Bounds of Location-2-Domination Number for Products of Graphs

G.Rajasekar¹ and A.Venkatesan²

¹Department of Mathematics, Jawahar Science College, Neyveli,
Tamilnadu, India. Email: grsmaths@gmail.com
²Department of Mathematics, St. Joseph’s College of Arts and Science College
(Autonomous) Cuddalore. Tamilnadu, India.
Email: suresh11venkat@gmail.com

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Abstract. In this paper Location-2-Domination set and their properties are being studied. A subset \( S \subseteq V \) is Location-2-Dominating set of \( G \) if \( S \) is 2-Dominating set of \( G \) and for any two vertices \( u, v \in V \setminus S \) such that \( N(u) \cap S \neq N(v) \cap S \), its denoted by \( R^2(G) \). Based on this definition the bounds of the Location-2-domination number for direct product, Cartesian product and semi-strong product of graphs namely \( P_n \bowtie C_m \), \( C_n \bowtie S_m \), \( C_n \times S_m \), \( P_n \times P_m \), \( C_n \bowtie P_m \), \( C_n \bowtie C_m \) have been found.

Keywords: 2-Domination, Location Domination, Product of Graphs

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1. Introduction

Throughout this paper let us follow the terminology and notation of Harary [11]. Cockayne and Hedetniemi [7] introduce the concept dominating set. A subset \( S \) of vertices from \( V \) is called a dominating set for \( G \) if every vertex of \( G \) is either a member of \( S \) or adjacent to a member of \( S \). A dominating set of \( G \) is called a minimum dominating set if \( G \) has no dominating set of smaller cardinality. The cardinality of minimum dominating set of \( G \) is called the dominating number for \( G \) and it is denoted by \( \gamma(G) \) [6].

Harary and Haynes [5] introduced the concepts of double domination in graphs. A dominating set \( S \) of \( G \) is called double dominating set if every vertex in \( V-S \) is adjacent to at least two vertices in \( S \). Given a dominating set \( S \) for graph \( G \), for each \( u \) in \( V-S \) let \( S(u) \) denote the set of vertices in \( S \) which are adjacent to \( u \). The set \( S \) is called locating dominating set, if for any two vertices \( u \) and \( w \) in \( V-S \) one has \( S(u) \) not equal to \( S(w) \) and the minimum cardinality of Location Domination set is denoted by \( RD(G) \) [7]. The Cartesian product \( G \bowtie H \) of graphs \( G \) and \( H \) is the graph with vertex set \( V(G) \times V(H) \) and edge set is \( (u,a)(v,b) \in E(G \bowtie H) \) if and only if \( a = b \) and...
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$uv \in E(G)$ or $u = v$ and $ab \in E(H)$ [3]. The direct product $G \times H$ of graphs $G$ and $H$ is the graph with vertex set $V(G) \times V(H)$ and edge set is $(u, a)(v, b) \in E(G \times H)$ if and only if $uv \in E(G)$ and $ab \in E(H)$ [14]. The Semi-Strong Product of two graphs $G$ and $H$ is the graph $G \bowtie H$ with vertices $V(G \bowtie H) = V(G) \times V(H)$ and edges $E(G \bowtie H) = \{(a, x)(b, y)\}$ if and only if $(a, b) \in E(G)$ and $x = y$ or $(a, b) \in E(G)$ and $(x, y) \in E(H))$ [12].

2. Preliminaries
2.1 Location-2-domination

**Definition 2.1.1.** [8] A subset $S \subseteq V$ is Location – 2 -Dominating set of $G$ if $S$ is a 2 Dominating set of $G$ and if for any two vertices $u, v \in V - S$ such that $N(u) \cap S \neq N(v) \cap S$.

The minimum cardinality of Location-2-Dominating is denoted by $R_2^D(G) = |S|$.

2.2. Location-2-domination for simple graphs

**Theorem 2.2.1.** [9] In Location-2-Domination for any graph the vertex $\{v\}$ is a pendent vertex then $\{v\} \in R_2^D(G)$ only.

