

Further Results on Prime Cordial Labeling

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Abstract. A prime cordial labeling of a graph G with vertex set $V(G)$ is a bijection $f : V(G) \rightarrow \{1, 2, 3, \dots, |V(G)|\}$ such that for every edge uv is assigned the label 1 if $\gcd(f(u), f(v)) = 1$ and 0 if $\gcd(f(u), f(v)) > 1$, then the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. A graph which admits prime cordial labeling is called prime cordial graph. In this paper, we prove that the butterfly graph, W-graph, H-graph and duplication of edges of an H-graph are prime cordial graphs.

Keywords: Labeling of graphs, prime cordial labeling

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1. Introduction

In this paper, we consider only finite simple undirected graph. Throughout the paper, we denote $V(G)$ and $E(G)$ are the vertex set and edge set of the graph G respectively. A graph labeling is an assignment of integers to vertices or edges or both based on certain criteria. In graph theory various labeling have been studied in recent years, and interestingly they have applications in various fields such as communication network, coding theory, database management, data mining, circuit design, cryptography, etc. For a detailed survey on various graph labeling one can refer Gallian [4].

A binary vertex labeling is labeling of vertices with either 0 or 1. The concept of cordial labeling was introduced by Cahit [1]. In cordial labeling, each edge $e = uv$ in $E(G)$ is assigned by $|f(u) - f(v)|$. Thus binary labeling is called a cordial labeling, if the number of vertices (edges) labeled with 0 and number of vertices (edges) labeled with 1 differ by at most 1. The concept of cordial labeling is further extended to various labeling such as divisor cordial labeling, product cordial labeling, total product cordial labeling, prime cordial labeling etc.

Vaidya and Shah [7] proved that the graph helm, flower and gear are divisor cordial graphs. Further, they proved that bistar, shadow graph and square graph admit divisor cordial labeling in [9]. Sugumaran and Rajesh [5] proved that swastik graph, jelly fish graph and Petersen graph admit sum divisor cordial labeling. Deshmukh and Shaik [3] proved that the tadpole and olive tree graphs are mean cordial graphs.

In this paper we focus our attention on prime cordial labeling of graphs. Baskar Babuji and Shobana [2] proved that the barycentric subdivision of cycle C_n is prime and full binary tree is prime cordial labeling. Vaidya and Shah [8] proved that the square graph of $B_{n,n}$, wheel graph are prime cordial graphs. Sugumaran and Suresh [6] proved that duplication of pair of vertices in path, jelly fish, bistar and actina graph admits prime cordial labeling.

2. Basic definitions

In this section, we provide a brief summary of the definitions and notations which are useful for the present work.

Definition 2.1. A *prime cordial labeling* of a graph G with vertex set $V(G)$ is a bijection $f : V(G) \rightarrow \{1, 2, 3, \dots, |V(G)|\}$ such that for every edge uv is assigned the label 1 if $\gcd(f(u), f(v)) = 1$ and 0 if $\gcd(f(u), f(v)) > 1$, then the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. A graph which admits prime cordial labeling is called *prime cordial graph*.

Definition 2.2. A *cycle butterfly graph* $B_{n,am}$ consists of two cycles of same order n sharing a common vertex together with an arbitrary number of m pendant vertices attached at the common vertex.

Definition 2.3. Consider the two copies of the star graph $K_{1,n}$. If the last pendant vertex in the first copy of $K_{1,n}$ is merged with the initial pendant vertex in the second copy of $K_{1,n}$ then the resulting graph is called the *W-graph* and it is denoted by W_{2n+1} . Note that $|E(W_{2n+1})| = 2n$ and $|V(W_{2n+1})| = 2n+1$.

