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Interval–valued Fuzzy Bridges and Interval–valued Fuzzy Cutnodes

Ann Mary Philip

Department of Mathematics, Assumption College
Changanassery – 686101, Kerala, India.
E-Mail: anmaryphilip@gmail.com

Abstract. In this paper, we define strength of a path, strength of connectedness between any two vertices and strongest path joining any two vertices in an interval–valued fuzzy graph (IVFG). Then we define interval–valued fuzzy bridges (IVF bridges) and interval–valued fuzzy cutnodes (IVF cutnodes) Also, we obtain necessary and sufficient conditions for an arc to be an IVF bridge and a vertex to be an IVF cutnode.

Keywords: Interval-valued fuzzy bridge, Interval-valued fuzzy cutnode.

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1. Introduction

Graph theory has so many applications in almost all real world problems. But since the world is full of uncertainty, fuzzy graph has a separate importance in many real life applications. The first definition of fuzzy graph was by Kaufmann [16] in 1973. But it was Rosenfeld [35] who considered fuzzy relations on fuzzy sets and developed the theory of fuzzy graphs as a generalization of Eulers graph theory in 1975. The works of Bhattacharya[9], Bhutani [10], Bhutani and Battou [11], Bhutani and Rosenfeld [12,13, 14], Mordeson [18], Mordeson and Nair [19,20], Mordeson and Peng [21], Sunitha and Vijayakumar [43-46], Nagoor Gani and Ahmed [22], Nagoor Gani and Malarvizhi [23], Nagoor Gani and Radha [24,25] form the foundation of all researches in fuzzy graph theory. In [42], Sunitha and Sunil Mathew made a very good survey of the researches done so far in fuzzy graph theory. Samanta and Pal introduced fuzzy tolerance graphs [36], fuzzy k-competion graphs and p-competition fuzzy graphs [37], fuzzy threshold graphs [38] and bipolar fuzzy hypergraphs [39].

In 2009, Hongmei and Lianhua [15] gave the definition of IVFG which is a generalization of fuzzy graph.. Since then, IVFG is growing fast and has wide applications in many fields. Akram and Dudek [5], in their paper Interval valued fuzzy graphs defined the operations of Cartesian product, composition, union and join on

IVFGs and investigated some properties. They also introduced the notion of interval-valued fuzzy complete graphs and presented some properties of self complementary and self weak complementary interval-valued fuzzy complete graphs. Akram also introduced interval–valued fuzzy line graphs [2] and bipolar fuzzy graphs [1]. Talebi and H. Rashmanlou [47] studied on isomorphism of IVFGs. Rashmanlou and Jun [29] defined the three new operations, direct product, semi strong product and strong product of IVFGs and discussed its properties on complete IVFGs. Debnath [28] introduced domination in IVFGs. Rashmanlou and Pal defined Irregular IVFG [26], Balanced IVFG [30] and Antipodal IVFG [31] and studied its properties. Also, they studied on the properties of highly irregular IVFG [33] and defined isometry on IVFG [32]. Akram, Alshehri and Dudek [4] introduced certain types of IVFG such as balanced IVFGs, neighbourly irregular IVFGs, neighbourly total irregular IVFGs, highly irregular IVFGs, highly total irregular IVFGs. Again Akram, Yousaf and Dudek [7] studied on the properties of self centered IVFGs. Pal, Samanta and Rashmanlou [27] defined the degree and total degree of an edge in the Cartesian product and composition of two IVFG and obtained some results. Mohideen [8] studied on strong and regular IVFGs. Narayanan and Maheswari [34] introduced strongly edge irregular and strongly edge totally irregular IVFG and made a comparative study between the two. Talebi, Rashmanlou and Ameri [48] studied on product IVFGs. Total regularity of the join of two IVFGs was discussed in [40]. Again regular and edge regular IVFGs were studied in [41].

Bridges and cutnodes is a very important concept in graph theory. Rosenfeld [35] obtained fuzzy analogs of bridges and cutnodes. Further it was studied by Bhattacharya [9]. Again Sunitha and Vijayakumar studied about the properties of fuzzy bridges and fuzzy cutnodes [44]. It was also studied by Mordeson and Nair [20]. Strength of the paths in IVFGs were discussed by Rashmanlou and Pal [31]. Again it was studied by Akram, Yousaf and Dudek [7]. Akram and Alsheri defined intuitionistic fuzzy bridges and intuitionistic fuzzy cutnodes in [3]. Again Akram and Farooq defined bipolar fuzzy bridges and bipolar fuzzy cutnodes in [6]. Bipolar fuzzy bridges and bipolar fuzzy cutnodes were also characterized by Mathew, Sunitha and Anjali [17].