**Theorem 2.2.2.** [8] Location-2-Domination number of a Path $P_n$ is

$$R_2^D(P_n) = \begin{cases} \frac{n - 1}{2} + 1, & \text{n is odd} \\ \frac{n}{2} + 1, & \text{n is even} \end{cases}$$

**Theorem 2.2.3.** [8] Location-2-Domination for any cycle $C_n$, for $n \neq 4$ is

$$R_2^D(C_n) = \begin{cases} \frac{n}{2}, & \text{n is even} \\ \frac{n - 1}{2} + 1, & \text{n is odd} \end{cases}$$

2.3 Location-2-domination for Cartesian product of graphs

**Theorem 2.3.1.** [10] For any graph $G = (P_n \boxtimes S_m)$,

$$R_2^D(G) = \begin{cases} R_2^D(P_n) + \frac{m(n - 1)}{2}, & \text{n is odd} \\ \frac{n}{2} (m + 1), & \text{n is even} \end{cases}$$

**Theorem 2.3.2.** [10] Location-2-Domination for any graph $G = (P_n \boxtimes P_m)$ is
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\[ R_2^D(G) = \begin{cases} \frac{nm}{2} & \text{if } n \text{ is even, } m \text{ is either even or odd} \\ \frac{nm-1}{2} & \text{if } n \text{ is odd, } m \text{ is odd} \end{cases} \]

**Theorem 2.3.3.** [10] Location-2-domination for any graph \( G = (C_n \sqcap C_m) \) is

\[ R_2^D(G) = \begin{cases} \frac{nm}{2} & \text{if } n \text{ is even, } m \text{ is either even or odd} \\ \frac{nm-1}{2} & \text{if } n \text{ is odd, } m \text{ is odd} \end{cases} \]

**2.4. Location-2-domination for direct product of graphs**

**Theorem 2.4.1.** [10] For graphs \( P_n \) (\( n \neq 3 \)) and \( S_m \), \( R_2^D(P_n \times S_m) = nm, n, m = 1, 2, 3, \ldots \)

**Theorem 2.4.2.** [10] Location-2-Domination for \( P_n \) and \( P_m, m \neq 3 \),

\[ R_2^D(P_n \times P_m) = \begin{cases} \frac{nm}{2} + 2 & \text{if } n, m \text{ is even} \\ \frac{nm}{2} + 2 & \text{if } \text{either } n \text{ is odd, } m \text{ is even} \\ \frac{n(m+1)}{2} & \text{if } n \text{ is odd but } n < m \end{cases} \]

**Theorem 2.4.3.** [10] For \( n, m \geq 5 \),

\[ R_2^D(C_n \sqcap C_m) = \frac{nm}{2}, \ n, m \text{ is even}, \]
\[ R_2^D(C_n \sqcap C_m) = \frac{(n-1)m}{2}, \ n \text{ is odd } m \text{ is even}, \]
\[ R_2^D(C_n \sqcap C_m) = \frac{n(m-1)}{2}, \ n \text{ is even } m \text{ is odd}, \]
\[ R_2^D(C_n \sqcap C_m) = \frac{n(m-1)}{2}, \ n, m \text{ is odd but } n > m, \]
\[ R_2^D(C_n \sqcap C_m) = \frac{m(n-1)}{2}, \ n, m \text{ is odd but } n < m. \]
3. Location -2-domination of products of graph

3.1. Location -2-domination (Cartesian product) of $C_n \square P_m$, $C_n \square S_m$

Theorem 3.1.1. For any graph $P_m$ and $C_n$ we have

$$R_2^D(G) = \left\{ \begin{array}{ll}
\frac{nm}{2} & \text{n is even, m is either even (or) odd} \\
\frac{nm}{2} + 1 & \text{n is odd m is even} \\
\frac{nm + 1}{2} & \text{n is odd, m is odd}
\end{array} \right.$$

Proof: Consider path of $m$ vertices and Cycle of $n$ vertices. The Vertex set of $P_m$ and $C_n$ are $\{1, 2, ..., m\}$ and $\{1, 2, ..., n\}$ respectively. Clearly $|G| = nm$ in which $2n$ vertices are degree 3 and $(m-2)n$ vertices are degree 4 and let $S$ - Set denote Location-2-Domination of $G$.