Definition 2.4. Let P_n^1, P_n^2 be any two paths with n vertices. Let $V(P_n^1) = \{u_1, u_2, u_3, \dots, u_n\}$ and $V(P_n^2) = \{v_1, v_2, v_3, \dots, v_n\}$. We join the vertices $u_{\frac{n+1}{2}}$ and $v_{\frac{n+1}{2}}$ by an edge, if n is odd. Otherwise join the vertices $u_{\frac{n}{2}}$ and $v_{\frac{n}{2}+1}$, then the resulting graph is called a *H-graph* on $2n$ vertices.

Definition 2.5. The *duplication of an edge* $e = uv$ of a graph G is the graph G' obtained from G by adding a new vertex v' to G such that v' is adjacent to both u and v .

3. Main results

Lemma 3.1. Cycle butterfly graph $B_{n,m}$ admits prime cordial labeling, where $n = 3$ and for all integer values of $m > 2$ and $m = 1$.

Proof: Let G be a cycle butterfly graph with two cycle of same order n . Let u be the common vertex and let u_2, u_3, \dots, u_n be the vertices of left wing and v_2, v_3, \dots, v_n be the vertices of right wing and let x_1, x_2, \dots, x_m be the pendant vertices of G . Clearly $|V(G)| = 2n+m-1$ and $|E(G)| = 2n+m$. We define a vertex labeling $f : V(G) \rightarrow \{1, 2, 3, \dots, 2n+m-1\}$ as follows.

Let us consider the following three cases.

Case 1. When $n = 3$ and $m = 1$.

$f(u) = 2, f(u_2) = 4, f(u_3) = 6, f(v_2) = 1, f(v_3) = 3$ and $f(x_1) = 5$.

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Case 2. When $n = 3$ and $m = 3$.

$f(u) = 2, f(u_2) = 4, f(u_3) = 6, f(v_2) = 1, f(v_3) = 3,$ and $f(x_1) = 5, f(x_2) = 7, f(x_3) = 8$.

Case 3. When $n = 3$ and $m \geq 4$.

$f(u) = 2, f(u_2) = 4, f(u_3) = 6, f(v_2) = 3, f(v_3) = 9$ and we label the remaining pendant vertices uniquely from the set $\{1, 5, 7, 8\} \cup \{10, 11, 12, \dots, 2n + m - 1\}$ in ascending order.

In view of the above labeling pattern we have $|e_f(0) - e_f(1)| \leq 1$.

Hence G is a prime cordial graph.

Theorem 3.2. Cycle butterfly graph $B_{n,m}$ admits prime cordial labeling, where $n \geq 4$ and $m \geq 1$.

Proof: Let G be a cycle butterfly graph with two cycle of same order n . Let u be the common vertex and let u_2, u_3, \dots, u_n be the vertices of left wing and v_2, v_3, \dots, v_n be the vertices of right wing and let x_1, x_2, \dots, x_m be the pendant vertices of G . Clearly $|V(G)| = 2n+m-1$ and $|E(G)| = 2n+m$. We define a vertex labeling $f : V(G) \rightarrow \{1, 2, 3, \dots, 2n+m-1\}$ as follows.

Let us consider the following five cases.

Case 1. When $n = 4$ and $m = 1$.

$f(u) = 2,$

$f(u_i) = 2i ; 2 \leq i \leq 4,$

$f(v_2) = 1, f(v_3) = 3, f(v_4) = 5, f(x_1) = 7$.

Case 2. When $n = 4$ and $m \geq 2$.

$f(u) = 2,$

$f(u_i) = 2i ; 2 \leq i \leq 4,$

$f(v_2) = 1, f(v_3) = 3, f(v_4) = 9$. We label the remaining pendant vertices from the set $\{5, 7\} \cup \{10, 11, 12, \dots, 2n + m - 1\}$ in ascending order.

Case 3. When $n = 5$ and $m \geq 1$.

$f(u) = 2,$

$f(u_i) = 2i ; 2 \leq i \leq 5,$

$f(v_2) = 1, f(v_3) = 3, f(v_4) = 9, f(v_5) = 5$. We label the remaining pendant vertices uniquely from the set $\{7\} \cup \{11, 12, 13, \dots, 2n + m - 1\}$ in ascending order.