In this paper, we define IVF bridges and IVF cutnodes and study its various properties.

2. Basic concepts

Graph theoretic terms and results used in this work are either standard or are explained as and when they first appear. We consider only simple graphs. That is, graphs with multiple edges and loops are not considered.

Definition 2.1. [35] Let V be a non empty set. A fuzzy graph is a pair of functions $G: (\sigma, \mu)$ where σ is a fuzzy subset of V and μ is a symmetric fuzzy relation on σ. That is, $\sigma: V \to [0,1]$ and $\mu: V \times V \to [0,1]$ such that $\mu(u,v) \leq \sigma(u) \wedge \sigma(v)$ for all u, v in V where $\sigma(u) \wedge \sigma(v)$ denotes minimum of $\sigma(u)\ and\ \sigma(v)$.

Definition 2.2. [5] An interval number D is an interval $[a^-, a^+]$ with $0 \leq a^- \leq a^+ \leq 1$.

Remark 2.1. (i) The interval number $[a, a]$ is identified with the number $a \in [0,1]$. (ii) $D[0,1]$ denotes the set of all interval numbers.

Definition 2.3. [5] For interval numbers $D_1 = [a_1^-, b_1^+]$ and $D_2 = [a_2^-, b_2^+]$

- $rmin(D_1, D_2) = [min\{a_1^-, a_2^-\}, min\{b_1^+, b_2^+\}]$
- $rmax(D_1, D_2) = [max\{a_1^-, a_2^-\}, max\{b_1^+, b_2^+\}]$
- $D_1 + D_2 = [a_1^- + a_2^- - a_1^- . a_2^-, b_1^+ + b_2^+ - b_1^+ . b_2^+]$
- $D_1 \leq D_2 \Leftrightarrow a_1^- \leq a_2^-$ and $b_1^+ \leq b_2^+$
- $D_1 = D_2 \Leftrightarrow a_1^- = a_2^-$ and $b_1^+ = b_2^+$
- $D_1 < D_2 \Leftrightarrow D_1 \leq D_2$ and $D_1 \neq D_2$
- $kD = k[a_1^-, b_1^+] = [ka_1^-, kb_1^+]$ where $0 \leq k \leq 1$.

Then $(D[0,1], \leq, \vee, \wedge)$ is a complete lattice with $[0,0]$ as the least element and $[1,1]$ as the greatest. Here $\vee$ denotes *maximum* and $\wedge$ denotes *minimum*.

Definition 2.4. [5] The **i**nterval–valued fuzzy set A in V is defined by $A = \{(x, [\mu_A^-(x), \mu_A^+(x)]) : x \in V\}$ where $\mu_A^-(x)$ and $\mu_A^+(x)$ are fuzzy subsets of V such that $\mu_A^-(x) \leq \mu_A^+(x)$ for all $x \in V$. We shall sometimes denote the IVFS A by $[\mu_A^-(x), \mu_A^+(x)]$.
For any two IVFSs $A = [\mu_A^-(x), \mu_A^+(x)]$ and $B = [\mu_B^-(x), \mu_B^+(x)]$ in V, we define

- $A \cup B = \left\{\left(x, max\left(\mu_A^-(x), \mu_B^-(x)\right), max\left(\mu_A^+(x), \mu_B^+(x)\right)\right) : x \in V\right\}$
- $A \cap B = \left\{\left(x, min\left(\mu_A^-(x), \mu_B^-(x)\right), min\left(\mu_A^+(x), \mu_B^+(x)\right)\right) : x \in V\right\}$

Definition 2.5. [5] If $G^* = (V, E)$ is a graph, then by an interval–valued fuzzy relation B on the set E we mean an IVFS such that $\mu_B^-(xy) \leq min\left(\mu_A^-(x), \mu_A^-(y)\right)$ and $\mu_B^+(xy) \leq min\left(\mu_A^+(x), \mu_A^+(y)\right)$ for all $xy \in E$.

Definition 2.6 [5] By an interval – valued fuzzy graph of a graph $G^* = (V, E)$, we mean a pair $G = (A, B)$, where $A = [\mu_A^-, \mu_A^+]$ is an IVFS on V and $B = [\mu_B^-, \mu_B^+]$ is an IVFR on E.

Definition 2.7 [26] The negative degree of a vertex $u \in V$ is defined by $d^-(u) = \sum_{uv \in E} \mu_B^-(uv)$. Similarly, positive degree of a vertex $u \in V$ is defined by $d^+(u) = \sum_{uv \in E} \mu_B^+(uv)$. Then the degree of the vertex $u \in V$ is defined as $d(u) = [d^-(u), d^+(u)]$.