Case (i): Suppose $n$ is even and $m$ is either even or odd, in this situation $|V(G)| = nm$ even number of vertices. In $G$ fix any vertex from $C_1$ and form open path through vertices of $P_1$, continue the open path starts with $C_2$ through $P_2$, continue the same process till $C_n$ through $P_m$, each time the process of continuation of open path from $C_1$ to $C_n$ done only by either towards right or left direction only not alternatively. Finally the collection of vertices from $C_1$ to $C_n$ through $P_1$ to $P_m$ forms a cycle of length even with $nm$ vertices. So by the Theorem: 2.2.3, $|S| = \frac{nm}{2}$, i.e. $R_2^D(G) = \frac{nm}{2}$.

Case (ii): Suppose $n$ is odd, but $m$ is even, in this situation $|V(G)| = nm$ is even, in $G$ fix any vertex from $C_1$ and form open path through vertices of $P_1$, continue the open path starts with $C_2$ through $P_2$, continue the same process till $C_n$ through $P_m$, each time in the process of continuation open path from $C_1$ to $C_n$ done only by either towards right or left direction only not for alternatively. Finally the collection of vertices from $C_1$ to $C_n$ through $P_1$ to $P_m$ forms a path of length even with $nm$ vertices. So by the Theorem: 2.2.2 $|S| = \frac{nm}{2} + 1$, i.e. $R_2^D(G) = \frac{nm}{2} + 1$.

Case (iii): Suppose $n$ is odd and $m$ is odd. Clearly $|G| = nm$ odd number of vertices, From $G$, let us consider $|S| = |S_1| + |S_2|$ where $|S_1|$ denote the Location-2-Domination for $\{C_1, C_3, ..., C_m\}$ and $|S_2|$ denote the Location-2-Domination for $\{C_2, C_4, ..., C_{m-1}\}$, but
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\[ |C_1|=|C_2|=...=|C_{m-1}|=|C_m|=n. \text{ And for } S_1=\{C_1,C_3,...,C_m\} \text{ the vertex set of } S_1 \text{ are } \{C_{11},C_{12},...,C_{1n},C_{31},C_{32},...,C_{3n},...,C_{m1},C_{m2},...,C_{mn}\}. \text{ By the Theorem: 2.2.3 Location-2-Domination for Cycle of length odd is } \frac{n-1}{2}+1=\frac{n+1}{2}. \text{ Therefore } |C_i|=\frac{n+1}{2}, i=1,3,...,m. \text{ Clearly } S_1 \text{-set contains } \frac{m+1}{2} \text{ times of cycle with odd length. Therefore } |S_1|=\left(\frac{m+1}{2}\right)\left(\frac{n+1}{2}\right) \text{ and } S_2=\{C_2,C_4,...,C_{m-1}\} \text{ the vertex set of } S_2 \text{ are } \{C_{21},C_{22},...,C_{2n},C_{41},C_{42},...,C_{4n},...,C_{(m-1)1},C_{(m-1)2},...,C_{(m-1)n}\}, \text{ now collect the vertex from } C_2 \text{ as } N(V-S_1)-S_1 \text{ in } C_1. \text{ This gives } \frac{n-1}{2} \text{ vertices in } C_2. \text{ Continuing the same process for } \{C_4,C_6,...,C_{m-1}\}, \text{ i.e. collect the vertex for } C_{i+1} \text{ as } N(V-S_i)-S_i \text{ from } C_i \text{ for } i=1,2,...,m-2. \]

Clearly \(S_2\)-set contains \(\frac{m-1}{2}\) times of \(C_{i+1}\), \(i=1,2,...,m-2\). i.e. \(|S_2|\)=\(\left(\frac{m-1}{2}\right)\left(\frac{n-1}{2}\right)\)

\[|S|=\left(\frac{m+1}{2}\right)\left(\frac{n+1}{2}\right)+\left(\frac{m-1}{2}\right)\left(\frac{n-1}{2}\right)=\frac{nm+1}{2}. \text{ Therefore, } R_2^D(G)=|S|=\frac{nm+1}{2}.\]

**Theorem 3.1.2.** For graphs \(P_n\) and \(S_m\), \(R_2^D(C_n\triangle S_m)=\begin{cases} \frac{n(m+1)}{2} & n \text{ is even} \\ \frac{n+1}{2}+\frac{m(n-1)}{2} & n \text{ is odd} \end{cases}\)

