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On the Spectrum and Energy of Concatenated Singular Graphs

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Abstract. The roots of the characteristic polynomial of the adjacency matrix A(G) of a graph G are called eigenvalues. The eigenvalues together with their multiplicities constitute the spectrum of G. Graphs having zero as an eigenvalue are called singular graphs. Nullity η of G is the multiplicity of the eigenvalue zero. The null spread of the edge e is defined as $\eta_e(G) = \eta(G) - \eta(G-e)$. Null spread of the edges of singular graphs depends on the null spread of its pendant vertices. The concatenation or edge gluing of two graphs G_1 and G_2 is the graph obtained by identifying two edges of G_1 and G_2. In this paper we study on the spectrum of the concatenation of two graphs. The effect of concatenation on energy is also a part of the investigation.

Keywords: Singular graph, spectrum, nullity, concatenation, energy of graph.

AMS Mathematics Subject Classification (2010): 05C50, 05C76

1. Introduction

Let G = (V(G), E(G)) be a finite, undirected simple graph of order n with vertex set V(G) and edge set E(G). The adjacency matrix A(G) of the graph G is a n x n matrix whose entries a_{ij} are the number of edges from vertex v_i to the vertexv_j. The characteristic polynomial of the adjacency matrix A(G) of the graph G is the characteristic polynomial of G and is denoted by $\phi(G, x)$. The roots of the equation $\phi(G, x) = 0$ are called the eigenvalues of the graph G. The collection of the eigenvalues together with their multiplicities constitute the spectrum of G denoted by spec(G). Graphs having zero as an eigenvalue are called singular graphs. The nullity $\eta(G)$ of the graph G is the multiplicity of zero in the graph's spectrum.

Definition 1.1. [3] Let $G - u$ be the induced sub graph of the graph G obtained on deleting the vertex u. The null spread of the vertex u is :$\eta_u(G) = \eta(G) - \eta(G - u)$.
Obviously the null spread satisfies $-1 \le n_u(G) \le 1$. If u is a core vertex, then $n_u(G) = 1$. There are vertices with $n_u(G) = 0$ and $n_u(G) = -1$. Such vertices are called noncore

vertices of null spread zero and noncore vertices of null spread −1 respectively(See Figure 1).

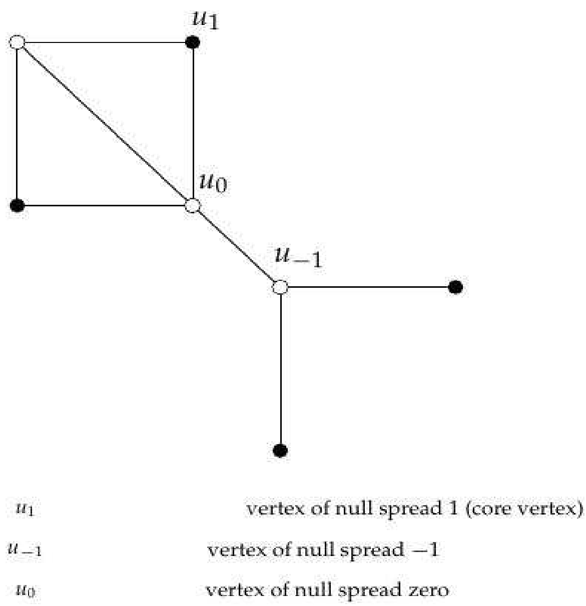

Figure 1: Three types of vertices

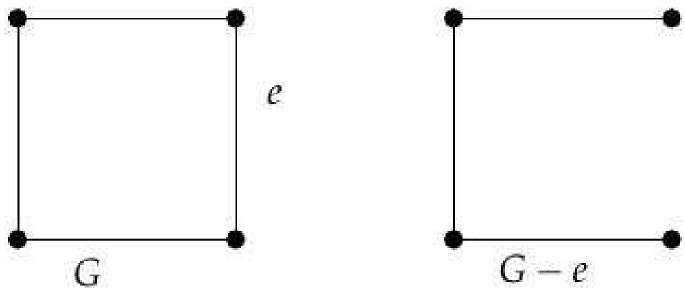

Figure 2: The graphs of G and G – e.

Definition 1.2. [3] Let $G - e$ be the induced subgraph of the graph G obtained on deleting an edge e from G. The null spread of the edge e is defined as $\eta_e(G) = \eta(G) - \eta(G-e)$.

If G is any nonempty graph, then for each $e \in E(G)$, $|\eta(G) - \eta(G-e)| \leq 2$.In Figure 2, the graph G has nullity two and $G-e$ has nullity zero. Thus $\eta_e(G) = 2$. Deletion of edges with positive null spread decreases the nullity of the graph.

Definition 1.3. [9] Let G_1 and G_2 be two graphs with disjoint vertex sets. If a vertex $u \in G_1$ is identified with a vertex $v \in G_2$, then the graph $G_1 \circ G_2$ obtained of order $|G_1| + |G_2| - 1$, is said to be the coalescence of G_1 and G_2 with respect to u and v.

The characteristic polynomial φ (G, x) of the graph $G = G_1 \circ G_2$ is given by the following theorem.

Theorem 1.1. [9] The characteristic polynomial of the coalescence $G_1 \circ G_2$ of two rooted graphs (G_1,u) and (G_2,w) obtained by identifying the vertices u and w so that the vertex v = u = w become a cut vertex of $G_1 \circ G_2$ is given by
$\varphi(G_1 \circ G_2) = \varphi(G_1)\, \varphi(G_2 - w) + \varphi(G_1 - u)\, \varphi(G_2) - x\, \varphi(G_1 - u)\, \varphi(G_2 - w)$(1.1)
We have the following results about the coalescence of graphs:

Theorem 1.2. [6] The coalescence of two singular graphs with nullity η_1 and η_2 coalesced at a core vertex yield a singular graph of nullity $\eta_1 + \eta_2 - 1$.

Theorem 1.3. [7] Let G_1 be a nonsingular graph and G_2 be a singular graph with nullity η_2. If G_1 and G_2 are coalesced at a vertex u of G_1 and a core vertex v of G_2, then the nullity of $G_1 \circ G_2$ is $\eta_2 - 1$.

Theorem 1.4. [7] Let G_1 and G_2 be two singular graphs of order n_1 and n_2 respectively. If$G_1 \circ G_2$is the coalescence of G_1 and G_2 at noncore vertices of null spread -1, then $\eta(G_1 \circ G_2) = \eta_1 + \eta_2 + 1$.

Theorem 1.5. [7] Let G_1 and G_2 be two singular graphs with nullity η_1 and η_2 respectively. The nullity of the coalescence of G_1 and G_2 at noncore vertices of null spread zero is $\eta_1 + \eta_2$.

Theorem 1.6. [7] Let G_1 and G_2 be two singular graphs with nullity η_1 and η_2 respectively. The coalescence of G_1 and G_2 at a core vertex of G_1 and at a noncore vertex (null spread 0 or -1) of G_2 or vice versa yield a singular graph of nullity $\eta_1 + \eta_2 - 1$.

Theorem 1.7. [7] Let G_1 and G_2 be two singular graphs with nullity η_1 and η_2 respectively. The coalescence of G_1 and G_2 at a noncore vertex of null spread zero of G_1 and at a noncore vertex of null spread -1 of G_2 or vice versa yield a singular graph of nullity $\eta_1 + \eta_2$.

Theorem 1.8. [7] Let G_1 be a non singular graph and G_2 be a singular graph with nullity η_2. Then the nullity of the coalescence of G_1 and G_2 with respect to any vertex of G_1 and a noncore vertex of zero null spread of G_2 is η_2.

Theorem 1.9. [7] Let G_1 be a nonsingular graph and G_2 be a singular graph of nullity η_2. Then the nullity of the coalescence of G_1 and G_2 with respect to any vertex u of G_1 and a noncore vertex w of G_2 of null spread -1 is

1. $\eta_2 + 1$, if $G_1 - u$ is singular.
2. η_2 , if $G_1 - u$ is nonsingular.

Theorem 1.10. [8] Let G_1 and G_2 be two nonsingular graphs and G be the colescence of $G_1$and G_2 with respect to a vertex uof G_1 and w ofG_2 . If $G_1 - u$ and $G_2 - w$ are singular, then G is singular.

Theorem 1.11. [8] A singular graph with noncore vertices always satisfies the following conditions.

1. If one ore more neighbours of a noncore vertex v is the only neighbours of another vertex v', then v' will be a noncore vertex.
2. the vertices having core or noncore vertex neighbours whose neighbours are noncore vertices will be noncore vertices.

Theorem 1.12. [7] Let G_1 be a non-singular graph and G_2 be a singular graph with nullity η_2. Let G be the coalescence of G_1 and G_2 with respect to any vertex u ϵ G_1 and a core vertex w of G_2. Then in G the coalesced vertex and its neighbours in G_1 will be noncore vertices of null spread zero or -1 according as $G_1 - u$ is non-singular or singular.

Theorem 1.13. [7] Let G_1 be a non-singular graph and G_2 be a singular graph with nullity η_2. Let G be the coalescence of G_1 and G_2 with respect to any vertex u of G_1 and a noncore vertex w of G_2. If $G_1 - u$ is non-singular, then in G the coalesced vertex and its neighbours in G_1will be noncore vertices of null spread zero or -1 according as w is of null spread zero or -1.

Theorem 1.14. [7] Let G_1 be a non-singular graph and G_2 be a singular graph with nullity η_2. Let G be the coalescence of G_1 and G_2 with respect to any vertex u of G_1 and a noncore vertex w of $G_2$of null spread -1. If $G_1 - u$ is singular, then in G the coalesced vertex is a noncore vertex of null spread -1 and its neighbours in G_1 will be core vertices.

Theorem 1.15. [7] Let G_1 be a non-singular graph and G_2 be a singular graph with nullity η_2. Let G be the coalescence of G_1 and G_2 with respect to any vertex u of G_1 and a noncore vertex w of G_2 of null spread zero. If $G_1 - u$ is singular, then in G the coalesced vertex is a noncore vertex of null spread -1 and its neighbours corresponding to G_1 will be noncore vertices of null spread zero.

Theorem 1.16. [7] Let G_1 and G_2 be two singular graphs and G be the coalescence of them with respect to any vertex u of G_1 and w of G_2.

(i) If G_1 is a core graph and u,w are core vertices, then in G the coalesced vertex and its neighbours corresponding to G_1 are core vertices.

(ii) If G_1 is a core graph, u is a core vertex and w is a noncore vertex of null spread -1, then in G the coalesced vertex is a noncore vertex of null spread -1 and its neighbours corresponding to G_1 are core vertices.

(iii) If G_1 is a core graph, u is a core vertex and w is a noncore vertex of null spread zero, then in G the coalesced vertex is a noncore vertex of null spread zero and its neighbours corresponding to G_1 are core vertices.