Definition 2.8. [26] If $d^-(u) = k_1, d^+(u) = k_2$ for all $u \in V$ and k_1, k_2 are real numbers, then the graph G is called $[k_1, k_2]$ - regular interval–valued fuzzy graph or regular interval – valued fuzzy graph of degree $[k_1, k_2]$.

Definition 2.9. [5] An IVFG $G = (A, B)$ is said to be a complete interval–valued fuzzy graph if $\mu_B^-(xy) = min(\mu_A^-(x), \mu_A^-(y))$ and $\mu_B^+(xy) = min(\mu_A^+(x), \mu_A^+(y))$ for all $x, y \in V$.

Theorem 2.1 [4] Let $G = (A, B)$ be an IVFG on a graph $G^* = (V, E)$ such that $G^* = (V, E)$ is an odd cycle. Then G is a RIVFG if and only if $B = [\mu_B^-, \mu_B^+]$ is a constant function.

Theorem 2.2. [26] Let $G = (A, B)$ be an IVFG on a graph $G^* = (V, E)$ such that $G^* = (V, E)$ is an even cycle. Then G is a RIVFG if and only if either $B = [\mu_B^-, \mu_B^+]$ is a constant function or alternate edges have same membership values.

3. Strongest path, IVF bridge, IVF cutnode

Definition 3.1. A path P in an interval – valued fuzzy graph G is a sequence of distinct vertices $v_1, v_2, v_3 \dots v_n$ such that either one of the following conditions is satisfied.

1. $\mu_B^-(v_i v_j) > 0$ and $\mu_B^+(v_i v_j) > 0$ for some i, j.
2. $\mu_B^-(v_i v_j) = 0$ and $\mu_B^+(v_i v_j) > 0$ for some i, j.

A path $P: v_1, v_2, v_3 \dots v_n$ in G is called a cycle if $v_1 = v_{n+1}$ and $n \geq 3$. When $n = 3$, we have $v_1 v_2 v_3 v_1$.

Definition 3.2. An IVFG G is said to be connected if any two nodes are joined by a path.

Definition 3.3. Let $P: v_0, v_1, v_2 \dots v_n$ be a path in an IVFG G. The μ^- strength of the path P is defined as $S_{\mu^-}(P) = \bigwedge_{i=1}^n \mu_{B^-}(v_{i-1}, v_i)$ and the μ^+ strength of the path P is defined as $S_{\mu^+}(P) = \bigwedge_{i=1}^n \mu_{B^+}(v_{i-1}, v_i)$. Then the strength ($\mu^- \mu^+$ strength) of the path P is defined as

$S_{\mu^- \mu^+}(P) = [S_{\mu^-}(P), S_{\mu^+}(P)]$ where $\bigwedge$ stands for minimum.

Definition 3.4. Let u and v be any two vertices of an IVFG G. Then the maximum of the μ^- strength of various paths connecting u and v is called the μ^- strength of connectedness between u and v and is denoted by $(\mu_{B^-})^\infty(u, v)$. Similarly, the maximum of the μ^+ strength of various paths connecting u and v is called the μ^+ strength of connectedness between u and v and is denoted by $(\mu_{B^+})^\infty(u, v)$.

Notation. Let G be an IVFG. The μ^- strength of connectedness between any two vertices u and v can also be denoted as $NCONN_G(u, v)$ and the μ^+ strength of connectedness between u and v can also be denoted as $PCONN_G(u, v)$.

Definition 3.5. The strongest path joining any two vertices u and v is that path which has μ^- strength equals $(\mu_{B^-})^\infty(u, v)$ and μ^+ strength equals $(\mu_{B^+})^\infty(u, v)$.

Remark 3.1. In crisp graph theory and fuzzy graph theory, strongest path joining any two vertices always exist. But in interval- valued fuzzy graph theory, the strongest path joining any two vertices u *and* v does not always exist. It exists if and only if the μ^- strength of connectedness and μ^+ strength of connectedness between u *and* v corresponds to the same path.

Example 3.1. The following diagram presents an IVFG.
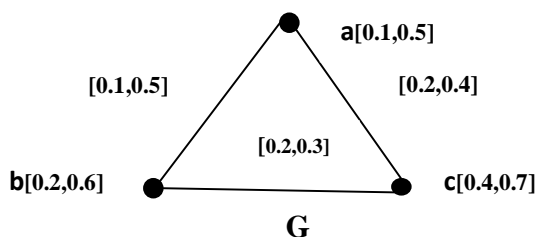
Figure 3.1: An example to show that strongest path between any two vertices does not always exist in the case of IVFGs.