Case 4. When $n = 6$ and $m \geq 1$.

$f(u) = 2,$

$f(u_i) = 2i ; 2 \leq i \leq 6,$

$f(v_2) = 1, f(v_3) = 3, f(v_4) = 9, f(v_5) = 5, f(v_6) = 7$. We label the remaining pendant vertices uniquely from the set $\{11\} \cup \{13, 14, 15, \dots, 2n + m - 1\}$ in ascending order.

Case 5. When $n \geq 7$ and $m \geq 1$.

$f(u) = 2,$

$f(u_{i+1}) = 2i+2 ; 1 \leq i \leq n-1,$

$f(v_2) = 1, f(v_3) = 3, f(v_4) = 9, f(v_5) = 5, f(v_6) = 7,$

$f(v_{i+6}) = 2i+9 ; 1 \leq i \leq n-6$, where $n > 6$. We label the remaining pendant vertices uniquely from the set $\{2n-1\} \cup \{2n+1, 2n+2, 2n+3, \dots, 2n+m-1\}$ in ascending order.

In view of all the above cases, we have $|e_f(0) - e_f(1)| \leq 1$. Hence G is a prime cordial graph.

Example 1. Prime cordial labeling of cycle butterfly graph $B_{5,4}$ is shown in Figure 1.

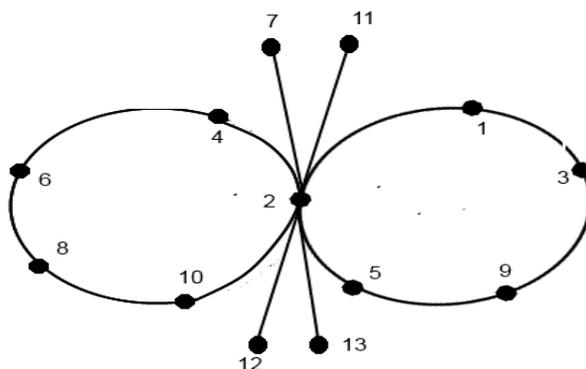


Figure 1: Prime cordial labeling of cycle butterfly graph $B_{5,4}$.

Theorem 3.3. The W -graph W_{2n+1} is a prime cordial graph.

Proof: Let $K_{1,n}^1$ and $K_{1,n}^2$ be the first and second copies of star graph with apex vertex a and b . Let $V(K_{1,n}^1) = \{a, x_1, x_2, x_3, \dots, x_n\}$ and $V(K_{1,n}^2) = \{b, y_1, y_2, y_3, \dots, y_n\}$. We obtain the W -graph by adjoining x_n and y_1 . Let G be the W -graph. Then $|V(G)| = 2n+1$ and $|E(G)| = 2n$.

The vertex labeling of $f : V(G) \rightarrow \{1, 2, 3, \dots, 2n+1\}$ is defined as follows.

$$f(x_n = y_1) = 1,$$

$$f(x_i) = 2i+2 ; 1 \leq i \leq n-1.$$

From the set $\{3, 5, 7, \dots, 2n+1\}$, choose a number p such that it divides only one of the number from this set other than p . In this case we assign vertex b as p and the remaining numbers from the above set are assigned to the vertices y_2, y_3, \dots, y_n uniquely in any order. Clearly f is a prime cordial labeling.

Hence the W -graph W_{2n+1} is prime cordial graph.

Example 2. Prime cordial labeling of the graph W_8 is shown in Figure 2.