Theorem 1.17. [7] Let G_1 and G_2 are two non-singular graphs and G be the coalescence of them with respect to any vertex u of G_1 and w of G_2. If $G_1 -$ u and $G_2 -$ w are singular, then in G the coalesced vertex will be a noncore vertex of null spread -1.

Definition 1.4. [13] Let (K ,u) and (H ,w) are two rooted graphs. The graph obtained by joining u and w by an edge is denoted by KH + uw (See Figure 3).

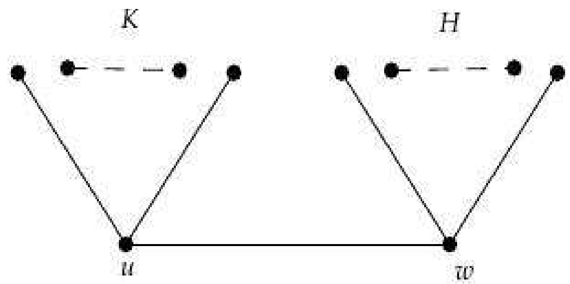

Figure 3: Joining (K ,u) and (H ,w) by an edge uw.

The characteristic polynomial of KH + uw [11] is

$$\varphi(KH + uw) = \varphi(K)\varphi(H) - \varphi(K - u)\varphi(H - w) \tag{1.2}$$

We have the following results:

Theorem 1.18. [13] Let the components of the graph obtained by deleting the edge uw from KH + uw be (K ,u) and (H ,w). If one of the following conditions is satisfied, then KH + uw is singular.

1. One component and its root-deleted subgraph are singular.
2. One component and the root-deleted subgraph of the other component are singular.

Theorem 1.19. [8] Let (K ,u) and (H ,w) be the components of the graph obtained by deleting an edge uw from KH + uw.

1. Let K and H be singular graphs with nullity η_1 and η_2 respectively. If u and w are core vertices of K and H respectively, then nullity of KH + uw is $\eta_1 + \eta_2 - 2$.
2. Let K and H be singular graphs with nullity η_1 and η_2 respectively. If u and w are noncore vertices (of null spread 0 or −1) of K and H respectively, then the nullity of KH + uw is $\eta_1 + \eta_2$.
3. Let K and H be singular graphs with nullity η_1 and η_2 respectively. If u is a core vertex of K and w is a noncore vertex of null spread −1 or vice versa, then the nullity of KH + uw is $\eta_1 + \eta_2$.
4. Let K and H be singular graphs with nullity η_1 and η_2 respectively. If u is core vertex of K and w is a noncore vertex of H of null spread 0 or vice versa, then the nullity of KH + uw is $\eta_1 + \eta_2 - 1$.
5. Let K and H be singular graphs with nullity η_1 and η_2 respectively. If u is a noncore vertex of K of null spread 0 and w is a noncore vertex of H of null spread −1 or vice versa, then the nullity of KH + uw is $\eta_1 + \eta_2$.
6. Let K be singular with nullity $\eta, \eta > 1$ and H be nonsingular. If u is a core vertex and H − w is nonsingular, then nullity of KH + uw is $\eta - 1$.
7. Let K be singular with nullity η, $\eta > 1$ and H be nonsingular. If u is a core vertex and H− w is singular, then nullity of KH + uw is η.
8. Let K be singular with nullity η and H be nonsingular. If u is a noncore vertex (of null spread 0 or −1), then nullity of KH + uw is η.

The spectrum of c_n and p_n are respectively given by

$2\cos(2\pi j/n),\ j = 0, \ldots, n-1$ and $2\cos\left(\frac{\pi j}{n+1}\right), j = 1, \ldots, n.$

The following theorem gives a useful basic property of characteristic polynomial of graphs.

Theorem 1.20. [2] Let uv be an edge of G. Then

$$\emptyset(G) = \emptyset(G-uv) - \emptyset(G-u-v) - 2\sum_{C\epsilon\mathcal{C}(uv)}\emptyset(G-C)$$

where $\mathcal{C}(uv)$ is the set of cycles containing uv. In particular, if uv is a pendant edge with pendant vertex v, then $\emptyset(G) = x\emptyset(G-v) - \emptyset(G-u-v)$.

Gutman in 1978 gave the following definition for energy of a graph

Definition 1.5. [20] If G is a graph on n vertices and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are its eigenvalues, then the energy of G is

$$\mathbf{E} = \mathbf{E}(G) = \sum_{j=1}^{n} \lambda j$$

A graph with energy, $\mathbf{E}(G) < n$, is said to be hypoenergetic and graph for which $\mathbf{E}(G) \geq n$ are called nonhypoenergetic. If $\mathbf{E}(G) < n-1$ and G is connected, G is called strongly hypoenergetic [20].

We have the following basic theorems about energy of graphs.

Theorem 1.21. [20] If the graph G is non-singular, then G is nonhypoenergetic.

Theorem 1.22. [20] Let G and H be two graphs with disjoint vertex sets and G o H be the coalescence of G and H at u ϵ H and v ϵ G. Then $\mathbf{E}(G \text{ o } H) \leq \mathbf{E}(G) + \mathbf{E}(H)$. Equality is attained if and only if either u is an isolated vertex of G or v is an isolated vertex of H or both.

2. Null spread of edges of graphs

In this section first we discuss the null spread of edges of P_n and C_n. First of all we have the obvious result.

Theorem 2.23. A path P_n of n vertices is singular if n is odd and non-singular if n is even.

Theorem 2.24. Let P_n be a path of n vertices and e be an edge of P_n. Then

(i) $\eta_e(P_n) = 0$, if n is odd.
(ii) $\eta_e(P_n) = 1$, if n is odd and e is a pendant edge.
(iii) $\eta_e(P_n) = -1$, if n is even and e is a pendant edge.
(iv) $\eta_e(P_n) = -2$, if n is even and $P_n - e$ has two components having odd number of vertices.
(v) $\eta_e(P_n) = 0$, if n is even and $P_n - e$ has two components having even number of vertices.

Proof: We prove part (ii). If n is odd, by theorem 2.23, we have $(P_n) = 1$.Since e is a pendant edge, its removal result in the removal of pendant vertex. So $P_n - e$ is a path of even number of vertices. Thus $\eta_e(P_n) = (P_n) - (P_{n-1}) = 1 - 0 = 1$.

Similarly, using theorem 2.23 we can prove the other parts too.

Theorem 2.25. A cycle C_n of n vertices is non-singular if and only if n is not divisible by 4.

Theorem 2.26. Let C_n be a cycle of n vertices and e be an edge of C_n. Then

(i) $\eta_e(C_n) = -1$, if and only if n is odd.

(ii) $\eta_e(C_n) = 0$, if and only if n is even and not divisible by 4.

(iii) $\eta_e(C_n) = 2$, if and only if n is divisible by 4.

Proof: (i) If n is odd, $C_n - e$ is a path of odd number of vertices and so $\eta(C_n - e) = 1$.Since $\eta(C_n) = 0$ for odd n, we have $\eta_e(C_n) = 0 - 1 = -1$. Conversely, if $\eta_e(C_n) = -1$, we have $\eta(C_{n-1}) = \eta(C_n) - (-1) = 1$. This is true only if n is odd.
(ii) If n is even and not divisible by 4, then C_n is non-singular. Also $C_n - e$ is a path of even number of vertices and is non-singular. So $\eta_e(C_n) = 0 - 0 = 0$.Conversely $\eta_e(C_n) = 0$ implies that $\eta(C_{n-1}) = \eta(C_n)$. But for a cycle this is true only if $\eta(C_{n-1}) = \eta(C_n) = 0$. So n must be even and not divisible by four.
(iii)If n is even and is divisible by 4, then $\eta(C_n) = 2$. Also $C_n - e$ is a path of even number of vertices and is non-singular. Thus $\eta_e(C_n) = 2 - 0 = 2$. Conversely $\eta_e(C_n) = 2$ implies that $\eta(C_n) - \eta(C_{n-1}) = 2$.This is true only if n is even and is divisible by 4.
Next we will discuss the null spread of the pendant edge of a singular graph.

Theorem 2.27. Let G be a singular graph of nullity η and order n. Suppose that e = uv be a pendant edge of G such that v is a pendant vertex.

(i) If v is a core vertex of G, then $\eta_e(G) = 1$.

(ii) If v is a noncore vertex of null spread zero of G, then $\eta_e(G) = 0$.

(iii) If v is a noncore vertex of null spread -1 of G, then $\eta_e(G) = -1$.

Proof: Deletion of a pendant edge of a graph is same as deletion a pendant vertex.
(i) Since core vertex has null spread one, we have $\eta(G - e) = \eta(G - v) = \eta - 1$. So $\eta_e(G) = \eta - (\eta - 1) = 1$.
(ii) Since v is a noncore vertex of zero null spread, we have $\eta(G\text{-}e) = \eta(G\text{-}v) = \eta(G) = \eta$. So $\eta_e(G) = \eta - \eta = 0$.
(iii) Since v is a noncore vertex of null spread -1,we have $\eta(G-e) = \eta(G-v) = \eta + 1$. So $\eta_e(G) = \eta - (\eta + 1) = -1$.
Our next theorem gives the null spread of the cut edge of a graph G.

Theorem 2.28. Let G be a graph with a cut edge e = uw and K,H are singular components of G – e having nullity η_1 and η_2 respectively. Then

(i) $\eta_e(G) = -2$ if and only if u and w are core vertices of K and H respectively.

(ii) $\eta_e(G) = -1$ if and only if u is a core vertex of K and w is a noncore vertex of H with null spread zero or vice versa .

(iii) If u and w are noncore vertices (of null spread zero or -1) of K and H respectively, then $\eta_e(G) = 0$.

(iv) If u is a core vertex of K and w is a noncore vertex of null spread -1 or vice versa,then $\eta_e(G) = 0$.

(v) If u is a noncore vertex of K of null spread zero and w is a noncore vertex of H with null spread -1 or vice versa, then $\eta_e(G) = 0$.

Proof: (i) By theorem 1.19, $\eta(G) = \eta(HK + uw) = \eta_1 + \eta_2 - 2$.Since the nullity of H and K are respectively η_1 and η_2,it follows that $\eta(G - e) = \eta_1 + \eta_2$.Thus $\eta_e(G) = -2$.

Conversely suppose that $\eta_e(G) = -2$. This means that nullity increases by two when we remove the edge e. It follows now from the construction of the graph HK + uw that u and w are core vertices.
The proof of other parts follows similarlly.

Remark 2.29. The part (iii), (iv) and (v) of the above theorem exhibit three situations in which $\eta_e(G) = 0$. So when$\eta_e(G) = 0$, it is impossible to find the type of end vertices of e = uv in these cases uniquely.Thus if $\eta_e(G) = 0$, then either the conditions in the hypothesis of part (iii) or (iv) or (v) holds.

Theorem 2.30. Let G be a graph with a cut edge e = uw and K, H be the components of G – e.