The following table gives all possible $a - b$ paths in G and its μ^- strengths and μ^+ strengths.

Path	μ^- strength	μ^+ strength
$P_1 : a, b$	0.1	0.5
$P_2 : a, c, b$	0.2	0.3

Table 3.1: Table showing various $a - b$ paths in G and its μ^- strengths and μ^+ strengths

Now, $(\mu_{B^-})^\infty(a, b) = \text{NCONN}_G(a, b) = 0.2$ and $(\mu_{B^+})^\infty(a, b) = \text{PCONN}_G(a, b) = 0.5$ and this corresponds to two different paths. So by definition there does not exist a strongest path between a *and* b.

Definition 3.6. An arc (u, v) of an IVFG G is called a μ^- bridge if the deletion of (u, v) reduces the μ^- strength of connectedness between some pair of vertices of G and is called a μ^+ bridge if the deletion of (u, v) reduces the μ^+ strength of connectedness between some pair of vertices of G.
An arc (u, v) of an IVFG G is called an interval valued fuzzy bridge (IVF bridge) if it is both a μ^- bridge and a μ^+ bridge.

Definition 3.7. A node w of an IVFG G is called a μ^- cutnode if the deletion of w reduces the μ^- strength of connectedness between some other pair of vertices of G and is

called a μ^+ cutnode if the deletion of w reduces the μ^+ strength of connectedness between some other pair of vertices of G.

A node w of an IVFG G is called an interval valued fuzzy cutnode (IVF cutnode) if it is both a μ^- cutnode and a μ^+ cutnode.

Example 3.2. Consider the IVFG G given below.
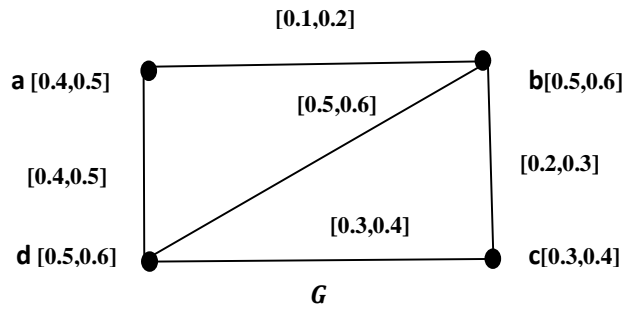
Figure 3.2: Example to explain terms in definitions 3.1- 3.7

The following table gives all possible $a - b$ paths in G and its μ^- strengths and μ^+ strengths.

Path	μ^- strength	μ^+ strength
$P_1: a, b$	0.1	0.2
$P_2: a, d, b$	0.4	0.5
$P_3: a, d, c, b$	0.2	0.3

Table 3.2: Table showing various $a - b$ paths in G and its μ^- strengths and μ^+ strengths

Now, $(\mu_{B^-})^\infty(a, b) = \text{NCONN}_G(a, b) = 0.4$ and $(\mu_{B^+})^\infty(a, b) = \text{PCONN}_G(a, b) = 0.5$.

Clearly, $P_2: a, d, b$ is the strongest $a - b$ path in G. Also, we can see that the removal of the arc (a, d) reduces the μ^- and μ^+ strength of connectedness between a and b. $\therefore (a, d)$ is an IVF bridge. Again the removal of the node d reduces the μ^- and μ^+ strength of connectedness between a and b. $\therefore d$ is an IVF cutnode.

Remark 3.2. An IVF bridge of an IVFG G need not be a bridge of G^*. In example 3.2, (a, d) is an IVF bridge, but it is not a bridge of G^* since even after its removal G^* is still connected.

Definition 3.8. A maximum spanning tree of a connected IVFG $G = (A, B)$ is an interval valued fuzzy spanning subgraph $T = (A, C)$ such that T^* is a tree and for which $\sum_{u \neq v} \mu_{C^-}(u, v)$ and $\sum_{u \neq v} \mu_{C^+}(u, v)$ are maximum.

Theorem 3.1. Let G be an IVFG and if $[\mu_{B^-}, \mu_{B^+}]$ is a constant for every edge of G, then $\text{NCONN}_G(u,v)$ and $\text{PCONN}_G(u,v)$ will be constants for any two vertices $u \ and\ v$ of G .
Proof: Let G be an IVFG and let $[\mu_{B^-}, \mu_{B^+}](e) = [c_1, c_2]$ for every edge e of G where $c_1\ and\ c_2$ are constants. Let $u \ and\ v$ be any two vertices of G. Then for every $u-v$ path P of G, $S_{\mu^-}(P) = c_1$ and $S_{\mu^+}(P) = c_2$. Then by the definition of the μ^- and μ^+ strength of connectedness, $\text{NCONN}_G(u,v) = c_1$, a constant and $\text{PCONN}_G(u,v) = c_2$, a constant.

