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Modified Hermite Interpolation on the Unit Circle

***Swarnima Bahadur*[1] and *Sariya Bano*[2]**
Department of Mathematics and Astronomy, University of Lucknow
Lucknow -226007, Lucknow, India, [1]Email: swarnimabahadur@ymail.com
[2]Email: sariya0507@gmail.com
[1]Corresponding author.

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Abstract. In this paper, we consider modified Hermite interpolation on the nodes, which are obtained by projecting vertically the zeros of the $(1-x^2)P_n(x)P_n'(x)$ onto the unit circle, where $P_n(x)$ stands for n^{th} Legendre polynomial. We obtain the explicit forms and establish a convergence theorem for that interpolatory polynomial.

Keywords: Legendre polynomial, explicit representation, convergence.

Mathematics Subject Classification: 41A05, 30E10.

1.Introduction

In 1990, Tu [10] obtained the divergence and mean convergence of the Hermite interpolation operator. Further, in 1991, Wang and Tian [12] considered the zeros of $(1-x^2)P_{n-1}'(x)$, where, $P_{n-1}'(x)$ is the derivative of $(n-1)^{th}$ Legendre polynomial and obtained the estimates for the same. In 1992, Min [8] obtained the mean convergence of the derivatives of Hermite interpolation operator $H_n(f,x)$ based on the zeros of the Chebyshev polynomial of the first kind. Also, Vertesi and Xu [11] considered the Hermite interpolating polynomial $H_{nm}(\omega,f)$ be defined at the zeros of the Jacobi polynomial $p_n(\omega,x)$, which are orthogonal on $[-1,1]$ with weight function,

$$\omega(x)=(1-x)^{\alpha}(1-x)^{\beta}\,,(\alpha,\beta>-1)\,.$$

Later on, Goodman et al. [6] considered the behavior of Hermite interpolanton the roots of unity. In 1998, author[1](with Mathur) [1] proved the convergence of Quasi-Hermite interpolation on the nodes obtained by projecting vertically the zeros of $(1-x^2)\,P_n(x)$ onto the unit circle, where $P_n(x)$ stands for the n^{th} Legendre polynomial. In another paper author[1] [3] considered the convergence of Hermite interpolation polynomial on the unit circle. Also, Berriochoa at al. [4] studied the convergence of the Hermite-Fejér and the Hermite interpolation on polynomials, which are constructed by taking equally spaced nodes on the unit circle. In 2014, author[1] and (with Shukla) [2] considered Hermite-interpolation on the nodes, which are vertically projected on the zeros of $(1-$

$x^2)P_n^{(\alpha,\beta)}(x)$ the onto the unit circle, where $P_n^{(\alpha,\beta)}(x)$ stands for Jacobi polynomial, obtained the explicit forms and established a convergence theorem for the interpolatory polynomial. Further, Berriochoa at al. [5] studied generalise Hermite interpolation problems on the unit circle considering nodal points equally spaced and using the values for the first two derivatives.

These have motivated us to consider different types of Hermite interpolation on some set of nodes on the unit circle .In this paper, we consider the non- uniformly distributed zeros on the unit circle, which are obtained by projecting vertically the zeros of $(1-x^2)P_n(x)P_n'(x)$ onto the unit circle, where $P_n(x)$ stands for n^{th} Legendre polynomial . We obtain the explicit forms of the interpolatory polynomials and establish a convergence theorem for the same. In section 2, we give some preliminaries and in section 3, we describe the problem and its existence. In section 4, we give the explicit formulae of the interpolatory polynomials. In sections 5 and 6, estimation and convergence of interpolatory polynomials are given, respectively.

2. Preliminaries

In this section, we shall give some well-known results, which we shall use.

The differential equation satisfied by $P_n(x)$is :

(2.1) $(1-x^2)P_n''(x)-2xP_n'(x)+n(n+1)P_n(x)=0$

(2.2) $R_{2n}(z)=\prod_{k=1}^{2n}(z-z_k)=K_nP_n\left(\frac{1+z^2}{2z}\right)z^n$

(2.3) $R_{4n-2}(z)=\prod_{k=1}^{4n-2}(z-z_k)=K_n^*P_n\left(\frac{1+z^2}{2z}\right)P_n'\left(\frac{1+z^2}{2z}\right)z^{2n-1}$