(i) Let K be singular with nullity η and H nonsingular. Then $\eta_e(G) = -1$ if and only if u is a core vertex and H − w is nonsingular.
(ii) Let K be singular with nullity η and H is nonsingular. If u is a core vertex and H − w is singular, then $\eta_e(G) = 0$.
(iii) Let K be singular with nullity $\eta$and H be nonsingular. If u is a noncore vertex(of null spread 0 or −1), then $\eta_e(G) = 0$.

Proof: (i) Part 6 of theorem 1.19shows that if u is a core vertex and H – w is non-singular, then $\eta_e(G) = -1$. Conversely, when $\eta_e(G) = -1$, the nullity of the graph increases on deleting the edge e. It now follows from the construction of the graph KH + uw that u is a core vertex and H – w is non-singular.
The proof of part (ii) and (iii) follows from part 7 and 8 of theorem 1.19.

Remark 2.31. In part (ii) and (iii), there are two different situations which leads to $\eta_e(G) = 0$. Here also it is impossible to find the type of end vertices u and w of the edge e = uw uniquely when $\eta_e(G) = 0$. So if $\eta_e(G) = 0$, then either the conditions in the hypothesis of part (ii) or part (iii) holds.

3. Concatenation of two graphs

Definition 3.6. Let G_1 and G_2 be two graphs of orders n_1 and n_2 respectively. Then the graph having $e(G_1)+e(G_2) - 1$ edges and n_1+n_2-2 vertices obtained by identifying an edge from G_1 and another from G_2 is called the concatenation or edge gluing of G_1 and G_2.

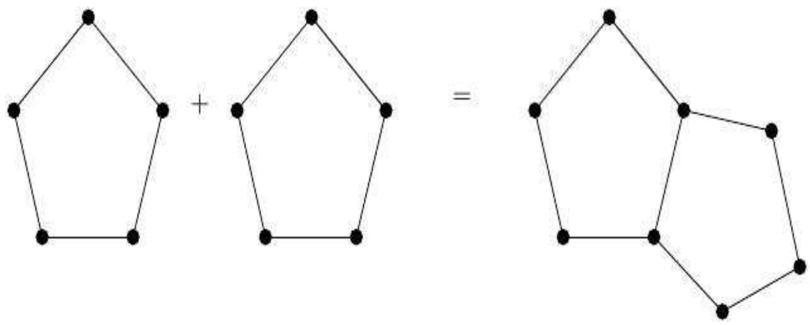

Figure 4: Concatenation of G_1 and G_2.

3.1. Concatenation of paths and cycles

We have the following simple results about the concatenation of paths:

Theorem 3.32. The concatenation of two paths of odd number of vertices concatenated at pendant edges is non-singular.

Proof: The concatenation of two paths of odd number of vertices concatenated at pendant edges is a path of even number of vertices. As paths of even number of vertices are non-singular, the theorem follows.

Theorem 3.33. The concatenation of two paths of even number of vertices concatenated at pendant edges is non-singular.

Proof: The concatenation of two paths of even number of vertices concatenated at pendant edges is a path of even number of vertices. As paths of even number of vertices are non-singular, the theorem follows.

Theorem 3.34. The concatenation of a path of odd number of vertices and a path of even number of vertices concatenated at pendant edges is singular.

Proof: The concatenation of a path of odd number of vertices and a path of even number of vertices concatenated at pendant edges is a path of odd number of vertices. As paths of odd number of vertices are singular, the theorem follows.

Theorem 3.35. The concatenation of two paths of odd number of vertices concatenated at nonpendant edges is either singular with nullity two or non-singular.

Proof: The concatenation of two paths concatenated at nonpendant edges can be regarded as the graph obtained by joining two paths by an edge at nonpendant vertices (see figure 5).Let G be the graph obtained by the concatenation of two paths of odd number of vertices at nonpendant edges. Then G is of the form $P_n P_m$+ uw, where both n and m are either even or odd. If both n and m are even, then by equation (1.2) we see that G is nonsingular. If n and m are odd, then both P_n and P_m are singular graphs of nullity one. There arise three cases. First of all if both u and w are core vertices, then by part 1 of theorem 1.19 we get G is non-singular. If both u and w are noncore vertices of null spread -1, then by part 2 of theorem 1.19 we get G is singular of nullity two. Finally if u is a core vertex and w is a noncore vertex of null spread -1, then by part 3 of theorem 1.19 we see that G is singular with nullity two.

Theorem 3.36. The concatenation of two paths of even number of vertices concatenated at nonpendant edges is either singular with nullity two or non-singular.

Proof is similar to the proof of theorem 3.35.

Theorem 3.37. The concatenation of two paths of even and odd number of vertices concatenated at non pendant edges is singular with nullity one.

Proof: As in the proof of theorem 3.35, the concatenation of two paths concatenated at nonpendant edges can be regarded as the graph obtained by joining two paths by an edge at nonpendant vertices. Let G be the graph obtained by the concatenation of two paths of odd and even number of vertices at nonpendant edges. Then G is of the form $P_n P_m$+ uw, where either n is odd and m is even or n is even and m is odd. Let us fix m as odd and n

as even. SoP_n, P_m are paths of even and odd number of vertices respectively.Since P_n is non-singular such that $P_n - u$ is singular and P_m issingular of nullity one, it follows by part 7 and 8 of theorem 1.19 that G is singular with nullity one.
Next we will discuss the concatenation of two cycles.

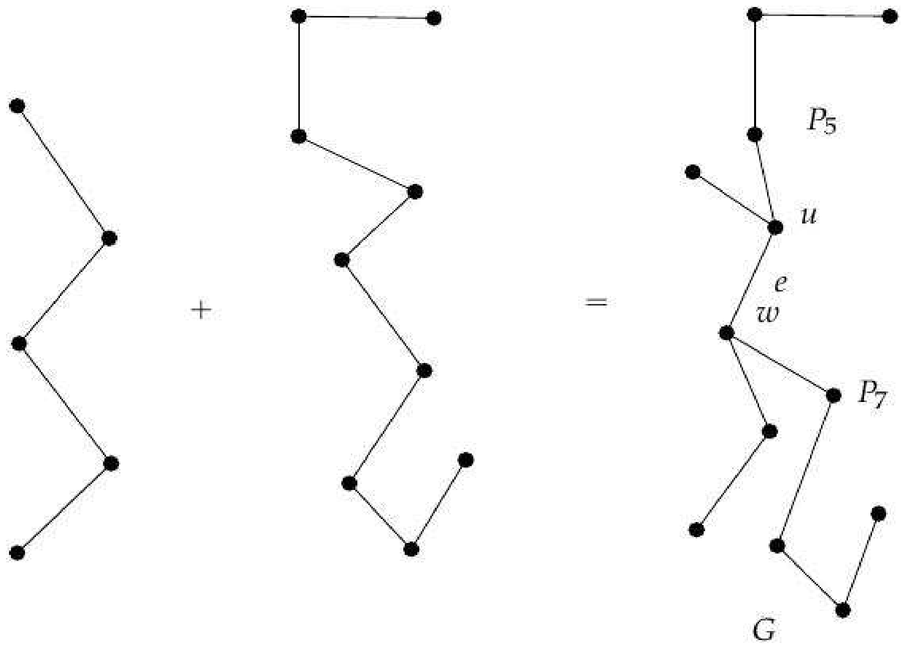

Figure 5: Concatenation of two odd paths at nonpendant edges

Theorem 3.38. Let G be the concatenation of two cycles C_k and C_l, where k + l = n + 2.Then the product of the eigenvalues of G is given by

$$\prod_{j=0}^{n-1} 2\cos\left(\frac{2\pi j}{n}\right) - \prod_{j=1}^{k-2} 2\cos\left(\frac{\pi j}{k-1}\right) \prod_{j=1}^{l-2} 2\cos\left(\frac{\pi j}{l-1}\right) - 2\left[\prod_{j=1}^{k-2} 2\cos\left(\frac{\pi j}{k-1}\right) + \prod_{j=1}^{l-2} 2\cos\left(\frac{\pi j}{l-1}\right)\right]$$

Proof: By theorem 1.20,the characteristic polynomial of G is given by

$$\emptyset(G) = \emptyset(G - uv) - \emptyset(G - u - v) - 2\sum_{C\epsilon\mathcal{C}(uv)} \emptyset(G - C)$$
$$= \emptyset(C_n) - \emptyset(P_{k-2})\emptyset(P_{l-2}) - 2[\emptyset(P_{k-2}) + \emptyset(P_{l-2})]$$

Product of the eigenvalues of G
= The coefficient of x^0 in $\emptyset$ (G)

$$= \prod_{j=0}^{n-1} 2\cos\left(\frac{2\pi j}{n}\right) - \prod_{j=1}^{k-2} 2\cos\left(\frac{\pi j}{k-1}\right) \prod_{j=1}^{l-2} 2\cos\left(\frac{\pi j}{l-1}\right) - 2\left[\prod_{j=1}^{k-2} 2\cos\left(\frac{\pi j}{k-1}\right) + \prod_{j=1}^{l-2} 2\cos\left(\frac{\pi j}{l-1}\right)\right].$$

Example 3.39. The product of eigenvalues of the graph in figure 6 is

$$\prod_{j=0}^{7-1} 2\cos\left(\frac{2\pi j}{7}\right) - \prod_{j=1}^{3} 2\cos\left(\frac{\pi j}{4}\right) \prod_{j=1}^{2} 2\cos\left(\frac{\pi j}{3}\right) - 2\left[\prod_{j=1}^{3} 2\cos\left(\frac{\pi j}{4}\right) + \prod_{j=1}^{2} 2\cos\left(\frac{\pi j}{3}\right)\right]$$

$$= 2\cos(0)\, 2\cos\left(\frac{2\pi}{7}\right) 2\cos\left(\frac{4\pi}{7}\right) 2\cos\left(\frac{6\pi}{7}\right) 2\cos\left(\frac{8\pi}{7}\right) 2\cos\left(\frac{10\pi}{7}\right) 2\cos\left(\frac{12\pi}{7}\right) -$$
$$2\cos\left(\frac{\pi}{4}\right) 2\cos\left(\frac{2\pi}{4}\right) 2\cos\left(\frac{3\pi}{4}\right) 2\cos\left(\frac{\pi}{3}\right) 2\cos\left(\frac{2\pi}{3}\right) -$$

$2[2\cos(\frac{\pi}{4})\, 2\cos(\frac{2\pi}{4})\, 2\cos(\frac{3\pi}{4}) + 2\cos(\frac{\pi}{3}) 2\cos(\frac{2\pi}{3})]$
$= 2 - 2 = 0.$

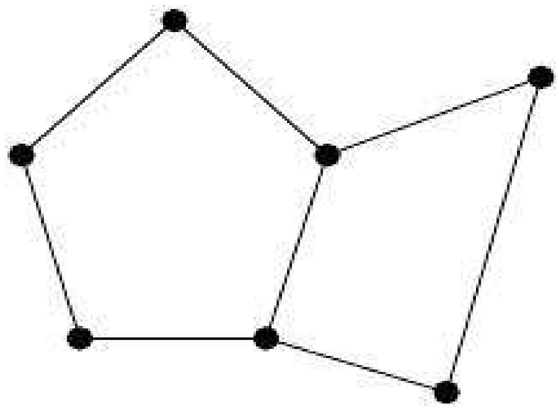

Figure 6: Concatenation of C_5 and C_4

Lemma 3.40. If n_1 is odd and n_2 is a multiple of 4,then
$\prod_{j=0}^{n_1+n_2-3} 2\cos(\frac{2\pi j}{n_1+n_2-2}) = 2\prod_{j=1}^{n_2-2} 2\cos(\frac{\pi j}{n_2-1})$

Theorem 3.41. Let C_{n_1} and C_{n_2} be cycles with n_1 and n_2 vertices respectively. G be the concatenation of C_{n_1} and C_{n_2} with respect to an edge e_1 of C_{n_1} and e_2 of C_{n_2} and e be the concatenated edge.

