Annals of Pure and Applied Mathematics
Vol. 14, No. 3, 2017, 497-508
ISSN: 2279-087X (P), 2279-0888(online)
Published on 13 November 2017
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v14n3a17

Annals of
Pure and Applied Mathematics

r-Relatively Prime Sets and *r*-Generalization of *Phi* Functions for Subsets of {1,2,…,n}

G. Kamala[1] **and *G. Lalitha***[2]

[1]Department of Mathematics, Osmania University, Hyderabad (TS), India
[2]Department of Mathematics, SRR & CVR Govt. Degree College
Vijayawada (AP) India
Email: [1]gkamala369@gmail.com, [2]lalitha.secpg@gmail.com
[1]Corresponding author.

Received 24 October 2017; accepted 9 November 2017

Abstract. A non-empty subset A of positive integers $\{1,2,\ldots,n\}$ is said to be relatively prime if $\gcd(A)=1$. Let r be a positive integer ≥ 1. A nonempty subset $A \subseteq \{1,2,\ldots,n\}$ is *r*-relatively prime if greatest r^{th} power common divisor of elements of A is 1. In this case we write $\gcd_r(A)=1$. Note that $\gcd(A)=1$ implies $\gcd_r(A)=1$ but the converse need not be true. Let $f^{(r)}(n)$ denotes the number of *r*-relatively prime subsets of $\{1, 2, \ldots, n\}$ and $f_k^{(r)}(n)$ denotes the number of *r*-relatively prime subsets of $\{1, 2, \ldots, n\}$ of Cardinality *k*. $\Phi(n)$ denotes the number of non empty subsets A of $\{1, 2, \ldots, n\}$ such that $\gcd(A)$ is relatively prime to *n*. $\Phi_k(n)$ denotes the number of non-empty subsets A of Cardinality *k* of $\{1,2,\ldots,n\}$ such that $\gcd(A)$ is relatively prime to *n*. We define $\Phi^{(r)}(n)$ to be the number of non-empty subsets *A* of $\{1, 2, \ldots, n\}$ such that greatest r^{th} power common divisor of elements of *A* and *n* is 1. $\Phi_k^{(r)}(n)$ is defined as the number of subsets *A* of $\{1, 2, \ldots, n\}$ such that $\mathrm{Card}(A)=k$ and $\gcd_r(A)$ is *r*-relatively to *n*. $\Phi^{(r)}(n)$ and $\Phi_k^{(r)}(n)$ are *r*-generalizations of $\Phi(n)$ and $\Phi_k(n)$ defined by Nathanson [2]. Exact formulas and asymptotic estimates are obtained for these functions. These results are extensions of results of Nathanson [2]. Some of our proofs use the r-Generalization of Mobius inversion formula.

Keywords: *r*-relatively prime sets, *r*-generalization of Euler *Phi* function.

AMS Mathematics Subject Classification (2010): 1BXX, 11B75

1. Introduction

For a nonempty subset A of {1,2,…,n } let $\gcd(A)$ denote the $\gcd$ of the elements of A. Nathanson [2] defined a non empty subset A of {1,2,…,n} is relatively prime if $\gcd(A)$ =1.Let $f(n)$ denote the number of relatively prime subsets of {1,2,…,n} and for $k \geq 1$, $f_k(n)$ denote the number of relatively prime subsets of {1,2,…,n} of cardinality k. Let $\Phi(n)$ denote the number of non empty subsets A of {1,2,…,n} such that $\gcd(A)$ is relatively prime to n and for integer $k \geq 1$, $\Phi_k(n)$ denote the number of non empty subsets A of {1,2,…,n} such that $\gcd(A)$ is relatively prime to n and card (A)= k. Nathanson[2] obtained the exact formulas and Asymptotic estimates for these four fuctions. In this paper ,we define the functions $f^{(r)}(n)$, $f_k^{(r)}(n)$, $\Phi^{(r)}(n)$ and $\Phi_k^{(r)}(n)$ and obtain exact formulas and Asympotic estimates for these four functions. We derive Mobius Inversion Formula r- Generalized Version to obtain Exact formulas.

Definition 1.