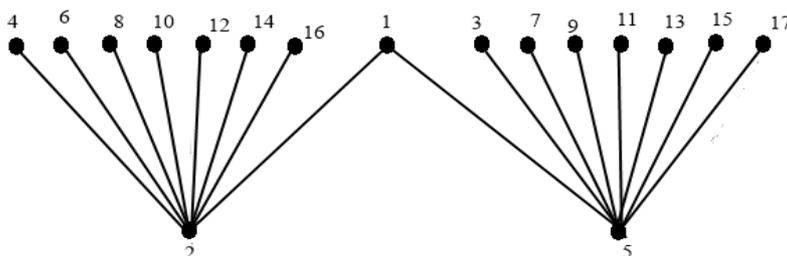


Figure 2: Prime cordial labeling of the graph W_8 .

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Theorem 3.4. H-graph admits prime cordial labeling.

Proof: Let G be a H-graph with $2n$ vertices and $2n-1$ edges. Let P_n^1 and P_n^2 be the parallel paths in G with n vertices and let $V(P_n^1) = \{u_i : 1 \leq i \leq n\}$ and $V(P_n^2) = \{v_i : 1 \leq i \leq n\}$. We consider the following two cases for the labeling of G .

Case 1. When n is even.

Here the vertices $u_{\frac{n}{2}}$ and $v_{\frac{n}{2}+1}$ are connected by an edge.

The vertex labeling of $f : V(G) \rightarrow \{1, 2, 3, \dots, 2n\}$ is defined as follows.

$$f(u_i) = 2i-1 ; 1 \leq i \leq n,$$

$$f(v_{\frac{n}{2}+1}) = 2,$$

$$f(v_j) = 2j+2 ; 1 \leq j \leq \frac{n}{2},$$

$$f(v_k) = 2k ; \frac{n}{2} + 1 < k \leq n.$$

Case 2. When n is odd.

Here the vertices $u_{\frac{n+1}{2}}$ and $v_{\frac{n+1}{2}}$ are connected by an edge.

The vertex labeling of $f : V(G) \rightarrow \{1, 2, 3, \dots, 2n\}$ is defined as follows.

$$f(u_i) = 2i-1 ; 1 \leq i \leq n,$$

$$f(v_{\frac{n+1}{2}}) = 2,$$

$$f(v_j) = 2j+2 ; 1 \leq j \leq \frac{n-1}{2},$$

$$f(v_k) = 2k ; \frac{n-1}{2} < k \leq n.$$

In view of the above labeling pattern, we have $|e_f(0) - e_f(1)| = 1$.

Hence the H-graph is a prime cordial graph.

Example 3. Prime cordial labeling of the graph H_5 is shown in Figure 3.

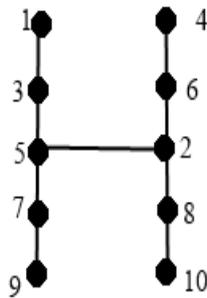


Figure 3: Prime cordial labeling of the graph H_5 .

Example 4. Prime cordial labeling of the graph H_6 is shown in Figure 4.

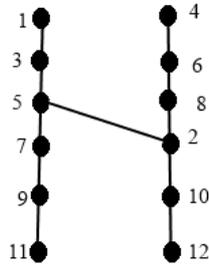


Figure 4: Prime cordial labeling of the graph H_6 .

Theorem 3.5. Duplicating of all edges of an H-graph H_n admits prime cordial labeling.

Proof: Consider an H-graph. Let $u_1, u_2, u_3, \dots, u_n$ and $v_1, v_2, v_3, \dots, v_n$ be the consecutive vertices of first and second vertical paths of the H-graph respectively. Let G be the graph obtained by duplicating of all the edges $u_1u_2, u_2u_3, u_3u_4, \dots, u_{n-1}u_n$, and $v_1v_2, v_2v_3, v_3v_4, \dots, v_{n-1}v_n$, by new vertices $u'_1, u'_2, u'_3, \dots, u'_{n-1}$ and $v'_1, v'_2, v'_3, \dots, v'_{n-1}$ respectively. Let w be a new vertex obtained by duplicating the edge $\frac{u_{n+1}v_{n+1}}{2}$, when n is odd or duplication of the edge $\frac{u_n v_{\frac{n}{2}+1}}$, when n is even. Then $|V(G)| = 4n-1$ and $|E(G)| = 6n-3$. We define $f : V(G) \rightarrow \{1, 2, 3, \dots, 4n-1\}$ is defined as follows, $f(w) = 1$.