Definition 3.9. An arc $(u,v)$of an IVFG G is called N $-$ weakest if $\mu_B^-(u,v) < \mu_B^-(x,y)$ where (x,y) is any other arc of G different from (u,v) and is called P $-$ weakest if $\mu_B^+(u,v) < \mu_B^+(x,y)$ where $(x,y)$is any other arc of G different from (u,v)
An arc (u,v) of an IVFG G is called the weakest arc of G if it is both N $-$ weakest and P $-$ weakest

Theorem 3.2. The following statements are equivalent
  1) (u,v) is an IVF bridge
  2) $\text{NCONN}_{G-(u,v)}(u,v) < \mu_B^-(u,v)$ and $\text{PCONN}_{G-(u,v)}(u,v) < \mu_B^+(u,v)$
  3) (u,v) is neither the N- weakest nor P-weakest arc of any cycle.

Proof.
$(2) \Longrightarrow (1)$
For that we show that $\sim(1) \Longrightarrow \sim(2)$. Suppose (u,v) is not an IVF bridge. Then we have 3 cases.
Case 1. (u,v) is not a μ^-bridge. Then $\text{NCONN}_{G-(u,v)}(u,v) = \text{NCONN}_G(u,v) \geq \mu_B^-(u,v)$
Case 2. (u,v) is not a μ^+bridge. Then $\text{PCONN}_{G-(u,v)}(u,v) = \text{PCONN}_G(u,v) \geq \mu_B^+(u,v)$
Case3. (u,v) is neither a μ^-bridge nor a μ^+bridge.
Then $\text{NCONN}_{G-(u,v)}(u,v) = \text{NCONN}_G(u,v) \geq \mu_B^-(u,v)$ and $\text{PCONN}_{G-(u,v)}(u,v) = \text{PCONN}_G(u,v) \geq \mu_B^+(u,v)$.
Cases (1), (2) and (3) together implies $\sim(2)$.
$(1) \Longrightarrow (3)$
For that we show that $\sim(3) \Longrightarrow \sim(1)$. Suppose $\sim(3)$ holds.Then there arises 3 cases.
Case 1. (u,v) is a N – weakest arc of a cycle. Then any path P involving the arc (u,v) can be coverted into a path P' not involving the arc (u,v) such that $S_{\mu^-}(P') \geq S_{\mu^-}(P)$ using the rest of the cycle as a path from u to v. Hence (u,v) cannot be a μ^-bridge and hence it cannot be an IVF bridge.
Case 2. (u,v) is a P – weakest arc of a cycle. Then any path P involving the arc (u,v) can be coverted into a path P' not involving the arc (u,v) such that $S_{\mu^+}(P') \geq S_{\mu^+}(P)$ using the rest of the cycle as a path from u to v. Hence (u,v) cannot be a μ^+bridge and hence it cannot be an IVF bridge.

Case 3. Arc (u,v) is both N – weakest and P - weakest.Then any path P involving the arc (u,v) can be coverted into a path P' not involving the arc (u,v) such that $S_{\mu^-}(P') \geq S_{\mu^-}(P)$ and $S_{\mu^+}(P') \geq S_{\mu^+}(P)$ using the rest of the cycle as a path from u to v. Hence (u,v) cannot be a μ^-bridge and μ^+bridge and hence it cannot be an IVF bridge.

(3) $\Rightarrow$ (2)

We show that $\sim$(2) $\Rightarrow$ $\sim$(3). When we consider $\sim$(2), 3 cases arise.

Case 1. Suppose $NCONN_{G-(u,v)}(u,v) \geq \mu_B^-(u,v)$. Then there is a $u-v$ path P not involving (u,v) such that $S_{\mu^-}(P) \geq \mu_B^-(u,v)$. This path P together with (u,v) forms a cycle of which (u,v) is the N-weakest arc.

Case 2. Suppose $PCONN_{G-(u,v)}(u,v) \geq \mu_B^+(u,v)$. Then there is a $u-v$ path P not involving (u,v) such that $S_{\mu^+}(P) \geq \mu_B^+(u,v)$. This path P together with (u,v) forms a cycle of which (u,v) is the P-weakest arc.