Mobius function r-analogue

$$\mu_r(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^s & \text{if } n = p_1^r p_2^r \dots p_s^r \text{ where } p_1,\ p_2,\ \dots,\ p_s \text{ are distinct primes} \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1. For all positive integers $n \geq 2^r$,

$$f^{(r)}(n) \leq 2^n - 2^{\left[\frac{n}{2^r}\right]}. \tag{1}$$

For positive integers $n \geq 2^r$ and *k*,

$$f_k^{(r)}(n) \leq \binom{n}{k} - \binom{\left[\frac{n}{2^r}\right]}{k}. \tag{2}$$

Proof: The set $\{1,\ 2,\ 3,\dots,\ 2^r,\ \dots,\ n\}$ contains the set $\left\{2^r,\ 2\times 2^r,\ \dots,\ \left[\frac{n}{2^r}\right]2^r\right\}$ which has no subset that is r-relatively prime. Therefore among the $2^n - 1$ non-empty subsets of $\{1,\ 2,\ ..,\ n\}$ those which contain any one of the $2^{[n/2^r]} - 1$ non-empty subsets of $\left\{2^r,\ 2\times 2^r,\ \dots,\ \left[\frac{n}{2^r}\right]2^r\right\}$ are not r-relatively prime. Hence

$f^{(r)}(n) \le \left(2^n-1\right)-\left(2^{\left[\frac{n}{2^r}\right]}-1\right) = 2^n-2^{\left[\frac{n}{2^r}\right]}$ which proves (1). Similarly, $f_k^{(r)}(n) \le \binom{n}{k}-\binom{\left[\frac{n}{2^r}\right]}{k}$. We now find a lower bound for $f^{(r)}(n)$ and $f_k^{(r)}(n)$.

If $1\in A,\ A\subseteq\{1,\ 2,\ ..,\ n\}$ then A is r-relatively prime. There are 2^{n-1} sets $A\subseteq\{1,\ 2,\ ..,\ n\}$ with $1\in A.$ Hence $f^{(r)}(n) \ \ge\ 2^{n-1}.$ (3)

Let $n\ge 3^r.$ If $1\notin A,\ 2^r\in A,\ 3^r\in A$ then A is r-relatively prime and hence

$$f^{(r)}(n) \ \ge\ 2^{n-1}+2^{n-3}. \tag{4}$$

Let $n\ge 5^r.$ If $1\notin A,\ 3^r\in A,$ but $2^r\in A,\ 5^r\in A$ then A is r-relatively prime and there are 2^{n-4} such subsets. Again $1\notin A,\ 2^r\in A,$ but $3^r\in A,\ 5^r\in A$ then A is r-relatively prime and there are 2^{n-4} such subsets. Hence

$$f^{(r)}(n) \ \ge\ 2^{n-1}+2^{n-3}+2\times 2^{n-4} \ =\ 2^{n-1}+2^{n-2}. \tag{5}$$

Similarly

$$f_k^{(r)}(n) \ \le\ \binom{n-1}{k-1}+\binom{n-3}{k-2}+2\binom{n-4}{k-2}. \tag{6}$$

Exact formulas and asymptotic estimates

Let $[x]$ denotes the greatest integer less than or equals to *x*. If $x\ge 1$ and $n=[x]$ then $\left[\frac{x}{d}\right]=\left[\frac{[x]}{d}\right]=\left[\frac{n}{d}\right]$ for all positive integers *d*.

2. Mobius inversion formula r- generalized version

Theorem 2. Let $F^{(r)}(x)$ be a function defined for $x\ge 1$ and define the function $G^{(r)}(x)$ for $x\ge 1$ as $G^{(r)}(x)=\sum\limits_{1\le d^r\le x} F^{(r)}\left(\frac{x}{d^r}\right)$ where the summation is over all positive integers *d* where $d^r\le x$

and $F^{(r)}(x)=0=G^{(r)}(x)$ if $x\in(0,\ 1).$ Then for all inters *d* where $d^r\le x,$

$$G^{(r)}(x)=\sum_{1\le d^r\le x} F^{(r)}\left(\frac{x}{d^r}\right) \Leftrightarrow F^{(r)}(x)=\sum_{1\le d^r\le x}\mu_r\left(d^r\right)G^{(r)}\left(\frac{x}{d^r}\right) \tag{6a}$$

Proof: Assume $G^{(r)}(x) = \sum_{1 \le d^r \le x} F^{(r)}\left(\frac{x}{d^r}\right)$.