**Proof:** Consider the vertex set of \(G\) namely \(\{v_j\}\) for \(1\leq i \leq n, 1 \leq j \leq m+1.\) Clearly \(|G|=nm\). Let \(S-\text{set denote Location-2-Dominating set, by observing } G, d_G(v_{ij})=m+1\) for \(i=1,n,\text{ and } 1 \leq j \leq m+1\) also \(d_G(v_{ij})=4\) for \(2 \leq i \leq n-1, 2 \leq j \leq m+1.\) i.e. \(v_{ij}, i=1,2,...,n\) is adjacent with \(v_{ij}, 2 \leq j \leq m+1.\)

**Case (i):** Suppose \(n\) is even and \(m\) is either even or odd. Clearly \(|G|=nm\) has even number of vertices, in this sense now collect \(S-\text{set possibly by } \{v_{ij}\}, i=1,3,5,...,n-1\) and \(\{v_{ij}\}\) for \(i=2,4,...,n, 2 \leq j \leq m+1,\) or \(\{v_{ij}\}, i=2,4,6,...,n\) and \(\{v_{ij}\}\) for \(i=1,3,...,n-1, 2 \leq j \leq m+1,\) this gives \(\frac{n}{2}\) times a single vertex and \(\frac{n}{2}\) times \(m\) vertices or \(\{v_{ij}\}\) for \(i=1,3,5,...,n-1, 1 \leq j \leq m+1,\) this gives \(\frac{n}{2}\) times \(m+1\) vertices.
\[ |S| = \frac{n}{2} + \frac{nm}{2} = \frac{n(m+1)}{2} \] 

therefore \( R^b_2(G) = \frac{n(m+1)}{2} \).

Suppose, \( \{v_{ij}\} \notin S \) for \( i = 2, 4, \ldots, n \), \( 2 \leq j \leq m+1 \) or some \( \{v_{ij}\} \notin S \) for \( i = 2, 4, \ldots, n \), \( 2 \leq j \leq m+1 \). Clearly this contradicts the definition of Location-2-Domination or minimum cardinality of \( S^- \) set or some \( \{v_{ij}\} \notin S \) for \( i = 2, 4, \ldots, n \), \( 2 \leq j \leq m+1 \), in this situation \( \{v_{ij}\}, i = 1, 3, \ldots, n-1, 1 \leq j \leq m+1 \) needs additional vertex, clearly it also contradicts the minimum cardinality of \( S^- \) set.

**Case (ii):** Suppose \( n \) is odd, \( m \) is either even or odd. Clearly \( |V(G)| = nm \) gives even number of vertices, in this sense now collect \( S^- \) set possibly by \( \{v_{ij}\}, i = 1, 3, \ldots, n \) and \( \{v_{ij}\} \) for \( i = 2, 4, \ldots, n-1 \), \( 2 \leq j \leq m+1 \), this gives \( \frac{n+1}{2} \) times a single vertex and \( \frac{n-1}{2} \) times \( m+1 \) vertices.

That is, \( |S| = \frac{n+1}{2} + \frac{(n-1)m}{2} \) and therefore \( R^b_2(G) = \frac{n+1}{2} + \frac{m(n-1)}{2} \).

Suppose, \( \{v_{ij}\} \notin S \) for \( i = 1, 3, \ldots, n \), \( 2 \leq j \leq m+1 \) this gives \( |S| = \frac{n-1}{2} + \frac{(n+1)m}{2} \) contradicts minimum cardinality of \( S^- \) set or some \( \{v_{ij}\} \notin S \) for \( i = 2, 4, \ldots, n \), \( 2 \leq j \leq m+1 \), clearly it contradicts the definition of Location-2-Domination.