We consider the following two cases for the labeling of remaining vertices in G .

Case 1. When $n = 3$.

$f(u_1) = 3, f(u_2) = 5, f(u_3) = 11, f(v_1) = 4, f(v_2) = 2, f(v_3) = 10, f(u'_1) = 9, f(u'_2) = 7, f(v'_1) = 6$ and $f(v'_2) = 8$.

Case 2. When $n > 3$.

$f(u_1) = 3, f(u_2) = 5, f(u_3) = 11, f(u'_1) = 9,$
 $f(u_{i+3}) = 11+4i ; 1 \leq i \leq n-3,$
 $f(u'_i) = f(u_i)+2 ; 2 \leq i \leq n-1.$

Subcase 2(i). When n is even.

$f(v_{\frac{n}{2}+1}) = 2,$
 $f(v_j) = 2j+2 ; 1 \leq j \leq \frac{n}{2},$
 $f(v_k) = 2k ; \frac{n}{2} + 1 < k \leq n,$
 $f(v'_j) = 2n + 2j ; 1 \leq j \leq n-1.$

Subcase 2(ii). When n is odd.

$f(v_{\frac{n+1}{2}}) = 2,$
 $f(v_j) = 2j+2 ; 1 \leq j \leq \frac{n-1}{2},$
 $f(v_k) = 2k ; \frac{n-1}{2} < k \leq n,$
 $f(v'_j) = 2n+2j ; 1 \leq j \leq n-1.$

In view of the above labeling pattern, we have $|e_f(0) - e_f(1)| = 1$, and hence f is a prime cordial labeling.

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Example 5. Prime cordial labeling of duplication of all edges of H_3 -graph is shown in Figure 5.

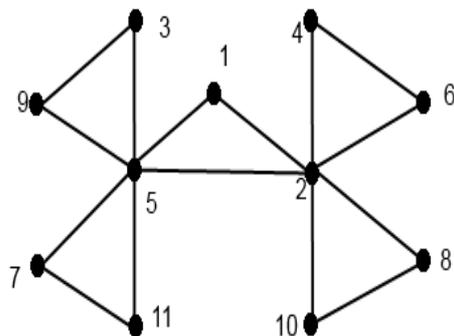


Figure 5: Prime cordial labeling of duplication of all edges of H_3 -graph.

Example 6. Prime cordial labeling of duplication of all edges of H_6 -graph is shown in Figure 6.

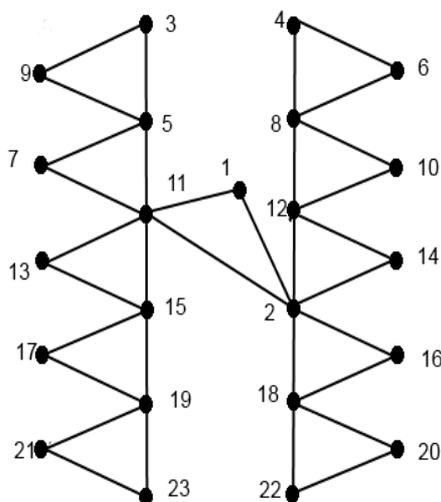


Figure 6: Prime cordial labeling of duplication of all edges of H_6 -graph.

4. Concluding remarks

Prime cordial labeling is explored for several graphs. In this paper we proved that the butterfly graph $B_{n,m}$, W -graph, H -graph and duplication of all edges of an H -graph are prime cordial graphs. It is an interesting open area of research for applying prime cordial labeling for the remaining unproven graphs.

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