Consider

$$G^{(r)}(x) = \sum_{1 \le d^r \le x} \mu_r\left(d^r\right) G^{(r)}\left(\frac{x}{d^r}\right) = \sum_{1 \le d^r \le x} \mu_r\left(d^r\right)\left[\sum_{1 \le t^r \le \frac{x}{d^r}} F^{(r)}\left(\frac{x}{t^r d^r}\right)\right]$$

$$= \sum_{1 \le u^r \le x} F^{(r)}\left(\frac{x}{u^r}\right)\left[\sum_{d^r / u^r} \mu_r\left(d^r\right)\right] = F^{(r)}(x).$$

Conversely, Assume $F^{(r)}(x) = \sum_{1 \le d^r \le x} \mu_r\left(d^r\right) G^{(r)}\left(\frac{x}{d^r}\right)$.

Consider $$\sum_{1 \le d^r \le x} F^{(r)}\left(\frac{x}{d^r}\right) = \sum_{1 \le d^r \le x} \mu_r\left(d^r\right)\left[\sum_{1 \le t^r \le \frac{x}{d^r}} \mu_r\left(d^r\right) G^{(r)}\left(\frac{x}{t^r d^r}\right)\right]$$

$$= \sum_{1 \le u^r \le x} G^{(r)}\left(\frac{x}{u^r}\right)\left[\sum_{t^r / u^r} \mu_r\left(t^r\right)\right]$$

$$= G^{(r)}(x).$$

Theorem 3. For all positive integers $n \ge 2^r$,

(i) $$\sum_{1 \le d^r \le n} f^{(r)}\left(\left[\frac{n}{d^r}\right]\right) = 2^n - 1 \tag{7}$$

and (ii) $$f^{(r)}(n) = \sum_{1 \le d^r \le n} \mu_r\left(d^r\right)\left(2^{\left[\frac{n}{d^r}\right]} - 1\right). \tag{8}$$

For all positive integers $n \ge 2^r$, k and $r \ge 1$,

(i) $$\sum_{1 \le d^r \le n} f_k^{(\mathrm{r})}\left(\left[\frac{x}{d^r}\right]\right) = \binom{n}{k} \tag{9}$$

and (ii) $$f_k^{(r)}(n) = \sum_{1 \le d^r \le n} \mu_r\left(d^r\right)\binom{\left[\frac{n}{d^r}\right]}{k} \tag{10}$$

Proof: Let *A* be a non-empty subset of $\{1,\ 2,\ ..,\ n\}$ and greatest r^{th} power common divisor of *A* is d^r. Then $A^1 = \frac{1}{d^r} * A = \left\{ \frac{a}{d^r} \big/ a \in A \right\}$ is r-relatively prime subset of $\left\{1,\ 2,\ ..,\ \left[\frac{n}{d^r}\right]\right\}$. Conversely, if A^1 is r-relatively prime subset of $\left\{1,\ 2,\ ..,\ \left[\frac{n}{d^r}\right]\right\}$, then $A = d^r * A^1 = \left\{ d^r \cdot a \big/ a \in A^1 \right\}$ is a non-empty subset of $\{1,\ 2,\ ..,\ n\}$ with greatest r^{th} power common divisor of *A* equals to d^r. Therefore it follows that there are exactly $f^{(r)}\left(\left[\frac{n}{d^r}\right]\right)$ subsets of $\{1,\ 2,\ ..,\ n\}$ with greatest r^{th} power common divisor d^r and hence $\sum\limits_{1 \le d^r \le n} f^{(r)}\left(\left[\frac{n}{d^r}\right]\right) = 2^n - 1$ which proves (7). We apply Theorem (2) to the function $F^{(r)}(x) = f^{(r)}([x])$ for all $x \ge 1$ we define $G^{(r)}(x) = \sum\limits_{1 \le d^r \le x} F^{(r)}\left(\frac{x}{d^r}\right)$

$$= \sum_{1 \le d^r \le x} f^{(r)}\left(\left[\frac{x}{d^r}\right]\right) = 2^{[x]} - 1.$$

By Theorem (2) $f^{(r)}([x]) = F^{(r)}(x) = \sum\limits_{1 \le d^r \le x} \mu_r\left(d^r\right) G^{(r)}\left(\left[\frac{x}{d^r}\right]\right)$

$$= \sum_{1 \le d^r \le x} \mu_r\left(d^r\right)\left(2^{\left[\frac{x}{d^r}\right]} - 1\right)$$

$$f^{(r)}(n) = n \sum_{1 \le d^r \le n} \mu_r\left(d^r\right)\left(2^{\left[\frac{n}{d^r}\right]} - 1\right)$$ which proves (8).