(2.4) $R_{4n}(z)=(z^2-1)R_{4n-2}(z)$

We shall require the fundamental polynomial of Lagrange interpolation based on the zeros of$R_{4n}(z)$ and $R_{2n}(z)$ are respectively given as:

(2.5) $L_k(z)=\frac{R_{4n}(z)}{(z-z_k)R_{4n}'(z_k)}$, $k=0(1)4n-1$

(2.6) $l_k(z)=\frac{R_{2n}(z)}{(z-z_k)R_{2n}'(z_k)}$, $k=1(1)2n$

We will also use the following results

For, $k=1(1)n$

(2.7) $\begin{cases} R_{2n}'(z_k)=\frac{K_n}{2}\left(z_k^2-1\right)z_k^{n-2}P_n'(x_k) \\ R_{2n}'(z_{n+k})=\frac{K_n}{2}\left(z_{n+k}^2-1\right)z_{n+k}^{n-3}P_n''(x_k) \end{cases}$

(2.8) $\begin{cases} R_{2n}''(z_k)=K_n\left[(n-1)\left(z_k^2-1\right)-1\right]z_k^{n-3}P_n'(x_k) \\ R_{2n}''(z_{n+k})=K_n\left[(n-1)\left(z_{n+k}^2-1\right)-1\right]z_{n+k}^{n-3}P_n'(x_k) \end{cases}$

$$\begin{cases} R'_{4n}(z_k) = \dfrac{K_n{}^*}{2}\left(z_k^2-1\right)^2\{P'_n(x_k)\}^2\, z_k^{2n-3} & , \quad k = 1(1)2n \\ R'_{4n}(z_k) = \dfrac{K_n{}^*}{2}\left(z_k^2-1\right)^2 P_n(x_k)P''_n(x_k)\, z_k^{2n-3}, & k = 2n+1, \dots, 4n-2 \end{cases}$$

$$(2.9)\begin{cases} \qquad k = 1(1)2n \\ R'_{4n}(z_k) = \frac{K_n{}^*}{2}\left(z_k^2-1\right)^2\{P'_n(x_k)\}^2 z_k^{2n-3} \\ \qquad k = 2n+1, \dots\dots\dots\dots\dots, 4n-2 \\ R'_{4n}(z_k) = \frac{K_n{}^*}{2}\left(z_k^2-1\right)^2 P_n(x_k)P''_n(x_k) z_k^{2n-3} \end{cases}$$

$$(2.10)\begin{cases} \text{For, } k = 1(1)2n \\ R''_{4n}(z_k) = K_n{}^*\left(z_k^2-1\right)\left[(2n-1)\left(z_k^2-1\right)-2\right]\{P'_n(x_k)\}^2 z_k^{2n-4}, \\ \qquad \text{For , } \quad k = 2n+1, \dots\dots, 4n-2 \\ R''_{4n}(z_{n+k}) = K_n{}^*\left(z_k^2-1\right)\left[(2n-1)\left(z_k^2-1\right)-2\right]P_n(x_k)P''_n(x_k) z_k^{2n-4}, \end{cases}$$

We will also use the following well known inequalities

(2.11) $(1-x^2)|P'_n(x)| \sim n^{1/2}, \qquad -1 < x < 1$

For, $\quad -1 < x_k < 1$

(2.12) $\left(1-x_k^2\right)^{-1} \sim \left(\frac{k}{n}\right)^{-2}$

(2.13) $|P_n(x_k)| \sim k^{-1/2}$

(2.14) $|P'_n(x_k)| \sim k^{-3/2}\ n^2$

(2.15) $|P''_n(x_k)| \sim k^{-5/2}\, n^4$

For more details one can see [9] .

3. The problem and regularity

Let $Z_n = \{z_k : k = 0(1)4n-1\}$satisfying

$$(3.1)\quad Z_n = \begin{cases} z_0 = 1, \quad z_{4n-1} = -1, \\ z_k = cos\theta_k + i\, sin\theta_k,\ z_{n+k} = \overline{z_k}\,,\ k = 1(1)2n-1 \end{cases}$$

where, $\{x_k = cos\theta_k \ : k = 1(1)2n-1\}$ are the zeros of $P_n(x)P'_n(x)$ where $P_n(x)$ stands for n^{th}Legendre polynomial. Here we are interested in determine the interpolatory polynomial $Q_{6n-1}(z)$ of degree at most $6n-1$ satisfying the following conditions:

$$(3.2)\begin{cases} Q_{6n-1}(z_k) = \alpha_k\ , & k = 0(1)4n-1, \\ [Q_{6n-1}(z)]'_{z=z_k} = \beta_k\ , & k = 1(1)2n, \end{cases}$$

where α_k and β_k are arbitrary complex constants .We establish a convergence theorem for the same.