(i) If $\eta_{e_1}(C_{n_1}) = \eta_{e_2}(C_{n_2}) = 2$, then $\eta_e(G) = 0$.

(ii) If $\eta_{e_1}(C_{n_1}) = 0$ and $\eta_{e_2}(C_{n_2}) = 2$, then $\eta_e(G) = -2$.

(iii) If $\eta_{e_1}(C_{n_1}) = -1$ and $\eta_{e_2}(C_{n_2}) = -1$, then $\eta_e(G) = 1$ or 0 according as n_1+n_2-2 is divisible by 4 or divisible by only 2.

(iv) If $\eta_{e_1}(C_{n_1}) = -1$ and $\eta_{e_2}(C_{n_2}) = 0$(or vice versa), then $\eta_e(G) = 0$.

(v) If $\eta_{e_1}(C_{n_1}) = -1$ and $\eta_{e_2}(C_{n_2}) = 2$(or vice versa), then $\eta_e(G) = 1$.

(vi) If $\eta_{e_1}(C_{n_1}) = 0$ and $\eta_{e_2}(C_{n_2}) = 0$, then $\eta_e(G) = 0$.

Proof: (i) If $\eta_{e_1}(C_{n_1}) = \eta_{e_2}(C_{n_2}) = 2$, then C_{n_1} and C_{n_2} are singular graphs of nullity 2.So n_1 and n_2 are divisible by 4. The concatenated graph has n_1+n_2-2 vertices and G – e is a cycle of n_1+n_2-2 vertices. As n_1+n_2-2 is not divisible by 4, G-e is non-singular i.e. $\eta(G-e) = 0$.By theorem 1.17, the product of the eigenvalues of
$G = \prod_{j=0}^{n_1+n_2-3} 2\cos(\frac{2\pi j}{n_1+n_2-2}) -$
$\prod_{j=1}^{n_1-2} 2\cos(\frac{\pi j}{n_1-1}) \prod_{j=1}^{n_2-2} 2\cos(\frac{\pi j}{n_2-1}) - 2\left[\prod_{j=1}^{n_1-2} 2\cos(\frac{\pi j}{n_1-1}) + \right.$
$\left.\prod_{j=1}^{n_2-2} 2\cos(\frac{\pi j}{n_2-1})\right] \neq 0,$
As $n_1 + n_2 - 2$ is not a multiple of 4 and both $n_1 - 1$ and $n_2 - 1$ are odd numbers. So $\eta(G) = 0$. Hence $\eta_e = 0$.

(ii) If $\eta_{e_1}(C_{n_1}) = 0$ and $\eta_{e_2}(C_{n_2}) = 2$, then n_1 is divisible by 2 and n_2 is divisible by 4. So n_1+n_2-2 is divisible by 4.Since G – e is a cycle of n_1+n_2-2 vertices, we have $\eta_e(G-e) = 2$. The product of the eigenvalues of

$G= \prod_{j=0}^{n_1+n_2-3} 2\cos(\frac{2\pi j}{n_1+n_2-2}) - \prod_{j=1}^{n_1-2} 2\cos(\frac{\pi j}{n_1-1}) \prod_{j=1}^{n_2-2} 2\cos(\frac{\pi j}{n_2-1}) -$
$2\left[\prod_{j=1}^{n_1-2} 2\cos(\frac{\pi j}{n_1-1}) + \prod_{j=1}^{n_2-2} 2\cos(\frac{\pi j}{n_2-1})\right] \neq 0,$

as n_1-1 and n_2-1 are odd numbers. So $\eta_e = 0-2 = -2$.

(iii) If $\eta_{e_1}(C_{n_1}) = -1$ and $\eta_{e_2}(C_{n_2}) = -1$, then n_1 and n_2 are odd numbers. So n_1+n_2-2 is divisible by 2 or 4. If n_1+n_2-2 is divisible by 4, then $G-e$ is singular with nullity 2.i.e. η(G – e) = 2. If n_1+n_2-2 is divisible by 2, then $G-e$ is non-singular and so η(G – e) = 0. If n_1+n_2-2 is divisible by 4,then product of the eigenvalue of

$G=\prod_{j=0}^{n_1+n_2-3} 2\cos(\frac{2\pi j}{n_1+n_2-2}) - \prod_{j=1}^{n_1-2} 2\cos(\frac{\pi j}{n_1-1}) \prod_{j=1}^{n_2-2} 2\cos(\frac{\pi j}{n_2-1}) -$
$2\left[\prod_{j=1}^{n_1-2} 2\cos(\frac{\pi j}{n_1-1}) + \prod_{j=1}^{n_2-2} 2\cos(\frac{\pi j}{n_2-1})\right] = 0$,as n_1+n_2-2 is divisible by 4 and n_1-1, n_2-1 are divisible by 2. Also the coefficient of x in the characteristic polynomial of G is nonzero. On the other hand if n_1+n_2-2 is divisible by 2 only, then the product of the eigenvalues of G is nonzero as $\prod_{j=0}^{n_1+n_2-3} 2\cos(\frac{2\pi j}{n_1+n_2-2}) \neq 0$ and the other products vanishes. So η(G) =1, if n_1+n_2-2 is divisible by 4 and η(G) =0,if n_1+n_2-2 is divisible by 2 only. Hence η_e(G) = 1 – 0 =1, if n_1+n_2-2 is divisible by 4 and η_e(G) = 0 – 0 =1, if n_1+n_2-2 is divisible by only 2.

(iv) if $\eta_{e_1}(C_{n_1}) = -1$ and $\eta_{e_2}(C_{n_2}) = 0$, then n_1 is an odd number and n_2 is an even number not divisible by 4.So n_1+n_2-2 is an odd number. So η(G−e) = 0. The product of the eigenvalues of

$G = \prod_{j=0}^{n_1+n_2-3} 2\cos(\frac{2\pi j}{n_1+n_2-2}) - \prod_{j=1}^{n_1-2} 2\cos(\frac{\pi j}{n_1-1}) \prod_{j=1}^{n_2-2} 2\cos(\frac{\pi j}{n_2-1}) -$
$2\left[\prod_{j=1}^{n_1-2} 2\cos(\frac{\pi j}{n_1-1}) + \prod_{j=1}^{n_2-2} 2\cos(\frac{\pi j}{n_2-1})\right] \neq 0$,as$n_1+n_2-2$ is odd and n_2 is an even number. So η(G) = 0.Henceη_e(G)= 0−0 = 0.

(v) If $\eta_{e_1}(C_{n_1}) = -1$ and $\eta_{e_2}(C_{n_2}) = 2$, then n_1 is an odd number andn_2 is an even number divisible by 4.So n_1+n_2-2 is an odd number. So η(G-e)= 0. The product of the eigenvalues of

$G=\prod_{j=0}^{n_1+n_2-3} 2\cos(\frac{2\pi j}{n_1+n_2-2}) - \prod_{j=1}^{n_1-2} 2\cos(\frac{\pi j}{n_1-1}) \prod_{j=1}^{n_2-2} 2\cos(\frac{\pi j}{n_2-1}) -$
$2\left[\prod_{j=1}^{n_1-2} 2\cos(\frac{\pi j}{n_1-1}) + \prod_{j=1}^{n_2-2} 2\cos(\frac{\pi j}{n_2-1})\right] = 0,$

since $\prod_{j=0}^{n_1+n_2-3} 2\cos(\frac{2\pi j}{n_1+n_2-2}) = 2\prod_{j=1}^{n_2-2} 2\cos(\frac{\pi j}{n_2-1})$, by lemma 3.37. Also the coefficient of x in the characteristic polynomial of G is nonzero. So η(G) = 1.Thus η_e(G) = 1 – 0 = 1.

(vi) If $\eta_{e_1}(C_{n_1}) = 0$ and $\eta_{e_2}(C_{n_2}) = 0$,then both n_1 andn_2 are even numbers not divisible by 4.So n_1+n_2 is aneven number divisible by 4.Then n_1+n_2-2 is an even number divisible by 2 only. Thus η(G-e) = 0. The product of the eigenvalues of G = $\prod_{j=0}^{n_1+n_2-3} 2\cos(\frac{2\pi j}{n_1+n_2-2}) - \prod_{j=1}^{n_1-2} 2\cos(\frac{\pi j}{n_1-1}) \prod_{j=1}^{n_2-2} 2\cos(\frac{\pi j}{n_2-1}) -$

$2\left[\prod_{j=1}^{n_1-2} 2\cos\left(\frac{\pi j}{n_1-1}\right) + \prod_{j=1}^{n_2-2} 2\cos\left(\frac{\pi j}{n_2-1}\right)\right] \neq 0$,as $n_1 + n_2 - 2$ is divisible by 2 only and $n_1 - 1$, $n_2 - 1$ are odd numbers.So $\eta(G) = 0$.Thus $\eta_e(G) = 0 - 0 = 0$.

3.2. Concatenation of two graphs at pendant edges

Concatenation of two graphs with respect to pendant edges is same as joining two graphs by an edge. Figure 7 illustrate this.

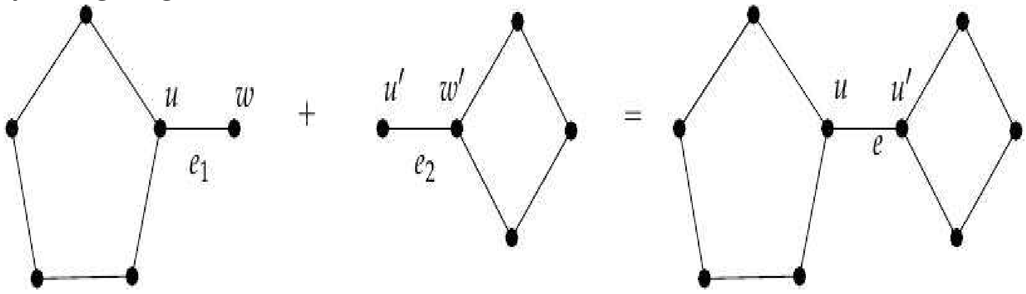

Figure 7: Concatenation of two graphs at pendant edges

The following two theorems can be proved using theorem 1.19.