Case 3. Suppose $NCONN_{G-(u,v)}(u,v) \geq \mu_B^-(u,v)$ and $PCONN_{G-(u,v)}(u,v) \geq \mu_B^+(u,v)$. Then there is a $u-v$ path P not involving (u,v) such that $S_{\mu^-}(P) \geq \mu_B^-(u,v)$ and $S_{\mu^+}(P) \geq \mu_B^+(u,v)$. This path P together with (u,v) forms a cycle of which (u,v) is the weakest arc.

Cases (1), (2) and (3) together implies $\sim$(3).

Remark 3.3. From theorem 3.2, we can conclude that N – weakest and P – weakest arcs of cycles cannot be IVF bridges and thus we have the following corollary.

Corollary 3.1. Let $G = (A,B)$ be an IVFG such that G^* is a cycle and let $t_1 = \wedge \mu_B^-(u,v)$ and $t_2 = \wedge \mu_B^+(u,v)$, then all the arcs (u,v) such that $\mu_B^-(u,v) > t_1$ and $\mu_B^+(u,v) > t_2$ are IVF bridges.

Example 3.3. Consider the IVFG G given below such that G^* is a cycle

a [0.2,0.8]   [0.2,0.6]   b[0.3,0.9]

[0.1,0.5]   [0.3,0.5]

[0.1,0.4]

d[0.4,0.9]   c[0.3,0.7]

G
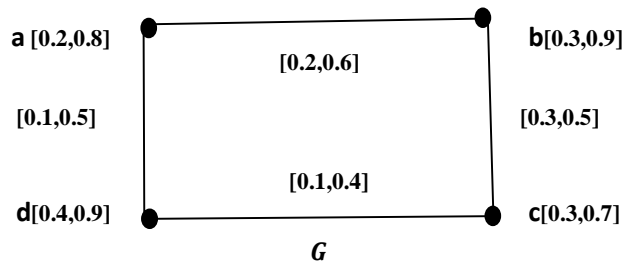
Figure 3.3: Example to illustrate Corollary 3.1

Here $t_1 = 0.1$ and $t_2 = 0.4$. Clearly (a,b) and (b,c) are IVF bridges.

Theorem 3.3. Let $G = (A,B)$ be an IVFG and let (u,v) be an IVF bridge. Then $NCONN_G(u,v) = \mu_B^-(u,v)$ and $PCONN_G(u,v) = \mu_B^+(u,v)$.

Proof: For any arc (u,v), we have $\text{NCONN}_G(u,v) \geq \mu_B^-(u,v)$ and $\text{PCONN}_G(u,v) \geq \mu_B^+(u,v)$. Suppose that (u,v) is an IVF bridge. Also suppose that $\text{NCONN}_G(u,v) > \mu_B^-(u,v)$ and $\text{PCONN}_G(u,v) > \mu_B^+(u,v)$. Then there exists a strongest $u-v$ path P with $S_{\mu^-}(P) > \mu_B^-(u,v)$ and $S_{\mu^+}(P) > \mu_B^+(u,v)$ and all the arcs (x,y) of this strongest path have $\mu_B^-(x,y) > \mu_B^-(u,v)$ and $\mu_B^+(x,y) > \mu_B^+(u,v)$. Now this path P together with the arc (u,v) forms a cycle in which (u,v) is the weakest arc contradicting that (u,v) is an IVF bridge. Hence, our assumption is wrong and the only possibility is that $\text{NCONN}_G(u,v) = \mu_B^-(u,v)$ and $\text{PCONN}_G(u,v) = \mu_B^+(u,v)$
The following example shows that the converse of the above theorem is not true

Example 3.4. In the following IVFG, (u,v) and (x,w) are the only IVF bridges and $\text{NCONN}_G(v,w) = \mu_B^-(v,w)$, $\text{PCONN}_G(v,w) = \mu_B^+(v,w)$ and $\text{NCONN}_G(u,x) = \mu_B^-(u,x)$, $\text{PCONN}_G(u,x) = \mu_B^+(u,x)$, but (v,w) and (u,x) are not IVF bridges.

u [0.3,0.5] [0.3,0.4] v[0.4,0.6]

[0.1,0.2] [0.1,0.2]

[0.3,0.4]

x [0.4,0.5] w[0.5,0.6]

G
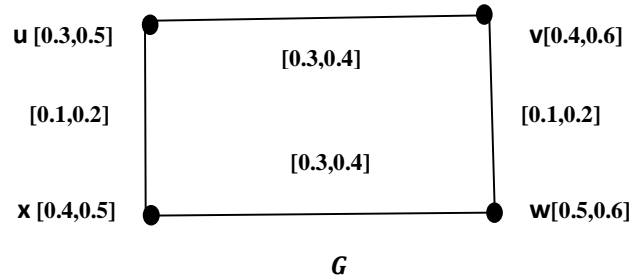
Figure 3.4: Example to show that converse of theorem 3.3 is not true

Remark 3.4. It follows from theorem 3.2 and theorem 3.3 that an arc (u,v) is an IVF bridge if and only if it is the unique strongest $u-v$ path.