We now prove (9) and (10).

Note that $f_k^{(r)}(n) = \#\left\{ A \subseteq \{1,\ 2,\ ...,\ n\} : \text{Card}\, A = k,\ \gcd_r(A) = 1 \right\}$.

Let $A \subseteq \{1,\ 2,\ ...,\ n\}$ with $\text{Card}\, A = k$ and greatest r^{th} power common divisor of *A* is equals to d^r. Let $A^1 = \frac{1}{d^r} * A = \left\{ \frac{a}{d^r} \big/ a \in A \right\}$. Then $A^1 \subseteq \left\{1,\ 2,\ ...,\ \left[\frac{n}{d^r}\right]\right\}$, Card $A^1 = \text{Card } A = k$ and $\gcd_r\left(A^1\right) = 1$. Conversely ,if $A^1 \subseteq \left\{1,\ 2,\ ...,\ \left[\frac{n}{d^r}\right]\right\}$ and Card $A^1 = k, \gcd_r\left(A^1\right) = 1$ then $A = d^r * A^1$ is such that Card $A = k$ and $\gcd_r(A) = d^r$.

There is $1-1$ correspondence between r-relatively prime subsets $A^1 \subseteq \left\{1,\ 2,\ ...,\ \left[\frac{n}{d^r}\right]\right\}$ of Cardinality k and the non-empty subsets A of $\{1,\ 2,\ ...,\ n\}$ with $\gcd_r(A) = d^r$ and Card $A = k$. Hence $\sum\limits_{1 \le d^r \le n} f_k^{(r)}\left(\left[\frac{n}{d^r}\right]\right) = \binom{n}{k}$ which proves (9).

By Theorem (1) we have $f_k^{(r)}(n) = \sum\limits_{1 \le d^r \le n} \mu_r\left(d^r\right)\binom{\left[\frac{n}{d^r}\right]}{k}$ which proves (10).

Theorem 4. For all positive integers $n \ge 2^r$, r we have

$$2^n - 2^{\left[\frac{n}{2^r}\right]} - n \cdot 2^{\left[\frac{n}{3^r}\right]} \le f^{(r)}(n) \le 2^n - 2^{\left[\frac{n}{2^r}\right]}. \tag{11}$$

Proof: For $n \ge 2^r$ we have by equation (7) $2^n - 1 = \sum\limits_{1 \le d^r \le n} f^{(r)}\left(\left[\frac{n}{d^r}\right]\right)$

This implies
$$\begin{aligned} 2^n &= f^{(r)}([n]) + f^{(r)}\left(\left[\frac{n}{2^r}\right]\right) + \sum_{3 \le d^r \le n} f^{(r)}\left(\left[\frac{n}{d^r}\right]\right) + 1 \\ &\le f^{(r)}(n) + 2^{\left[\frac{n}{2^r}\right]} + n \cdot 2^{\left[\frac{n}{3^r}\right]} \end{aligned}$$

combining this with equality (1), $2^n - 2^{\left[\frac{n}{2^r}\right]} - n \cdot 2^{\left[\frac{n}{3^r}\right]} \le f^{(r)}(n) \le 2^n - 2^{\left[\frac{n}{2^r}\right]}$.

Theorem 5. For all positive integers $n \ge 2^r$, k and r

$$\binom{n}{k} - \binom{\left[\frac{n}{2^r}\right]}{k} - n\binom{\left[\frac{n}{3^r}\right]}{k} \le f_k^{(r)}(n) \le \binom{n}{k} - \binom{\left[\frac{n}{2^r}\right]}{k}. \tag{12}$$

Proof: By equation (9) $\sum\limits_{1 \le d^r \le n} f_k^{(r)}\left(\left[\frac{n}{d^r}\right]\right) = \binom{n}{k}$

Therefore $\binom{n}{k} = f_k^{(r)}([n]) + f_k^{(r)}\left(\left[\frac{n}{2^r}\right]\right) + \sum_{3^r \le d^r \le n} f_k^{(r)}\left(\left[\frac{n}{d^r}\right]\right)$

$$\le f_k^{(r)}(n) + \binom{\left[\frac{n}{2^r}\right]}{k} + n\binom{\left[\frac{n}{3^r}\right]}{k}.$$

Therefore $\binom{n}{k} - \binom{\left[\frac{n}{2^r}\right]}{k} - n\binom{\left[\frac{n}{3^r}\right]}{k} \le f_k^{(r)}(n) \le \binom{n}{k} - \binom{\left[\frac{n}{2^r}\right]}{k}$ by equation (2).