### 3.2. Location-2-domination (Direct product) of \( P_n \times W_m, P_n \times C_m \)

**Theorem 3.2.1.** For any Graphs \( P_n, n \neq 2 \) and \( W_m, m \neq 5 \) we have

\[
R^b_2(G) = \begin{cases} 
\frac{nm}{2} & \text{if } n \text{ is even} \\
\frac{m(n+1)}{2} & \text{if } n \text{ is odd}
\end{cases}
\]

**Proof:** Label the vertices of \( G \) as \( \{v_{ij}\}, 1 \leq i \leq n, 1 \leq j \leq m \), clearly \( |G| = nm \) from \( G \). 
\( d_G(v_{11}) = d_G(v_{mn}) = m-1, \quad d_G(v_{ij}) = d_G(v_{ij}) = 3, 2 \leq j \leq m \) and 
\( d_G(v_{ij}) = 6, 2 \leq i \leq n-1, \quad 2 \leq j \leq m \). Now labels of \( G \) are partitioned into \( n \) different sets namely \( U_i, 1 \leq i \leq n \) are \( \{v_{ij}\}, 1 \leq i \leq n, 1 \leq j \leq m \) respectively. But there is no adjacency from \( v_{ij} \) to \( v_{ij} \) for \( 1 \leq i \leq n, 1 \leq j \leq m \). Clearly, \( u_i \) is adjacent to \( u_{i-1} \) and \( u_{i+1} \) for \( i = 1, 2, \ldots, n \).
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Case (i): Suppose \( n \) is even, then the collection of the sets \( U_i, i = 1, 3, ..., n-1 \) or \( i = 2, 4, ..., n \) will have \( \frac{n}{2} \) times \( m \) vertices i.e. \( |S| = \frac{nm}{2} \) therefore \( D_2^0(G) = \frac{nm}{2} \).

Case (ii): Suppose \( n \) is odd by based on Theorem 2.2.2, the collection of the sets \( U_i, i = 1, 3, ..., n-1 \) will have \( \frac{n-1}{2} \) times \( m \) vertices i.e. \( |S| = \frac{(n-1)m}{2} \) therefore \( D_2^0(G) = \frac{(n-1)m}{2} \). Suppose if we collect the sets \( U_i, i = 1, 3, ..., n \) this contradicts the minimum cardinality.

Result 3.2.1. \( D_2^0(P_2 \times W_m) = m \).

Result 3.2.2. \( D_2^0(P_n \times W_m) = \begin{cases} 3n, & n \text{ is even} \\ \frac{5(n+1)}{2} + \frac{n-1}{2}, & n \text{ is odd} \end{cases} \)

Theorem 3.2.2. For Graphs \( C_n \) and \( S_m \), \( D_2^0(C_n \times S_m) = nm n, m = 1, 2, 3, ... \)

Proof: The vertex set of \( G \) are \( \{v_i\}, 1 \leq i \leq n, 1 \leq j \leq m+1 \). Let \( S \) denote Location-2-Dominating set of \( G \). Clearly by observation of \( G \) \( d_{v_i}(v_{i+1}) = 2m, 1 \leq i \leq n \) and \( d_{v_i}(v_{i+1}) = 2, 1 \leq i \leq n, 2 \leq j \leq m+1 \). Now collect the \( S \)-set possibly by either \( v_i, 1 \leq i \leq n, 2 \leq j \leq m+1 \) or \( v_i, 1 \leq i \leq n \) and \( v_j, 1 \leq i \leq n, 3 \leq j \leq m+1 \) i.e. leaving anyone of the same base vertex of \( i = 1, 2, ... \) or \( j = 1, 2, ... \) clearly this \( n \) times \( m \) vertices. i.e. \( |S| = nm \). Suppose \( v_i \in S, i = 1, 2, 3, ..., n \) and \( v_j \not\in S, 1 \leq i \leq n, 2 \leq j \leq m+1 \) then this contradicts the definition of Location-2-Domination. Therefore \( D_2^0(G) = |S| = nm \).

3.3 Location-2-domination (semi-strong product) of \( P_n \bowtie P_m, C_n \bowtie P_m, C_n \bowtie C_m \)

Theorem 3.3.1. For any graphs \( P_n, n \neq 3 \) and \( P_m, m \neq 2, G = P_n \bowtie P_m \) is

\[
D_2^0(G) = \begin{cases} \frac{nm}{2}, & n \text{ is even, } m \text{ is even or odd} \\ \frac{nm+2}{2}, & n \text{ is odd, } m \text{ is even} \\ \frac{n(m+1)}{2}, & n, m \text{ is odd } n < m \\ \frac{m(n+1)}{2}, & n, m \text{ is odd } m < n \\ \frac{n(m+1)}{2}, & n, m \text{ is odd } n = m \end{cases}
\]
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**Proof:** Label the vertices of $G$ as $\{v_{ij}\}, 1 \leq i \leq n$ and $1 \leq j \leq m$. Let $S$-set be the Location-2-Domination set of $G$, then clearly $d_G(v_{ij}) \geq 2, i=1,2,...,n \ j=1,2,...,m$.

