-}T_6 \)-space.

**Proof:** Let \( C_1 \) and \( D_1 \) be two disjoint sets in \( X \). Since \( f \) is one-one and onto, there exists disjoint sets \( C_2 \) and \( D_2 \) of \( Y \) such that \( f(C_1) = C_2 \) and \( f(D_1) = D_2 \). Since \( f \) is an \( I^*\)-map, so that \( f(C_1^{x_1}) = f(C_2^{y_1}) = C_2^{y_1} \text{ and } f(D_1^{x_2}) = f(D_2^{y_2}) = D_2^{y_2} \). Since \( (Y, \sigma) \) is \( G^*\text{-}T_6 \)-space, there exists a continuous map \( g : Y \rightarrow [0, 1] \) such that \( C_2^{y_1} \neq g^{-1}([1]) \text{ and } D_2^{y_2} = g^{-1}([1]) \). This implies \( f(C_1^{x_1}) \neq g^{-1}([1]) \text{ and } f(D_1^{x_2}) = g^{-1}([1]) \). Now \( g(f(C_1^{x_1})) = (1) \text{ and } g(f(D_1^{x_2})) = (1) \). This implies \( h(C_1^{x_1}) = (1) \text{ and } h(D_1^{x_2}) = (1) \). Thus \( C_1^{x_1} \neq h^{-1}([1]) \text{ and } D_1^{x_2} = h^{-1}([1]) \text{ where } h = g \circ f : X \rightarrow [0, 1] \text{ is a continuous map. Hence by definition we have } (X, \tau) \text{ is a } G^*\text{-}T_6 \)-space.

**Theorem 3.8.2.** If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is one-one \( I^{**}\)-map of a space \( X \) onto \( G^*\text{-}T_6 \)-space \( Y \), then \( X \) is a \( G^*\text{-}T_6 \)-space.

**Proof:** Let \( C_1 \) and \( D_1 \) be two disjoint sets in \( X \). Since \( f \) is one-one and onto, there exists disjoint sets \( C_2 \) and \( D_2 \) of \( Y \) such that \( f(C_1) = C_2 \) and \( f(D_1) = D_2 \). Since \( (Y, \sigma) \) is \( G^*\text{-}T_6 \)-space, there exists a continuous map \( g : Y \rightarrow [0, 1] \) such that \( C_2^{y_1} \neq g^{-1}([1]) \text{ and } D_2^{y_2} = g^{-1}([1]) \). So that \( f^{-1}(C_2^{y_1}) \neq f^{-1}(g^{-1}([1])) \text{ and } f^{-1}(D_2^{y_2}) = f^{-1}(g^{-1}([1])) \). This implies \( f^{-1}(C_2^{y_1}) = h^{-1}([1]) \text{ and } f^{-1}(D_2^{y_2}) = h^{-1}([1]) \). Since \( f \) is an \( I^{**}\)-map, we have \( (f^{-1}(C_2))^{f^{-1}(y_1)} = h^{-1}([1]) \text{ and } (f^{-1}(D_2))^{f^{-1}(y_2)} = h^{-1}([1]) \). This implies
Gem-Separation Axioms in Topological Space

\( C_1^{x_1} \neq h^{-1}\{1\} \) and \( D_1^{x_2} = h^{-1}\{1\} \) where \( h = g \circ f : X \to [0, 1] \) is a continuous map. Thus by definition we have \( (X, \tau) \) is a \( G^*\)-\( T_6 \)-space.

**Theorem 3.8.3.** If \( f : (X, \tau) \to (Y, \sigma) \) is one-one \( A \)-map of a space \( X \) onto \( G^*-\!T_6 \)-space \( Y \), then \( X \) is a \( G^*\)-\( T_6 \)-space.