Theorem 3.42. Let G_1 and G_2 be two singular graphs with nullity η_1 and η_2 respectively and G be the concatenation of G_1 and G_2 with respect to the pendant edgee_1= uw of G_1 and e_2= u'w' of G_2 ,where w and w' are pendant vertices of G_1 and G_2 respectively. Then,

(i) If u and u' are core vertices of G_1 and G_2 respectively, then nullity of G is $\eta_1 + \eta_2 - 2$.
(ii) If u and u' are noncore vertices (of null spread 0 or -1) of G_1 and G_2 respectively, then the nullity of G is $\eta_1 + \eta_2$.
(iii) If u is a core vertex of G_1 and u' is a noncore vertex of null spread -1 of G_2 or vice versa, then the nullity of G is $\eta_1 + \eta_2$.
(iv) If u is core vertex of G_1 and u' is a noncore vertex of G_2 of null spread 0 or vice versa, then the nullity of G is $\eta_1 + \eta_2 - 1$.
(v) If u is a noncore vertex of G_1 of null spread 0 and u'is a noncore vertex of G_2 of null spread -1 or vice versa, then the nullity of G is $\eta_1 + \eta_2$.

Proof: Since G = G_1G_2 +uu', the theorem follows from part 1 to 5 of theorem 1.19

Corollary 3.43. Let G_1, G_2 and G be as in theorem 3.42 and e be the concatenated edge. Then

(i) If u and u' are core vertices of G_1 and G_2 respectively, then $\eta_e(G) = -2$.
(ii) If u and u' are noncore vertices (of null spread 0 or -1) of G_1 and G_2 respectively, then $\eta_e(G) = 0$.
(iii) If u is a core vertex of G_1 and u' is a noncore vertex of null spread -1 of G_2 or vice versa, then $\eta_e(G) = 0$.
(iv) If u is core vertex of G_1 and u' is a noncore vertex of G_2 of null spread 0 or vice versa, then $\eta_e(G) = -1$.

(v) If u is a noncore vertex of G_1 of null spread 0 and u'is a noncore vertex of G_2 of null spread −1 or vice versa, then$\eta_e(G) = 0$.

Theorem 3.44. Let G_1 be a singular graph of nullity η, G_2 be non-singular and G be the concatenation of G_1 and G_2 with respect to the pendant edge e = uw of G_1 and e' = u'w' of G_2 ,where w and w' are pendant vertices of G_1 and G_2 respectively. Then,

(i) If u is a core vertex of $G_1$and $G_2 - u'$ is nonsingular, then nullity of G is η−1.

(ii) If u is a core vertex of $G_1$and $G_2 - u'$ is singular, then nullity of G is η.

(iii) If u is a noncore vertex (of null spread 0 or −1) of G_1, then nullity of G is η.

Proof: Since G = G_1G_2 +uu', the theorem follows from part 6 to 8 of theorem 1.19.

Corollary 3.45. Let G_1, G_2 and G be as in theorem 3.44 and e be the concatenated edge.Then

(i) If u is a core vertex of G_1 and $G_2 - u'$ is nonsingular, then $\eta_e(G) = -1$.

(ii) If u is a core vertex of G_1 and $G_2 - u'$ is singular, then $\eta_e(G) = 0$.

(iii) If u is a noncore vertex (of null spread 0 or −1) of G_1, then $\eta_e(G) = 0$.

Example 3.46. In Figure 8,the graph G_1 is a singular graph with nullity one and G_2 is non-singular.The concatenated graph G is singular of nullity one.Note that u is a noncore vertex of null spread −1.

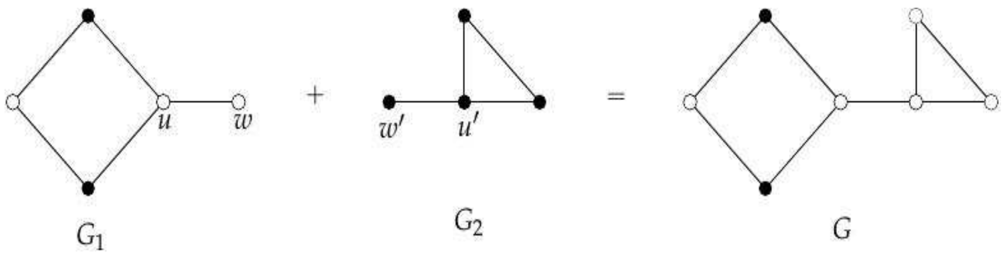

Figure 8: Concatenation of a singular and non-singular graph at pendant edges.

3.3. Concatenation of two graphs at cut edges

Let G_1 be a graph with cut edge e_1=uv, so that $G_1 - e_1$ has components H and K. Similarly, G_2 has cut edgee_2 = u'w' with H' and K' as the components of $G_2 - e_2$. The concatenation, G of G_1 and G_2 with respect to e_1 and $e_2$is same as (HoH′)(KoK′) + vv',where HoH′ and KoK′are the coalescence of H,H′ and K, K′respectively.

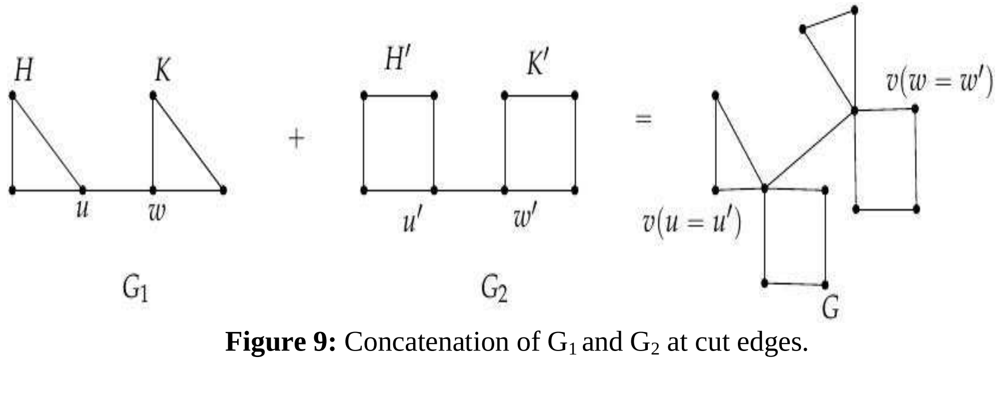

Figure 9: Concatenation of G_1 and G_2 at cut edges.

Theorem 3.47. Let G_1 and G_2be singular graphs with nullity η_1 and $\eta_2$respectively and G be the concatenation of them with respect to their cut edgese_1 = uw and e_2= u'w'. Assume that the components of $G_1 - e_1$ and$G_2 - e_2$ are singular.Then

(i) If $\eta_{e_1}(G_1) = \eta_{e_2}(G_2) = -2$, then G is singular with nullity $\eta_1 + \eta_2$.

(ii) If $\eta_{e_1}(G_1) = -1$ and $\eta_{e_2}(G_2) = -2$, then G is singular with nullity $\eta_1 + \eta_2$.

(iii) If $\eta_{e_1}(G_1) = \eta_{e_2}(G_2) = -1$, then G is singular of nullity $\eta_1 + \eta_2$.

(iv) If $\eta_{e_1}(G_1) = 0$ and $\eta_{e_2}(G_2) = -2$, then G is singular with nullity $\eta_1 + \eta_2$.

(v) If $\eta_{e_1}(G_1) = 0$ and $\eta_{e_2}(G_2) = -1$, then G is singular with nullity $\eta_1 + \eta_2$,provided u,u′are core vertices,w is a noncore vertex of null spread zeroand w′ is a noncore vertex of null spread -1.

(vi) If $\eta_{e_1}(G_1) = 0$ and $\eta_{e_2}(G_2) = -1$, then G is singular with nullity $\eta_1 + \eta_2 - 1$, provided u,w′are corevertices ,u′ is a noncore vertex of null spread zero and w is a noncore vertex of null spread -1.

Proof: (i) Since $\eta_{e_1}(G_1) = \eta_{e_2}(G_2) = -2$, by theorem 2.28 we have u, w, u′, w′ are core vertices. Let K, H are the components of $G_1 - e_1$and K',H' are the components of $G_2 - e_2$. Assume that η_K,η_H,$\eta_{K'}$and $\eta_{H'}$ are the nullities of K, H, K' and H' respectively. By definition, G can be regarded as (KoK′) (HoH′)+vv',where v = u = u' and v'= w = w′are the coalesced vertices. Then by theorem 1.2, KoK′ is singular of nullity $\eta_K+\eta_{K'} -1$ and H o H′ is singular of nullity $\eta_H+\eta_{H'} - 1$. Also v and v′ arecore vertices. So by part 1 of theorem 1.19, we have nullity of G is $\eta_K+\eta_{K'} - 1+\eta_H+\eta_{H'}- 1 - 2= \eta_K + \eta_H - 2 + \eta_{H'} + \eta_{K'}- 2 = \eta_1 + \eta_2$as $\eta_1= \eta_K + \eta_H - 2$ and $\eta_2= \eta_{K'} + \eta_{H'} -2$.

(ii) Since $\eta_{e_1}(G_1) = -1$, by theorem 2.28,we have u is core vertex of K and w is a noncore vertex of H of null spread zero or vice versa. Let us fix u as core vertex of K and v as noncore vertex of H of null spread zero. Also by theorem 2.28, we have u′ and w′ are core vertices of K' and H' respectively as $\eta_{e_2}(G_2) = -2$. By definition, G can be regarded as (KoK′) (HoH′) + vv', where v = u = u' and v' = w = w′are the coalesced vertices. Then we get by theorem 1.2 and 1.6 that K o K′ is a singular graph of nullity $\eta_K+\eta_{K'} -1$ and H o H′ is a singular graph of nullity $\eta_H+\eta_{H'} -1$. Note that the vertex v is a core vertex and v′ is a noncore vertex of null spread zero. So by part 3 of theorem 1.19, we see that nullity of G is $\eta_K+\eta_{K'} -1 + \eta_H+\eta_{H'} - 1 - 1 = \eta_1 + \eta_2$as $\eta_1= \eta_K+ \eta_H - 1$ and $\eta_2= \eta_{K'} + \eta_{H'}$ - 2.

(iii) Since $\eta_{e_1}(G_1)$ = -1 and $\eta_{e_2}(G_2) = -1$, we have by theorem 2.28 that u is a core vertex of K and w is a noncore vertex of H of null spread zero or vice versa. Also u′ is a core vertex of K′ and w′ is a noncore vertex of H′ of null spread zero or vice versa. Let us fix u, u′ are core vertices and w, w′ are noncore vertices of null spread zero. By definition, G can be regarded as (KoK′) (HoH′) + vv', where v = u = u' and v' = w = w′are the coalesced vertices. Then we get by theorem 1.2 and 1.5 that K o K′ is a singular graph of nullity $\eta_K+\eta_{K'} -1$ and H o H′ is a singular graph of nullity $\eta_H+ \eta_{H'}$. Note that v is a core vertex and v′ is anoncore vertex of null spread zero.Then by part 2 of theorem 1.19, we get nullity of G as $\eta_K + \eta_{K'} - 1 + \eta_H + \eta_{H'} - 1 = \eta_1 + \eta_2$ as $\eta_1= \eta_K+ \eta_H - 1$ and $\eta_2= \eta_{K'} + \eta_{H'} - 1$.If we take u,w′ as core vertices and u′,w as noncore vertices of null spread zero we see that nullity of G is again $\eta_1 + \eta_2$.