**Case (i):** Suppose $n$ is even and $m$ is even or odd, now let us collect the $S$-set possibly by $\{v_{ij}\} i=1,3,...,n-1, j=1,2,...,m$ or $\{v_{ij}\} i=2,4,...,n, j=1,2,3,...,m$ this gives $\frac{nm}{2}$ vertices i.e. $|S| = \frac{nm}{2}$.

**Case (ii):** Suppose $n$ is odd, $m$ is even in this sense let us collect the $S$-set possibly by $\{v_{ij}\} i=1,2,3,...,n, j=1,3,...,m-1$ and $v_{in}, v_{jm}$ and this gives $\frac{nm}{2} + 2$ vertices that is $|S| = \frac{nm}{2} + 2$ therefore $R_2^D(G) = \frac{nm}{2} + 2$. Suppose the vertices $v_{in}, v_{jm}$ does not belong to $S$-set or $\{v_{ij}\} i=1,3,...,n-1, j=1,2,3,...,m$ then this is a contradiction to minimum cardinality.

**Case (iii):** Suppose $n, m$ is odd but $n < m$, in this case let us collect the $S$-set possibly by $\{v_{ij}\} i=1,2,3,...,n, j=1,3,...,m-1$ and this gives $n$ times $\frac{m+1}{2}$ vertices that is $|S| = \frac{n(m+1)}{2}$ therefore $R_2^D(G) = \frac{n(m+1)}{2}$. Then the collection $\{v_{ij}\} i=1,3,...,n-1, j=1,2,3,...,m$ is not a minimum cardinality set.

**Case (iv):** Similar to the case (iii).

**Case (v):** Suppose $n, m$ is odd but $n = m$ in this case let us collect the $S$-set possibly by $\{v_{ij}\} i=1,2,3,...,n, j=1,3,...,m$ or $\{v_{ij}\} i=1,3,...,n, j=1,2,3,...,m$ then this gives $\frac{n(m+1)}{2}$ vertices that is $|S| = \frac{n(m+1)}{2}$ and hence $R_2^D(G) = \frac{n(m+1)}{2}$.

**Result 3.3.1.** $R_2^D(P_n \times P_2) = n$

**Result 3.3.2.** $R_2^D(P_3 \times P_m) = 2m$

**Observation 3.3.1.** The semi-strong product of $C_n \times P_m$ is not equal to $P_n \times C_m$

**Theorem 3.3.2.** For any graphs $C_n, n > 5$ and $P_m, m \neq 2$, $G=C_n \bowtie P_m$
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\[ R^0_2(G) = \begin{cases} 
2 \left( \frac{m}{2} + 1 \right) + \frac{m}{2} + m \left( \frac{n-4}{2} \right) & n, m \text{ is even} \\
(m+1) + m \left( \frac{n-3}{2} \right) & n, m \text{ is odd} \\
m \left( \frac{n-3}{2} \right) + 2 (m+2) & n \text{ is odd } m \text{ is even} \\
3 \left( \frac{m+1}{2} \right) + m \left( \frac{n-4}{2} \right) & n \text{ even } m \text{ odd}
\end{cases} \]

**Proof:** \[ |V(G)| = nm = \{v_{ij}\}, i = 1, 2, \ldots, n, \ j = 1, 2, \ldots, m \] Clearly \( d_G(v_{ij}) \geq 2 \) for \( i = 1, 2, \ldots, n, \ j = 1, 2, \ldots, m \)

**Case (i):** Suppose \( n, m \) is even, in this case cardinality of \( S \) - set contains the vertices are \( v_{ij}, i = 1, n; j = 1, 3, \ldots, m-1 \) \( m \) and \( v_{(n-1)j}, j = 1, 3, \ldots, m-1 \) also \( v_{ij} \) for \( i = 3, 5, \ldots, n-3, j = 1, 2, \ldots, m \). Clearly this gives \( 2 \) times \( \frac{m}{2} + 1 \) vertices and \( \frac{m}{2} \) times a single vertex. Also \( \frac{n-4}{2} \) times \( m \) vertices. That is \( |S| = 2 \left( \frac{m}{2} + 1 \right) + \frac{m}{2} + m \left( \frac{n-4}{2} \right) \) and therefore \( R^0_2(G) = 2 \left( \frac{m}{2} + 1 \right) + \frac{m}{2} + m \left( \frac{n-4}{2} \right) \).