Theorem 3.4. Let $G = (A,B)$ be an IVFG and let (u,v) be an IVF bridge. Then $\text{NCONN}_{G-(u,v)}(u,v) < \text{NCONN}_G(u,v)$ and $\text{PCONN}_{G-(u,v)}(u,v) < \text{PCONN}_G(u,v)$.
Proof: Clearly follows from Theorem 3.2 and Theorem 3.3.

In the next theorem, we give the conditions to be satisfied by an IVFG to have atleast one IVF bridge.

Theorem 3.5. Let $G = (A,B)$ be an IVFG and any two edges e_1 and e_2 of G are such that their membership degrees satisfies the following two conditions:
1. $\mu_{B^-}(e_1) \neq \mu_{B^-}(e_2)$ and $\mu_{B^+}(e_1) \neq \mu_{B^+}(e_2)$
2. If $\mu_{B^-}(e_1) < \mu_{B^-}(e_2)$, then $\mu_{B^+}(e_1) < \mu_{B^+}(e_2)$ and if $\mu_{B^-}(e_1) > \mu_{B^-}(e_2)$, then $\mu_{B^+}(e_1) > \mu_{B^+}(e_2)$.
Then G has atleast one IVF bridge.

Proof: Let G be an IVFG satisfying the conditions of the above theorem. Choose any edge (u_0, v_0) such that $\mu_B^-(u_0, v_0) = max\{\mu_B^-(u, v); (u, v) \ is \ an \ edge \ of \ G\}$.

Clearly, $\mu_B^+(u_0, v_0) = max\{\mu_B^+(u, v); (u, v) \ is \ an \ edge \ of \ G\}$ since we are considering only those IVFGs satisfying the above two conditions. Now there exists atleast one edge (u, v) distinct from (u_0, v_0) such that $\mu_B^-(u, v) < \mu_B^-(u_0, v_0)$ and $\mu_B^+(u, v) < \mu_B^+(u_0, v_0)$. We claim that (u_0, v_0) is an IVF bridge of G. For, the deletion of the edge (u_0, v_0) decreases the μ^- and μ^+ strength of connectedness between $u_0 \ and \ v_0$. In otherwords, $NCONN_{G-(u_0,v_0)}(u_0, v_0) < \mu_B^-(u_0, v_0)$ and $PCONN_{G-(u_0,v_0)}(u_0, v_0) < \mu_B^+(u_0, v_0)$. Then by theorem 3.2, (u_0, v_0) is an IVF bridge of G.

Corollary 3.2. Let $G = (A, B)$ be an IVFG as given in theorem 3.5. Then an edge (u, v) for which $\mu_B^-(u, v)$ and $\mu_B^+(u, v)$ are maximum is an IVF bridge of G.

The converse of corollary 3.2 is not true which is clear from the following example.

Example 3.5.
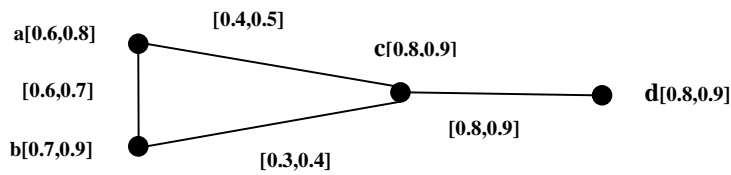
$$G$$

Figure 3.5: Example to show converse of corollary 3.2 is not true.

Here $(a; b)$ is an IVF bridge. But, $[\mu_B^-, \mu_B^+](a, b)$ is not maximum.

Next we give two theorems on IVF cutnode without proof.

Theorem 3.6. Let $G = (A, B)$ be an IVFG such that G^* is a cycle. Then a node of G is an IVF cutnode if and only if it is a common node of two IVF bridges.

Theorem 3.7. If w is common node of atleast two IVF bridges, then w is an IVF cutnode.

The following example shows that the converse of the above theorem is not true.