Proof of part (iv) and (v) follows similarly.

Corollary 3.48. In the above theorem

(i) If $\eta_{e_1}(G_1) = \eta_{e_2}(G_2) = -2$, then the concatenated edge has null spread -2.
(ii) If $\eta_{e_1}(G_1) = -1$ and $\eta_{e_2}(G_2) = -2$, then the concatenated edge has null spread -1.
(iii) If $\eta_{e_1}(G_1) = \eta_{e_2}(G_2) = -1$, then the concatenated edge has null spread -1 or zero according as the pendant vertices of e_1 and e_2 at each end are of same type or not.
(iv) If $\eta_{e_1}(G_1) = 0$ and $\eta_{e_2}(G_2) = -2$, then the concatenated edge has null spread zero.
(v) If $\eta_{e_1}(G_1) = 0$ and $\eta_{e_2}(G_2) = -1$, then the concatenated edge has null spread zero.

The case in which $\eta_{e_1}(G_1) = \eta_{e_2}(G_2) = 0$ is rather complicated. This is because $\eta_e(G) = 0$ for any cut edge e = uw do not uniquely determine the type of vertices u and w. So this case is specially treated in the next theorem.

Theorem 3.49. Let G_1 and G_2 be singular graphs with nullity η_1 and η_2 respectively and G be the concatenation of them with respect to their cut edges e_1= uw and e_2= u'w'. Suppose that $\eta_{e_1}(G_1) = \eta_{e_2}(G_2) = 0$ and the components of $G_1 - e_1$ and $G_2 - e_2$ are singular.

(i) If u, w, u', w' are noncore vertices of null spread zero, then G is singular with nullity $\eta_1 + \eta_2$.
(ii) If u, w, u', w' are noncore vertices of null spread -1, then G is singular with nullity $\eta_1 + \eta_2 + 2$.
(iii) If u,w are noncore vertices with null spread zero and u', w' are noncore vertices having null spread -1 or vice versa, then nullity of G is $\eta_1 + \eta_2$.
(iv) If u, u' are core vertices and w, w' are noncore vertices with null spread -1 or vice versa, then nullity of G is $\eta_1 + \eta_2$.
(v) If u, w' are core vertices and u', w are noncore vertices of null spread -1 or vice versa, then nullity of G is $\eta_1 + \eta_2 - 2$.
(vi) If u, u' are noncore vertices of null spread zero and w, w' are noncore vertices of null spread -1 or vice versa, then nullity of G is $\eta_1 + \eta_2 - 1$.
(vii) If u,w' are noncore vertices of null spread zero and w, u' are noncore vertices of null spread -1 or vice versa, then nullity of G is $\eta_1 + \eta_2$.

Proof: We prove part (vii). The proofs of other parts follows similarly. Let K, H be the components of $G_1 - e_1$ and K',H' are the components of $G_2 - e_2$. Assume that η_K, η_H, $\eta_{K'}$ and $\eta_{H'}$ are the nullities of K, H, K' and H' respectively. By definition, G can be regarded as (KoK') (HoH') + vv', where v = u = u' and v' = w = w' are the coalesced vertices. Since u,w' are noncore vertices of null spread zero and w, u' are noncore vertices of null spread -1, we have K o K' is singular with nullity $\eta_K + \eta_{K'}$ and H o H' is singular with nullity $\eta_H + \eta_{H'}$. Also w and w' are noncore vertices of null spread -1. So by part 5 of theorem 1.19, we have nullity of G is $\eta_K + \eta_{K'} + \eta_H + \eta_{H'} = \eta_K + \eta_H + \eta_{K'} + \eta_{H'} = \eta_1 + \eta_2$ as $\eta_1 = \eta_K + \eta_H$ and $\eta_2 = \eta_{K'} + \eta_{H'}$.

Corollary 3.50. In the above theorem, the concatenated edge has null spread zero.

Example 3.51. The graphs G_1 and G_2 in figure 10 has nullity 3 and 4 respectively. The vertices u,w, u', w' are noncore vertices of null spread -1. Also $\eta_{e_1}(G_1) = \eta_{e_2}(G_2) = 0$. The concatenated graph has nullity 9 and the concatenated edge has null spread zero.

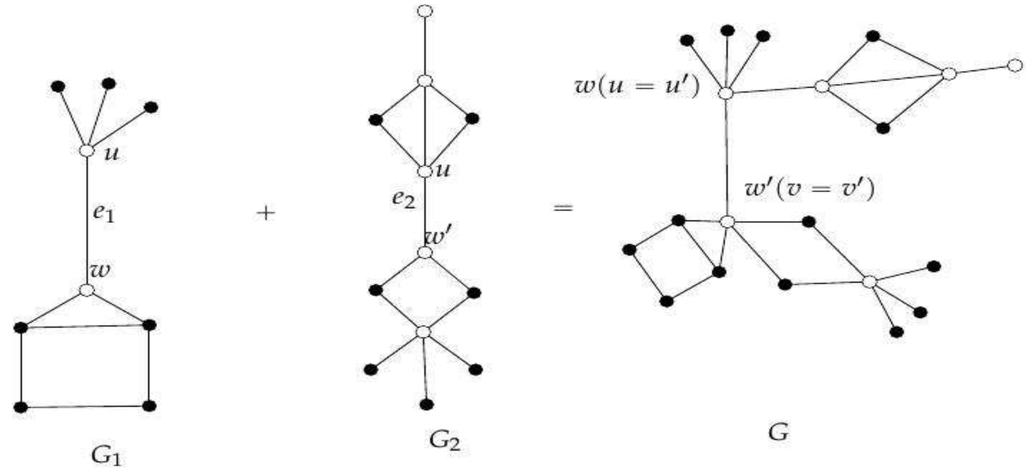

Figure 10: Concatenation of graphs with respect to cut edges.

Theorem 3.52. Let G_1 and G_2 be singular graphs having cut edges with nullities η_1 and η_2 respectively and G be the concatenation of them with respect to their cut edges e_1= uw and e_2 = u'w'. Suppose that one of the components K, H of $G_1 - e_1$ and K', H' of $G_2 - e_2$ is singular and the other is non-singular

(i) If $\eta_{e_1}(G_1) = \eta_{e_2}(G_2) = -1$ and the singular components of both $G_1 - e_1$ and $G_2 - e_2$ are on same side of the concatenated edge of G, then G is singular with nullity $\eta_1 + \eta_2$.

(ii) If $\eta_{e_1}(G_1) = \eta_{e_2}(G_2) = -1$ and the singular components of both $G_1 - e_1$ and $G_2 - e_2$ are on either side of the concatenated edge of G, then G is singular with nullity $\eta_1 + \eta_2$.

(iii) If $\eta_{e_1}(G_1) = -1$, $\eta_{e_2}(G_2) = 0$, the singular components K of $G_1 - e_1$ and K' of $G_2 - e_2$ are on same side of the concatenated edge of G, u' is a core vertex and $H' - w'$ is singular for non-singular component H' of $G-e_2$, then G is singular with nullity $\eta_1 + \eta_2$.

(iv) If $\eta_{e_1}(G_1) = -1$, $\eta_{e_2}(G_2) = 0$, the singular components K of $G_1 - e_1$ and H' of $G_2 - e_2$ are on either side of the concatenated edge of G, w' is a core vertex and $K' - u'$ is singular for non-singular component K' of $G_2 - e_2$, then G is singular with nullity $\eta_1 + \eta_2 - 1$.

(v) If $\eta_{e_1}(G_1) = -1$, $\eta_{e_2}(G_2) = 0$, the singular components K of $G_1 - e_1$ and K' of $G_2 - e_2$ are on same side of the concatenated edge of G and u' is a noncore vertex of null spread zero or -1, then G is singular with nullity $\eta_1 + \eta_2$.

(vi) If $\eta_{e_1}(G_1) = -1$, $\eta_{e_2}(G_2) = 0$, the singular components K of $G_1 - e_1$ and H' of $G_2 - e_2$ are on either side of the concatenated edge of G and w' is a noncore vertex of null spread 0 or -1 then G is singular with nullity $\eta_1 + \eta_2$.

Proof: (i) Let K, H are the components of $G_1 - e_1$ and K', H' are the components of $G_2 - e_2$. Suppose that K, K' are singular and H, H' are nonsingular. Since $\eta_{e_1}(G_1) = \eta_{e_2}(G_2) = -1$, by theorem2.30, we have u, u' are core vertices and H – w, $H' - w'$ are non-singular graphs. Assume that η_K, $\eta_{K'}$ are the nullities of K,K' respectively. By definition, G can be regarded as (KoK′) (HoH′) + vv′,where v = u = u′ and v′ = w = w′are the coalesced vertices. Theorem 1.2 shows thatKo K′ is singular with nullity $\eta_K + \eta_{K'} - 1$ and H o H′ is non-singular. Also $v$is a core vertex and H o H′ − v′ is non-singular. So by part 6 of theorem 1.19, we have nullity of G is$\eta_K + \eta_{K'} - 2 = \eta_K - 1 + \eta_{K'} - 1 = \eta_1 + \eta_2as\eta_1 = \eta_K - 1$ and $\eta_2 = \eta_{K'} - 1$ (Theorem 1.19).

(ii) Let K, H are the components of $G_1 - e_1$andK′, H′ are the components of $G_2 - e_2$. Suppose that K, H′ are singular and H, K′ are nonsingular. Since $\eta_e(G_1) = \eta_{e'}(G_2) = -1$,by theorem 2.30, we have u, w′are core vertices and H – w, K′ − u′are non-singular graphs. Assume that η_K, $\eta_{H'}$are the nullities of K, H' respectively. By definition, G can be regarded as (KoK′) (HoH′) + vv′, where v = u = u′ and v′ = w = w′are the coalesced vertices. Theorem1.3 shows that Ko K′ is singular with nullity $\eta_K - 1$ and H o H′ is singular with nullity $\eta_{H'} - 1$. Note that v and v′ are noncore vertices of null spread zero. So by part 2 of theorem 1.19, we have nullity of G is $\eta_K - 1 + \eta_{H'} - 1 = \eta_1 + \eta_2$as $\eta_1 = \eta_K - 1$ and $\eta_2 = \eta_{K'} - 1$ (Theorem 1.19).

Next we prove (vi). The proofs of other parts follow similarly.