Theorem 3.1. $Q_{6n-1}(z)$ is regular on Z_n.

Proof: Let

$Q_{6n-1}(z) = R_{4n-2}(z)q(z)$

where, $q(z)$ is polynomial of degree $\leq 2n+1$.

Obviously, $Q_{6n-1}(z_k) = 0$, for, k= $1(1)4n-2$

By,$[Q_{6n-1}(z)]'_{z=z_k} = 0$, for k= $1(1)2n$

we get,

$q(z_k) = 0$,

therefore, we have

(3.3) $q(z) = (a\,z+b)R_{2n}(z)$

Now for $z = 1\ \&\ -1$, we get $a = b = 0$.

Hence the theorem follows.

4. Explicit representation of interpolatory polynomials

We shall write $Q_n(z)$ satisfying (3.2) as:

$$(4.1)\quad Q_{6n-1}(z) = \sum_{k=0}^{4n-1} \alpha_k A_k(z) \quad + \sum_{k=1}^{2n} \beta_k B_k(z)$$

where,$A_k(z)$and $B_k(z)$ are unique polynomial , each of degree at most $6n-1$ satisfying the conditions :

$$(4.2)\ \begin{cases} A_k(z_j) = \delta_{jk}, & k = 0(1)4n-1 \\ [A_k(z)]'_{z=z_j} = 0, & k = 1(1)2n \end{cases}$$

$$(4.3)\ \begin{cases} B_k(z_j) = 0\ , & k = 0(1)4n-1 \\ [B_k(z)]'_{z=z_j} = \delta_{jk}\ , & k = 1(1)2n \end{cases}$$

Theorem 4.1. Let $A_k(z)$ satisfying the condition defined by

$$(4.4)\ A_0(z) = \frac{(1+z)}{2}\frac{R_{4n-2}(z)R_{2n}(z)}{R_{4n-2}(1)R_{2n}(1)}$$

$$(4.5)\ A_{4n-1}(z) = \frac{(1-z)}{2}\frac{R_{4n-2}(z)R_{2n}(z)}{R_{4n-2}(-1)R_{2n}(-1)}$$

$$(4.6)\ A_k(z) = \begin{cases} t_k(z)l_k(z)L_k(z)\ , & k = 1(1)2n \\ L_k(z)\frac{R_{2n}(z)}{R_{2n}(z_k)}\ , & k = 2n+1, \dots\dots.., 4n-2 \end{cases}$$

where,

(4.7) $t_k(z) = [1-(z-z_k)\{l'_k(z_k)+L'_k(z_k)\}]$

Proof: For, $k = 1(1)2n$, let

$A_k(z) = t_k(z) l_k(z) L_k(z)$,
where, $t_k(z)$ is a linear polynomial such that,
(4.8) $t_k(z) = a_k + b_k(z - z_k)$
Obviously, $A_k(z_j) = 0$, for $j \neq k$,
and for $j = k$, we must have
(4.9) $t_k(z_k) = 1$
Obviously, $A'_k(z_j) = 0$, for $j \neq k$,
and for $j = k$, we get
(4.10) $t'_k(z_k) = -\{l'_k(z_k) + L'_k(z_k)\}$
using (4.9) and (4.10) in (4.8), we get (4.7).

For, $k = 2n+1, \dots, 4n-2$, let

$$A_k(z) = L_k(z) \frac{R_{2n}(z)}{R_{2n}(z_k)},$$

Then obviously, $A_k(z_j) = \delta_{jk}$,
Similarly, one can find (4.4) and (4.5).
Hence the theorem follows.

Theorem 4.2: For $k = 1(1)2n$, we have
(4.11) $B_k(z) = (z - z_k) l_k(z) L_k(z)$
Proof: One can obtained $B_k(z)$, owing to conditions (4.3).