3. A Phi function for sets and its *r*-generalization

Nathanson [2] defined Phi function for sets, denoted by $\Phi(n)$ to be the number of non-empty subsets *A* of $\{1, 2, 3, ..., n\}$ such that $\gcd(A)$ is relatively prime to *n*. For example for distinct primes *p* and *q* we have

$$\Phi(p) = 2^p - 2, \quad \Phi\left(p^2\right) = 2^{p^2} - 2^p, \quad \Phi(pq) = 2^{pq} - 2^p - 2^q + 2.$$

Note that $\Phi_1(n) = \varphi(n)$ for all $n \ge 1$. We define, for a positive integer $r \ge 1$, $\Phi^{(r)}(n)$ to be the number of non-empty subsets *A* of $\{1, 2, ..., n\}$ such that greatest r^{th} power common divisor of *A* and *n* is 1. For example for distinct primes p and q,

$$\Phi^{(r)}\left(p^r\right) = 2^{p^r} - 2$$

$$\Phi^{(r)}\left(p^{2r}\right) = 2^{p^{2r}} - 2^{p^r}$$

$$\Phi^{(r)}\left(p^r q^r\right) = 2^{p^r q^r} - 2^{q^r} - 2^{p^r} + 2.$$

Corollary 6. If $F^{(r)}(n)$ and $G^{(r)}(n)$ are arithmetic functions, then

$$G^{(r)}(n) = \sum_{d^r / n} F^{(r)}\left(\frac{n}{d^r}\right) \Leftrightarrow F^{(r)}(n) = \sum_{d^r / n} \mu_r\left(d^r\right) G^{(r)}\left(\frac{n}{d^r}\right).$$

Proof: Assume $G^{(r)}(n) = \sum_{d^r / n} F^{(r)}\left(\frac{n}{d^r}\right)$.

Consider $\sum_{d^r / n} \mu_r\left(\frac{n}{d^r}\right) = \sum_{d^r / n} \mu_r\left(d^r\right)\left(\sum_{t^r / \frac{n}{d^r}} F^{(r)}\left(\frac{n}{t^r d^r}\right)\right)$

$$= \sum_{u^r/n} F^{(r)}\left(\frac{n}{u^r}\right) \sum_{d^r/u^r} \mu_r\left(d^r\right) = F^{(r)}(n).$$

Conversely, assume $F^{(r)}(n) = \sum_{d^r/n} \mu_r\left(d^r\right) G^{(r)}\left(\frac{n}{d^r}\right)$.

Consider $$\sum_{d^r/n} F^{(r)}\left(\frac{n}{d^r}\right) = \sum_{d^r/n} \left(\sum_{t^r/\frac{n}{d^r}} \mu_r\left(t^r\right) G^{(r)}\left(\frac{n}{t^r d^r}\right) \right)$$

$$= \sum_{u^r/n} G^{(r)}\left(\frac{n}{u^r}\right) \left(\sum_{t^r/u^r} \mu_r\left(t^r\right) \right) = G^{(r)}(n).$$

Theorem 7. For all positive integers n, $r \geq 1$,

$$\sum_{d^r/n} \Phi^{(r)}\left(\frac{n}{d^r}\right) = 2^n - 1. \tag{13}$$

Also $\Phi^{(r)}(1) = 1$ and for $n \geq 2^r$, $\Phi^{(r)}(n) = \sum_{d^r/n} \mu_r\left(d^r\right)\left(2^{\frac{n}{d^r}} - 1\right)$. (14)

Proof: For every r^{th} power divisor d^r of n we define the function $\psi^{(r)}(n, d)$ to be the number of non-empty subsets A of $\{1, 2, ..., n\}$ such that greatest r^{th} power common divisor of A and n is d^r, i.e.