(vi) Let K, H are the components of $G_1 - e_1$andK′, H′ are the components of $G_2 - e_2$. Suppose thatK, H′ are singular and H, K′ are nonsingular. Since $\eta_{e_1}(G_1) = -1$ and the singular components of both $G_1 - e_1$ and $G_2 - e_2$ are on either side of G, by theorem 2.30 we haveu is a core vertex and H − $w$is nonsingular. Assume that η_K, $\eta_{H'}$ are the nullities of K, H' respectively. By definition, G can be regarded as (KoK′) (HoH′) + vv′, where v = u = u′ and v′ = w = w′are the coalesced vertices. Theorem 1.3,1.8 and 1.9 shows that Ko K′ is singular of nullity $\eta_K - 1$ and H o H′ is singular of nullity $\eta_{H'}$. Thenv′ is a noncore vertex of null spread zero or −1 according as w′ is a noncore vertex of null spread zero or −1 and v is a noncore vertex of null spread zero. So part 2 of theorem 1.19 shows that nullity of G is $\eta_K + \eta_{H'} - 1 = \eta_1 + \eta_2$ as $\eta_K - 1 = \eta_1$ and $\eta_{H'} = \eta_2$(theorem 1.19).

Corollary 3.53. In the above theorem,

(i) If $\eta_{e_1}(G_1) = \eta_{e_2}(G_2) = -1$ and the singular components of both $G_1 - e_1$ and $G_2 - e_2$ are on one side of the concatenated edge of G, then the concatenated edge has null spread −1.

(ii) If $\eta_{e_1}(G_1) = \eta_{e_2}(G_2) = -1$ and the singular components of both $G_1 - e_1$ and $G_2 - e_2$ are on either side of the concatenated edge of G, then the concatenated edge has null spread zero.

(iii) If $\eta_{e_1}(G_1) = -1$, $\eta_{e_2}(G_2) = 0$, then the concatenated edge has null spread zero.

The case of $\eta_{e_1}(G_1) = \eta_{e_2}(G_2) = 0$ is separately treated in the next theorem.

Theorem 3.54. Let G_1 and G_2 be singular graphs with nullities η_1 and η_2 respectively and G be the concatenation of them with respect to their cut edges e_1 = uw and e_2 = u′w′. Suppose that one of the components K, H of $G_1 - e_1$and K′, H′ of $G_2 - e_2$ is singular and the other is non-singular.

(i) If $\eta_{e_1}(G_1) = \eta_{e_2}(G_2) = 0$, the singular components of both $G_1 - e_1$ and $G_2 - e_2$ are on same side of the concatenated edge of G and both e_1, e_2 are cut edges which satisfies the hypothesis in part 2 of theorem 2.30, then G is singular with nullity $\eta_1 + \eta_2$.
(ii) If $\eta_{e_1}(G_1) = \eta_{e_2}(G_2) = 0$, the singular components of both $G_1 - e_1$ and $G_2 - e_2$ are on either side of the concatenated edge of G and both e_1, e_2 are cut edges which satisfies the hypothesis in part 2 of theorem 2.30, then G is singular with nullity $\eta_1 + \eta_2 - 2$.
(iii) If $\eta_{e_1}(G_1) = \eta_{e_2}(G_2) = 0$, the singular components of both $G_1 - e_1$ and $G_2 - e_2$ are on same side of the concatenated edge of G and both e_1, e_2 are cut edges which satisfies the hypothesis in part 3 of theorem 2.30, then G is singular with nullity $\eta_1 + \eta_2 + 1$.
(iv) If $\eta_{e_1}(G_1) = \eta_{e_2}(G_2) = 0$, the singular components of both $G_1 - e_1$ and $G_2 - e_2$ are on either side of the concatenated edge of G and both e_1, e_2 are cut edges which satisfies the hypothesis in part 3 of theorem 2.30, then G is singular with nullity $\eta_1 + \eta_2$.

Proof: (i) Since the singular components of both $G_1 - e_1$ and $G_2 - e_2$ are on one side of the concatenated edge of G and both e_1, e_2 are cut edges which satisfies the hypothesis in part 2 of theorem 2.30, assume that K, K' are singular and H, H' are nonsingular. Then u, u' are core vertices and H – w, H'- w' are singular. Let η_K, $\eta_{K'}$ be the nullities of K, K' respectively. By definition, G can be regarded as (KoK′) (HoH′) + vv', where $v = u = u'$ and v' = w = w′ are the coalesced vertices. Theorem 1.3 and 1.10 shows that Ko K′ is singular of nullity $\eta_K + \eta_{K'} - 1$ and H o H′ is singular of nullity one. The coalesced vertex w is a core vertex and w′ is a noncore vertex of null spread -1. So part 3 of theorem 1.19 shows that nullity of G is $\eta_K + \eta_{K'} - 1 + 1 = \eta_1 + \eta_2$ as $\eta_K = \eta_1$ and $\eta_{K'} = \eta_2$.

Next we prove (iv). The proof of other parts follow similarly.

(iv) Since the singular components of both $G_1 - e_1$ and $G_2 - e_2$ are on either side of the concatenated edge of G and both e_1, e_2 are cut edges which satisfies the hypothesis in part 3 of theorem 2.30, assume that K, H' are singular and H, K' are non-singular. Then u,w' are noncore vertices of null spread -1 or zero and H – w, $K' -$ u' are nonsingular. Let η_K, $\eta_{H'}$ be the nullities of K, H' respectively. By definition, G can be regarded as (KoK′) (HoH′) + vv', where v = u = u' and v' = u = u′ are the coalesced vertices. Theorem 1.8 and 1.9 shows that Ko K′ is singular with nullity η_K and H o H′ is singular with nullity $\eta_{H'}$. Note that the coalesced vertices v and v′ are noncore vertices of null spread zero or -1 according as u and w' are noncore vertices of null spread zero or -1. So part 3 of theorem 1.19 shows that nullity of G is $\eta_K + \eta_{H'} = \eta_1 + \eta_2$ as $\eta_K = \eta_1$ and $\eta_{H'} = \eta_2$.

Corollary 3.55. In the above theorem, if $\eta_e(G_1) = \eta_{e'}(G_2) = 0$, then the concatenated edge has null spread zero.

Example 3.56. The graph G in figure 11 is the concatenation of G_1 and G_2 with respect to their cut edges e_1 and e_2 respectively. Here $\eta_{e_1}(G_1) = -1$ and $\eta_{e_2}(G_2) = 0$. The nullities of G_1 and G_2 are two. We have concatenated G_1 and G_2 with respect to their cut edges e_1 and e_2 in such a way that the singular components of both G_1 and G_2 are either sides of the concatenated edge of G. Here u and u′ core vertices. H−w is non-singular and H' − w' is singular. Note that the nullity of G is $2 + 2 - 1 = 3$. This is what we have said in part (iv) of theorem 3.49.

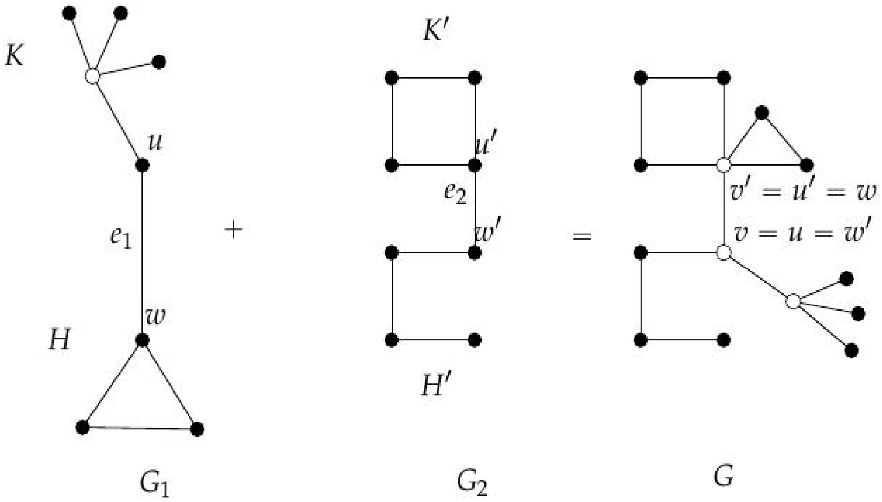

Figure 11: Concatenation of graphs with respect to cut edges.

3.4. Concatenation of a cycle and a graph having cut edge

Theorem 3.57. Let G_1 be a cycle with $\eta_{e_1}(G_1) = -1$, G_2 be a singular graph with nullity η having a cut edge $e_2 = u'w'$ and the components of $G_2 - e_2$ are singular. Let G be the concatenation of G_1 and G_2 with respect to e_1 and e_2.

(i) If $\eta_{e_2}(G_2) = -2$, then G is singular with nullity $\eta + 1$.

(ii) If $\eta_{e_2}(G_2) = -1$, then G is singular with nullity η.

(iii) If $\eta_{e_2}(G_2) = 0$ and u', w' are noncore vertices of null spread zero, then G is singular with nullity η.

(iv) If $\eta_{e_2}(G_2) = 0$ and u', w' are noncore vertices of null spread -1, then G is singular with nullity $\eta + 1$.

(v) If $\eta_{e_2}(G_2) = 0$, u' is a core vertex and w' is a noncore vertex of null spread -1, then G is singular with nullity $\eta - 1$.

(vi) If $\eta_{e_2}(G_2) = 0$, u' is a noncore vertex of null spread zero and w' is a noncore vertex of null spread -1, then G is singular with nullity η.

Proof: We prove part (v). The proof of other parts follow similarly. Since G_1 is a cycle with $\eta_{e_1}(G_1) = -1$, we see that G_1 is a cycle of odd number of vertices. So G_1 is nonsingular. Let K and H be singular components of $G_2 - e_2$ having nullities η_K and η_H respectively. Given that u' is a core vertex and w' is a noncore vertex of null spread -1. Let $e_1 = uw$. The concatenation of G_1 and G_2 with respect to $e_1 = uw$ and $e_2 = u'w'$ is same as taking coalescence of G_1 with K and H with respect to the end vertices of e_1 and e_2 (as in figure 12). Suppose that u' is the root of K and w' is the root of H. First coalesce G_1 and K with respect to u and u'. Since u' is a core vertex, $G_1 \circ K$ is a singular graph with nullity $\eta_K - 1$, by theorem 1.3. After coalescence the vertex w of G_1 becomes a noncore vertex of null spread zero (theorem 1.12). Next take coalescence of $G_1 \circ K$ and H with respect to w and w'. As w is a noncore vertex of null spread zero and w' is a noncore vertex of null spread -1, we see by theorem 1.7 that $(G_1 \circ K) \circ H$ is a singular graph with nullity $\eta_K - 1 + \eta_H = \eta - 1$, where $\eta_K + \eta_H = \eta$.

Corollary 3.58. In the above theorem the concatenated edge has null spread zero.