5. Estimation of fundamental polynomials

Lemma 5.1. Let $l_k(z)$ be given by (2.4). Then
(5.1) $max_{|z|=1} \sum_{k=1}^{2n} |l_k(z)| \leq c n^{1/2} \log n$,
where, c is a constant and independent of n and z.
Proof: Let $z = x + iy$ and $|z| = 1$,

$$\sum_{k=1}^{2n} |l_k(z)| \leq \sum_{k=1}^{2n} \left| \frac{R_{2n}(z)}{(z - z_k) R'_{2n}(z_k)} \right|$$

$$\leq \sum_{k=1}^{2n} \frac{|P_n(x)| (1 - x x_k)^{\frac{1}{2}}}{2\sqrt{2} |P'_n(x_k)| (x - x_k)}$$

using (2.12) and (2.14) we get the result.

Lemma 5.2. For $z = e^{i\theta}$, $(0 \leq \theta < 2\pi)$, we have
(5.2) $\sum_{k=0}^{4n-1} |A_k(z)| \leq c\, n^{3/2} \log n$,
where, $A_k(z)$ is given in theorem 4.1 and c is a constant independent of n and z.
Proof: Using the conditions from (2.11) to (2.15), we get the result.

Lemma 5.3. Let $B_k(z)$ be defined in theorem 4.2. Then, we have

(5.3)$\sum_{k=1}^{2n}|B_k(z)| \leq cn^{1/2}\log n$, $|z| \leq 1$

where, c is a constant independent of n and z.

Proof: Using (5.1) and (2.11) – (2.15) in theorem 4.2, we get the result.

6. Convergence

In this section, we prove the following:

Theorem 6.1. Let $f(z)$ be continuous for $|z| \leq 1$ and analytic for $|z| < 1$. Let the arbitrary β_k 's be such that

(6.1) $|\beta_k| = O\left(n^{1/2}\omega_2(f, n^{-1})\right)$

Then $\{Q_{6n-1}(z)\}$ defined by

(6.2) $$Q_{6n-1}(z) = \sum_{k=0}^{4n-1} f(z_k)A_k(z) + \sum_{k=1}^{2n} \beta_k B_k(z)$$

satisfies the relation,

(6.3) $|Q_{6n-1}(z) - f(z)| = O\left(n^{3/2}\omega_2(f, n^{-1})\log n\right)$,

where $\omega_2(f, n^{-1})$ be the second modulus of continuity of $f(z)$.

Remark 6.1. Let $f(z)$ be continuous for $|z| \leq 1$ and analytic for $|z| < 1$, and $f' \in Lip\alpha, \alpha > \frac{1}{2} + \epsilon$, then the sequence $\{Q_{6n-1}(z)\}$ converges uniformly to $f(z)$ in $|z| \leq 1$, which follows from (6.3) as

(6.4) $\omega_2(f, n^{-1}) = O\left(n^{-\frac{3}{2}-\epsilon}\right), \epsilon > 0$.

To prove the theorem (6.1), we shall need the followings:

Let $f(z)$ be continuous for $|z| \leq 1$ and analytic for $|z| < 1$. Then there exist a polynomial $F_n(z)$ of degree $\leq 6n - 1$, satisfying, **Jackson's inequality**.

(6.5) $|f(z) - F_n(z)| \leq c\omega_2(f, n^{-1}), \ z = e^{i\theta}(0 \leq \theta < 2\pi)$

And also an inequality due to [7].

(6.6) $\left|F_n^{(m)}(z)\right| \leq cn^m\omega_2(f, n^{-1})$, $m\epsilon I^+$.

Proof: Since $Q_{6n-1}(z)$ be is uniquely determined polynomial of degree $\leq 6n - 1$ and the polynomial $F_n(z)$ satisfying (6.5) and (6.6) can be expressed as :

$$F_{6n-1}(z) = \sum_{k=0}^{4n-1} F_n(z_k)A_k(z) + \sum_{k=1}^{2n} F_n'(z_k)B_k(z)$$

Then,

$$|Q_{6n-1}(z) - f(z)| \leq |Q_{6n-1}(z) - F_n(z)| + |F_{6n-1}(z) - f(z)|$$

$$\leq \sum_{k=0}^{4n-1} |f(z_k) - F_n(z_k)|\,|A_k(z)| + \sum_{k=1}^{2n} \{|\beta_k| + |F_n'(z_k)|\}\,|B_k(z)| + |F_n(z) - f(z)|$$

using (6.1),(6.2), (6.4), (6.5), Lemma 5.2 and Lemma 5.4, we get (6.3).

7. Conclusion

In this paper, we have defined the modified-Hermite interpolation on some set of nodes on the unit circle and established the convergence theorem for that interpolatory polynomial.

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