Example 3.6. In the following IVFG, a is an IVF cutnode. But (a, d) and (b, c) are the only IVF bridges.
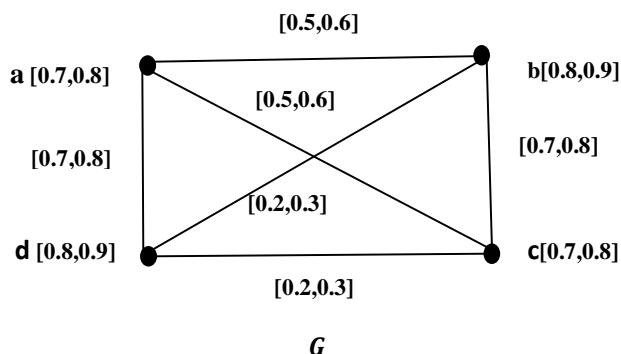
Figure 3.6. Example to show that converse of theorem 3.7 is not **true**

Theorem 3.8. Let G be an IVFG such that G^* is a cycle. If $[\mu_{B^-}, \mu_{B^+}]$ is a constant for every edge of G, then G does not have an IVF bridge. Also, it does not have an IVF cutnode.

Proof: Since G^* is a cycle, there exist two distinct paths between any two vertices. Again since $[\mu_{B^-}, \mu_{B^+}]$ is a constant for every edge of G, deletion of an edge does not reduce the strength of connectedness between any two vertices. So G does not have an IVF bridge and hence by theorem 3.6, G also does not have an IVF cutnode.

Theorem 3.9. A RIVFG on an odd cycle does not have an IVF bridge. Hence it does not have an IVF cutnode.

Proof: Let $G = (A, B)$ be a RIVFG on an odd cycle. Then by theorem 2.1, $[\mu_{B^-}, \mu_{B^+}]$ is a constant for every edge of G and by above theorem G does not have an IVF bridge and hence an IVF cutnode.

Theorem 3.10. Let $G = (A, B)$ be a RIVFG on an even cycle $G^* = (V, E)$. Then either G does not have an IVF bridge or it has $q/2$ IVF bridges where $q = |E|$. Also, G does not have an IVF cutnode.

Proof: Let $G = (A, B)$ be a RIVFG on an even cycle $G^* = (V, E)$. Then by theorem 2.2, either $[\mu_{B^-}, \mu_{B^+}]$ is a constant for every edge of G or alternate edges have same membership values.

Case 1. $[\mu_{B^-}, \mu_{B^+}]$ is a constant. Then by theorem 3.8, G does not have an IVF bridge and hence an IVF cutnode.

Case 2. Alternate edges have same membership values. Then by corollary 3.1, those edges with greater membership values are IVF bridges of G. There are $q/2$ such edges

483

where $q = |E|$. Hence G has $^q/_2$ IVF bridges. But then no vertex is a common vertex of two IVF bridges. So by theorem 3.6, G does not have an IVF cutnode.

Theorem 3.11. An arc (u, v) is an IVF bridge of $G = (A, B)$ if and only if (u, v) is in every MST of G.

Proof: Let (u, v) be an IVF bridge of G. Then arc (u, v) is the unique strongest $u - v$ path and hence is in every MST of G.

Conversely, let (u, v) be in every MST T of G and assume that (u, v) is not an IVF bridge. Then by theorem 3.2, three cases arise.

Case 1. (u, v) is the N- weakest arc of some cycle in G. Then $\text{NCONN}_G(u, v) > \mu_B^-(u, v)$ and $\text{PCONN}_G(u, v) \geq \mu_B^+(u, v)$ which implies that (u, v) is in no MST of G.

Case 2. (u, v) is the P- weakest arc of some cycle in G. Then $\text{PCONN}_G(u, v) > \mu_B^+(u, v)$ and Then $\text{NCONN}_G(u, v) \geq \mu_B^-(u, v)$ which implies that (u, v) is in no MST of G.

Case 3. (u, v) is both the N- weakest and P-weakest arc of some cycle in G Then $\text{NCONN}_G(u, v) > \mu_B^-(u, v)$ and $\text{PCONN}_G(u, v) > \mu_B^+(u, v)$ and hence (u, v) is in no MST of G.

Corollary 3.3. Let G be a connected IVFG with $|V| = n$. The G has atmost $n - 1$ IVF bridges.
Proof: Follows directly from the above theorem.

Theorem 3.12. A node w is an IVF cutnode of an IVFG G if and only if w is an internal node of every MST of G.

Corollary 3.4. Every IVFG has atleast two nodes which are not IVF cutnodes of G.

3. Conclusion

In this paper, we have defined the strength of connectedness between two vertices of an IVFG and extended the notion of bridges and cutnodes to IVFGs. Then we have obtained some conditions to be satisfied by an IVFG to have atleast one IVF bridge. Also we have obtained characterizations of IVF bridges and IVF cutnodes.

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