$$\psi^{(r)}(n, d) = \#\left\{A \subseteq \{1, 2, ..., n\} \;:\; A \neq \phi, \text{ and } \gcd_r\left(A \cup \{n\}\right) = d^r\right\}.$$

Then $\psi^{(r)}(n, d) = \Phi^{(r)}\left(\frac{n}{d^r}\right)$ and hence $2^n - 1 = \sum_{d^r/n} \psi^{(r)}(n, d) = \sum_{d^r/n} \Phi^{(r)}\left(\frac{n}{d^r}\right)$.

which proves (13). This implies $\Phi^{(r)}(n) = \sum_{d^r/n} \mu_r\left(d^r\right)\left(2^{\frac{n}{d^r}} - 1\right)$ (by using corollary (6)) which proves (14).

Theorem 8. For positive integers $n \geq 2^r$, k and r

$$\sum_{d^r/n} \Phi_k^{(r)}\left(\frac{n}{d^r}\right) = \binom{n}{k} \tag{15}$$

and $$\Phi_k^{(r)}(n) = \sum_{d^r/n} \mu_r\left(d^r\right)\binom{\frac{n}{d^r}}{k}. \tag{16}$$

Proof: For every r^{th} power divisor d^r of n we define $\psi_k^{(r)}(n,\ d)$ to be number of subsets A of $\{1,\ 2,\ ...,\ n\}$ of Cardinality k such that greatest r^{th} power common divisor of A and n is d^r.

That is

$$\psi_k^{(r)}(n,\ d) = \#\left\{A \subseteq \{1,\ 2,\ ...,\ n\}\ :\ \#A = k \text{ and } \left(\gcd_r(A) \cup \{n\}\right)_r = d^r\right\}.$$

Note that $\psi_k^{(r)}(n,\ d) = \Phi_k^{(r)}\left(\frac{n}{d^r}\right)$

$$\sum_{d^r/n} \psi_k^{(r)}(n,\ d) = \binom{n}{k} = \sum_{d^r/n} \Phi_k^{(r)}\left(\frac{n}{d^r}\right)$$ which proves (15).

Using Corollary (6), we get $\Phi_k^{(r)}(n) = \sum_{d^r/n} \mu_r\left(d^r\right)\binom{\frac{n}{d^r}}{k}$ which proves (16).

Theorem 9. Let $n \geq 2^r$. If n is odd then $$\Phi^{(r)}(n) = 2^n + \mathrm{O}\left(n.2^{\left[\frac{n}{3^r}\right]}\right). \tag{17}$$

If $2^r \mid n$, $$\Phi^{(r)}(n) = 2^n - 2^{\frac{n}{2^r}} + \mathrm{O}\left(n.2^{\left[\frac{n}{3^r}\right]}\right). \tag{18}$$

Proof: We have $$\Phi^{(r)}(n) = \sum_{\substack{d^r=1 \\ \gcd_r\left(d^r,\ n\right)=1.}}^{n} \#\left\{A \subseteq \{1,\ 2,\ ...,\ n\} / A \neq \phi,\ \gcd_r(A) = d^r\right\}$$ where

the summation is over r^{th} power of integer d which satisfy the condition $1 \leq d^r \leq n$

$$= \sum_{\substack{d^r=1 \\ \gcd_r\left(d^r,\ n\right)=1.}}^{n} f^{(r)}\left(\left[\frac{n}{d^r}\right]\right)$$

If n is odd, then $\Phi^{(r)}(n) = f^{(r)}(n) + f^{(r)}\left(\left[\frac{n}{2^r}\right]\right) + \sum\limits_{\substack{3^r \le d^r \le n \\ \gcd_r(d^r,\, n)=1}}^{n} f^{(r)}\left(\left[\frac{n}{d^r}\right]\right)$

$$= 2^n - 2^{\left[\frac{n}{2^r}\right]} + O\left(n.2^{\left[\frac{n}{3^r}\right]}\right) + 2^{\left[\frac{n}{2^r}\right]} + O\left(2^{\left[\frac{n}{4^r}\right]}\right) + O\left(n.2^{\left[\frac{n}{3^r}\right]}\right)$$

$= 2^n + O\left(n.2^{\left[\frac{n}{3^r}\right]}\right)$ (Using equation (11)) which proves (17).