**Case (ii):** Suppose \( n, m \) is odd, now the \( S \) - set contains the vertices \( v_{ij}, i = 1, n; j = 1, 3, \ldots, m \) and also \( v_{ij} \) for \( i = 3, 5, \ldots, n-2, j = 1, 2, \ldots, m \). Then this gives \( 2 \) times \( \frac{m+1}{2} \) vertices and \( \frac{n-3}{2} \) times \( m \) vertices. That is \( |S| = (m+1) + m \left( \frac{n-3}{2} \right) \) and therefore \( R^0_2(G) = (m+1) + m \left( \frac{n-3}{2} \right) \).

**Case (iii):** Proof is similar to Case (i) and hence \( R^0_2(G) = (m+2) + m \left( \frac{n-3}{2} \right) \).

**Case (iv):** Suppose \( n \) is even, \( m \) is odd, now the \( S \) - set contains the vertices \( v_{ij}, i = 1, n-1, n; j = 1, 3, \ldots, m \) and also \( v_{ij} \) for \( i = 3, 5, \ldots, n-3; j = 1, 2, \ldots, m \). Then this gives \( 3 \) times \( \frac{m+1}{2} \) vertices and \( \frac{n-4}{2} \) times \( m \) vertices. That is \( |S| = 3 (m+1) + m \left( \frac{n-4}{2} \right) \) and therefore \( R^0_2(G) = 3 (m+1) + m \left( \frac{n-4}{2} \right) \).
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Result 3.3.3. \( R^D_2(C_2 \times C_m) = m, m \neq 2, 3 \)

Result 3.3.4. \( R^D_2(C_3 \times C_m) = 2(m-1) \)

Result 3.3.5. \( R^D_2(C_4 \times C_m) = 2m \)

Theorem 3.3.3. For Graphs \( C_n, n \neq 2, 3, 4 \) and \( C_m \), \( G = C_n \bowtie C_m \) is 2

\[
R^D_2(G) = \begin{cases} 
\frac{nm}{2} & \text{n, m is even} \\
\left(\frac{m-1}{2}\right)n & \text{n is even or odd, m is odd} \\
\left(\frac{n-1}{2}\right)m & \text{n is odd, m is even}
\end{cases}
\]

Proof: Let the vertices of \( G \) be \( \{v_{ij}\}, i = 1, 2, ..., n, j = 1, 2, ..., m \). Let \( S \) denotes Location-2-Dominating set.

Case (i): Proof is followed by Theorem 3.5 Case (i)

Case (ii): Suppose \( n \) is odd, \( m \) is odd and let us collect the \( S \)-set possibly by \( \{v_{ij}\}, i = 1, 3, ..., n-2, j = 1, 2, 3, ..., m \). Clearly this gives \( \frac{m-1}{2} \) times \( n \) vertices, that is \( |S| = n\left(\frac{m-1}{2}\right) \) and therefore \( R^D_2(G) = n\left(\frac{m-1}{2}\right) \).

Case (iii): suppose \( n \) is odd, \( m \) is even and let us collect the \( S \)-set possibly by \( \{v_{ij}\}, i = 1, 2, 3, ..., n, j = 1, 3, ..., m-1 \). Clearly this gives \( \frac{n-1}{2} \) times \( m \) vertices that is \( |S| = m\left(\frac{n-1}{2}\right) \) and hence \( R^D_2(G) = m\left(\frac{n-1}{2}\right) \). Suppose if anyone the vertex collection as \( \{v_{ij}\}, i = 1, 3, ..., n-2, n-1, j = 1, 2, 3, ..., m \) then this is once again a contradiction to minimum cardinality.

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REFERENCES

Bounds of Location-2-Domination Number for Products of Graphs