Theorem 3.59. Let G_1 be a cycle with $\eta_{e_1}(G_1) = -1$ andG_2 be a singulargraph with nullity η with a cut edge e_2= $u'w'$. Assume that one component K of $G_2 - e_2$ is singular with nullity η and other component H is non-singular.

(i) If $\eta_{e_2}(G_2) = -1$, then G is singular with nullity $\eta - 1$.

(ii) If $\eta_{e_2}(G_2) = 0$,u' is a core vertex and $H - w'$ is singular, then G is singular with nullity $\eta - 1$.

(iii) If $\eta_{e_2}(G_2) = 0$ and u' is a noncore vertex (of null spread 0 or -1), then G is singular with nullity η.

Proof: Similar to theorem 3.57.

Corollary 3.60. In the above theorm the concatenated edge has null spread zero.

Example 3.61. The graph G in figure 12 is the concatenation of $G_1$and G_2 concatenated with respect to the edges $e_1$of G_1 and e_2 of G_2.The nullity of the graph G_2 is two. Also $\eta_{e_1}(G_1) = -1$ and $\eta_{e_2}(G_2) = 0$ with u' is a noncore vertex of null spread zero and w' is a noncore vertex of null spread-1. The nullity of the concatenated graph G is two.

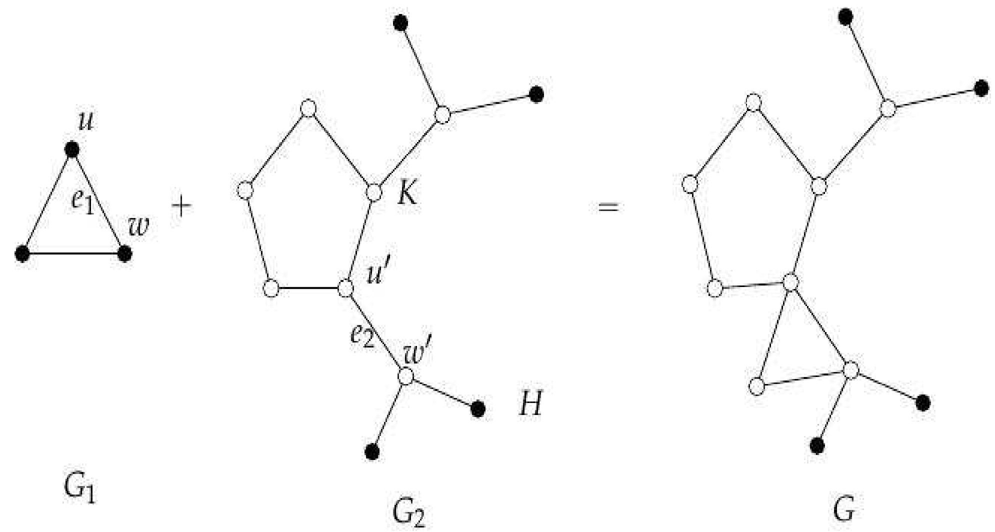

Figure 12: Concatenation of an odd cycle and a graph with a cut edge.

Theorem 3.62. Let G_1 be a cycle with $\eta_{e_1}(G_1) = 0$,G_2 be a singular graph with nullity ηhavinga cut edge$e_2 = u'w'$and the components of $G_2 - e_2$are singular. Let G be the concatenation of G_1 and G_2 with respect to $e_1$and e_2.

(i) If $\eta_{e_2}(G_2) = -2$,then G is singular with nullity η.

(ii) If $\eta_{e_2}(G_2) = -1$,then G is singular with nullity η.

(iii) If $\eta_{e_2}(G_2) = 0$ and u', w' are noncore vertices of null spread zero, then G is singular with nullity η.

(iv) If $\eta_{e_2}(G_2) = 0$ and u', w' are noncore vertices of null spread -1, then G is singular with nullity η.

(v) If $\eta_{e_2}(G_2) = 0$, u' is a core vertex and w' is a noncore vertex of null spread -1,then G is singular with nullity η.

(vi) If $\eta_{e_2}(G_2) = 0$, u' is a noncore vertex of null spread zero and w' is a noncore vertex of null spread -1, then G is singular with nullity η.

Proof: We prove only part (vi). The proof of other parts follow similarly. Since G_1 is a cycle with $\eta_{e_1}(G_1)= 0$, we see that $|G_1|$ = n, where n is an even number not divisible by four. So $G_1$is nonsingular. Let K and H be singular components of $G_2 - e_2$ having nullities η_K and η_H respectively. Given that u' is a noncore vertex of null spread zero and w' is a noncore vertex of null spread -1. Let e_1= uw. The concatenation of G_1 and G_2 with respect to $e_1 = uw$ and $e_2 = u'w'$ is same as taking coalescence of G_1 with K and H with respect to the end vertices of e_1 and e_2(as in figure 12). Suppose that $u'$is the root of K and w' is the root of H. First coalesce G_1 and K with respect to u and u'. Since u' is a noncore vertex of null spread zero, G_1o K is a singular graph with nullity η_K, by theorem 1.8.After coalescence the vertex w of G_1 becomes a noncore vertex of null spread zero (theorem 1.15). Next take coalescence of G_1o K and H with respect to w and w'. As w is a noncore vertex of null spread zero and w' is a noncore vertex of null spread -1, we see by theorem 1.7 that (G_1o K) o H is a singular graph with nullity $\eta_K + \eta_H = \eta$.

Corollary 3.63. In the above theorm the concatenated edge has null spread zero.

Theorem 3.64. Let G_1 be a cycle with $\eta_{e_1}(G_1) = 0$ and G_2 be a singular graph with nullity η having a cut edge $e_2 = u'w'$. Assume that one component K of $G_2 - e_2$ is singular with nullity η, $\eta > 1$ and other component H is non-singular.

(i) If $\eta_{e_2}(G_2) = -1$, then G is singular with nullity $\eta - 1$.

(ii) If $\eta_{e_2}(G_2) = 0$,u' is a core vertex and $H - w'$ is singular, then G is singular with nullity $\eta - 1$.

(iii) If $\eta_{e_2}(G_2) = 0$ and u' is a noncore vertex (of null spread 0 or-1), then G is singular with nullity η.

Proof: Similar to the proof of theorem 3.57.

Corollary 3.65. In the above theorm the concatenated edge has null spread zero.

Theorem 3.66. Let G_1 be a cycle with $\eta_{e_1}(G_1) = 2$, G_2 be a singular graph with nullity η having a cut edge $e_2 = u'w'$ and the components of $G_2 - e_2$ are singular. Let G be the concatenation of G_1 and G_2 with respect to $e_1$and e_2.

(i) If $\eta_{e_2}(G_2) = -2$, then G is singular with nullity $\eta + 2$.

(ii) If $\eta_{e_2}(G_2) = -1$, then G is singular with nullity $\eta + 1$.

(iii) If $\eta_{e_2}(G_2) = 0$ and u', w' are noncore vertices of null spread zero, then G is singular with nullity η.

(iv) If $\eta_{e_2}(G_2) = 0$ and u', w' are noncore vertices of null spread -1, then G is singular with nullity η.
(v) If $\eta_{e_2}(G_2) = 0$, u' is a core vertex and w' is a noncore vertex of null spread -1,then G is singular with nullity η.
(vi) If $\eta_{e_2}(G_2) = 0$, u' is a noncore vertex of null spread zero and w' is a noncore vertex of null spread -1, then G is singular with nullity η.

Corollary 3.67. In the above theorm, if $\eta_{e_2}(G_2) = -1$, then the concatenated edge has null spread one and in all other cases the concatenated edge has null spread zero.

Theorem 3.68. Let G_1 be a cycle with $\eta_{e_1}(G_1) = 2$ and G_2 be a singular graph with nullity η with a cut edge $e_2 = u'w'$. Assume that one component K of $G_2 - e_2$ is singular with nullity η and the other component H is non-singular.

(i) If $\eta_{e_2}(G_2) = -1$, then G is singular with nullity η.
(ii) If $\eta_{e_2}(G_2) = 0$,u' is a core vertex and $H - w'$ is singular, then G is singular with nullity η.
(iii) If $\eta_{e_2}(G_2) = 0$ and u' is a noncore vertex (of null spread 0 or -1), then G is singular with nullity η.

Proof: Similar to the proof of theorem 3.57.

Corollary 3.69. In the above theorm, if $\eta_{e_2}(G_2) = -1$, then the concatenated edge has null spread one and in all other casesthe concatenated edge has null spread zero.

We conclude this section with the following two results about the energy of graphs.

Theorem 3.70. Let G_1 be a singular graph having a cycle andG_2 be a singular graph with a cut edge $e_2 = u'w'$and the components of $G_2 - e_2$ are singular. Let G be the concatenation of G_1 and G_2 with respect to an edge $e_1$of the cycle of $G_1$and e_2 of G_2. If G_1 is hypoenergetic and the components of $G_2 - e_2$ are strongly hypoenergetic, then G is hypoenergetic.

Proof: Let $|G_1| = n_1$ and $|G_2| = n_2$. Let K, H are the components of $G_1 - e_2$.The concatenation of G_1 and G_2 with respect to $e_1$and e_2 is same as taking coalescence of G_1 with K and H with respect to the end vertices of e_1 and e_2. So by theorem 1.22, we have $\mathbf{E}(G) \leq \mathbf{E}(G_1) + \mathbf{E}(K) + \mathbf{E}(H) < |G_1| + |K| - 1 + |H| - 1 = n_1 + n_2 - 2$.

Theorem 3.71. Let G_1 and G_2 be singular graphs with nullity η_1 and η_2 respectively and G be the concatenation of them with respect to their cut edges $e_1 = uw$ and $e_2 = u'w'$. If the components of $G_1 - e_1$ and $G_2 - e_2$ are strongly hypoenergetic, then G is hypoenergetic.

Proof: Let $|G_1| = n_1$ and $|G_2| = n_2$. Let K, H be the components of $G_1 - e_1$ and K′, H′ be the components of $G_2 - e_2$. The concatenation, G of G_1 and G_2 with respect to e_1 and e_2 is same as $(HoH')(KoK') + vv'$, where v, v' are the coalesced vertices and HoH', KoK' are the coalescence of H,H′ and K, K′respectively. So by theorem 1.22,$\mathbf{E}$(G) ≤ $\mathbf{E}$(K) + $\mathbf{E}$(H) + $\mathbf{E}$(K') +$\mathbf{E}$(H') + $\mathbf{E}(K_2) < |K| - 1 + |H| - 1 + |K'| - 1 + |H'| - 1 + 2 = n_1 + n_2 - 2$.

4.Conclusion

Theory of large graphs are widely applicable not only in mathematics but also in computer science, statistical physics, biology, engineering, and many other fields. Concatenation or edge gluing is a technique used in the construction of larger graphs. In this paper we made a humble attempt to construct a theoretical basis for the study of concatenation of graphs. Some of the basic results are stated and proved using the techniques we have developed in our earlier research. There remains several areas to be explored in the study of spectral properties of concatenated graphs both theoretical and applied.

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