If $2^r \mid n$, then $\Phi^{(r)}(n) = f^{(r)}(n) + f^{(r)}\left(\frac{n}{2^r}\right) + \sum\limits_{\substack{3^r \le d^r \le n \\ \gcd_r(d^r,\, n)=1}}^{n} f^{(r)}\left(\left[\frac{n}{d^r}\right]\right)$

$$= \left(2^n - 2^{\frac{n}{2^r}} + O\left(n.2^{\left[\frac{n}{3^r}\right]}\right)\right) + O\left(n.2^{\left[\frac{n}{3^r}\right]}\right)$$

$= 2^n - 2^{\frac{n}{2^r}} + O\left(n.2^{\left[\frac{n}{3^r}\right]}\right)$ which proves (18)

Theorem 10. If $2^r \nmid$ n and n is sufficiently large then

$$\Phi_k^{(r)}(n) = \binom{n}{k} + O\left(n\binom{\left[\frac{n}{3^r}\right]}{k}\right). \tag{19}$$

If $2^r \mid n$ then $\Phi_k^{(r)}(n) = \binom{n}{k} - \binom{\frac{n}{2^r}}{k} + O\left(n\binom{\left[\frac{n}{3^r}\right]}{k}\right).$ (20)

Proof: We have

$$\Phi_k^{(r)}(n) = \sum_{\substack{1 \le d^r \le n \\ \gcd_r(d^r,\, n)=1}} \#\left\{A \subseteq \{1,\ 2,\ ...,\ n\} :\ \#A = k,\ \gcd_r(A) = d^r\right\}$$

$$= \sum_{\substack{1\le d^r \le n \\ \gcd_r\left(d^r,\, n\right)=1}} f_k^{(r)}\left(\left[\frac{n}{d^r}\right]\right).$$

By theorem (4), $f_k^{(r)}(n) = \binom{n}{k} - \binom{\left[\frac{n}{2^r}\right]}{k} + O\left(n\binom{\left[\frac{n}{3^r}\right]}{k}\right)$

Therefore $\Phi_k^{(r)}(n) = \binom{n}{k} - \binom{\frac{n}{2^r}}{k} + O\left(n\binom{\left[\frac{n}{3^r}\right]}{k}\right)$

$$+\binom{\frac{n}{2^r}}{k} - \binom{\left[\frac{n}{4^r}\right]}{k} + O\left(n\binom{\left[\frac{n}{6^r}\right]}{k}\right) + O\left(n\binom{\left[\frac{n}{3^r}\right]}{k}\right)$$

$$= \binom{n}{k} - \binom{\frac{n}{2^r}}{k} + O\left(n\binom{\left[\frac{n}{3^r}\right]}{k}\right) \text{ if } 2^r \mid n.$$

If $2^r \nmid$ n $\quad \Phi_k^{(r)}(n) = \binom{n}{k} + O\left(n\binom{\left[\frac{n}{3^r}\right]}{k}\right).$

4. Conclusion

In conclusion we have obtained effective formulas to get the exact number of r-relatively prime subsets of {1,2,…,n} and number of non empty subsets *A* of {1,2,…,n} such that *A* is r-relatively prime to n by deriving Mobius inversion formula r- generalised version.

Acknowledgement. We are grateful to the referees for suggesting some corrections and valuable comments.

REFERENCES

1. M.B.Nathanson, Elementary Methods in Number Theory, Graduate Texts in Mathematics, Vol. 195. Springer-Verlag, New York, (2000).
2. Melvyn B.Nathanson, Affine invariants, relatively prime sets, and a Phi function for subsets of {1,2,…,n}, *Integers,* 7(2007), A1, 7pp.
3. V.Kavitha and R.Govindarajan, A study on Ramsey numbers and its bounds, *Annals of Pure and Applied Mathematics*, 8(2) (2014) 227-236.
4. M.Ayad and O.Kihel, The number of relatively prime subsets of {1,2,…,n}, *Integers,* 9 (2009) 163-166.

5. M.El.Bachraoui, On relatively prime subsets, combinatorial identities, and diaphontine equations, *J. Integers Seq.*, 15 (2012) Article 12.3.6, 9 pp.
6. C.Jayasekaran and A.Jancy Vini, Results on relatively prime dominating sets in graphs, *Annals of Pure and Applied Mathematics*, 14(3) (2017) 359-369.