**Proof:** Let \( C_1 \) and \( D_1 \) be two disjoint sets in \( X \). Since \( f \) is one-one and onto, there exists disjoint sets \( C_2 \) and \( D_2 \) of \( Y \) such that \( f(C_1) = C_2 \) and \( f(D_1) = D_2 \). Since \( f \) is an \( A \)-map, so that \( f(C_1^{x_1}) \subseteq C_2^{f(x_1)} = C_2^{y_1} \) and \( f(D_1^{x_2}) \subseteq D_2^{f(x_2)} = D_2^{y_2} \). Since \( (Y, \sigma) \) is \( G^*\)-\( T_6 \)-space, there exists a continuous map \( g : Y \to [0, 1] \) such that \( C_2^{y_1} \neq g^{-1}\{1\} \) and \( D_2^{y_2} = g^{-1}\{1\} \). This implies \( f(C_1^{x_1}) \subseteq C_2^{y_1} \neq g^{-1}\{1\} \) and \( f(D_1^{x_2}) \subseteq D_2^{y_2} = g^{-1}\{1\} \). This implies \( f(C_1^{x_1}) \neq g^{-1}\{1\} \) and \( f(D_1^{x_2}) = g^{-1}\{1\} \). Now \( g(f(C_1^{x_1})) \neq (1) \) and \( g(f(D_1^{x_2})) = (1) \). Thus \( C_1^{x_1} \not\subseteq h^{-1}\{1\} \) and \( D_1^{x_2} \subseteq h^{-1}\{1\} \) where \( h = g \circ f : X \to [0, 1] \) is a continuous map. Hence by definition we have \( (X, \tau) \) is a \( G^*\)-\( T_6 \)-space.

**Theorem 3.8.4.** If \( f : (X, \tau) \to (Y, \sigma) \) is one-one \( AO \)-map of a space \( X \) onto \( G^*\)-\( T_6 \)-space \( Y \), then \( X \) is a \( G^*\)-\( T_6 \)-space.

**Proof:** Let \( C_1 \) and \( D_1 \) be two disjoint sets in \( X \). Since \( f \) is one-one and onto, there exists disjoint sets \( C_2 \) and \( D_2 \) of \( Y \) such that \( f(C_1) = C_2 \) and \( f(D_1) = D_2 \). Since \( (Y, \sigma) \) is \( G^*\)-\( T_6 \)-space, there exists a continuous map \( g : Y \to [0, 1] \) such that \( C_2^{y_1} \neq g^{-1}\{1\} \) and \( D_2^{y_2} = g^{-1}\{1\} \). This implies \( f(C_1^{x_1}) \subseteq C_2^{y_1} \neq g^{-1}\{1\} \) and \( f(D_1^{x_2}) \subseteq D_2^{y_2} = g^{-1}\{1\} \). This implies \( f(C_1^{x_1}) \neq g^{-1}\{1\} \) and \( f(D_1^{x_2}) = g^{-1}\{1\} \). Now \( g(f(C_1^{x_1})) \neq (1) \) and \( g(f(D_1^{x_2})) = (1) \). This implies \( h(C_1^{x_1}) \neq (1) \) and \( h(D_1^{x_2}) = (1) \). Thus \( C_1^{x_1} \neq h^{-1}\{1\} \) and \( D_1^{x_2} = h^{-1}\{1\} \) where \( h = g \circ f : X \to [0, 1] \) is a continuous map. Hence by definition we have \( (X, \tau) \) is a \( G^*\)-\( T_6 \)-space.

**4. Conclusion**

In this article, we studied some basic concepts and relations involving Gem-separation axioms. We also rename \( I^*\)-\( T_0 \)-space, \( I^*\)-\( T_1 \)-space, \( I^*\)-\( T_2 \)-space by Gem-Kolmogorov space(G-\( T_0 \)-space), Gem-accessible space or Gem-Frechlet space(G-\( T_1 \)-space) and Gem-Hausdorff space(G-\( T_2 \)-space) and \( I^*\)-spaces by \( G^*\)-\( T_i \)-spaces. In future the concepts used in nano-topology can be adopted to prove that Gem-set in nano topological space.

**REFERENCES**