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Characteristic Subgroups of a finite Abelian Group $Z_n \times Z_n$

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Abstract. We consider the following questions: (i) number of characteristic subgroups of a finite abelian p-group $Z_{p^n} \times Z_{p^n}$ (ii) number of characteristic subgroups of a finite abelian group $Z_n \times Z_n$ and (iii) characteristic subgroup lattice of $Z_n \times Z_n$ is isomorphic to subgroup lattice of Z_n .

Keywords: Subgroup; cyclic subgroup; characteristic subgroup; group of all automorphism

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1. Introduction

In 1939, Baer [1] considered the following question "When two groups have isomorphic subgroups lattices?" Since this is a very difficult problem. Here authors consider a related question "When two groups have isomorphic lattices of characteristic subgroups?" In general problem considered by Baer [1] or related question consider by authors seems to very difficult. We will consider only the particular case of finite Abelian group of rank two i.e., $Z_n \times Z_n$.

A subgroup N of a group G is called a Characteristic Subgroup if Φ (N)=N for all Automorphism Φ of G. This term was first used by *Frobenius* in 1895.

Theorem 1.1. If $\gcd(|H|,|K|) = 1$. $H \times K$ is characteristic subgroup of G if and only if H and K are characteristic subgroup of G.

Proof: Let $x \in H \times K$

∴ x is uniqly expreesed as product of $h \in H$ and $k \in K$ such that $x = hk$.

Then $f(x) = f(hk) = f(h)f(k) \quad \forall f \in Aut(G)$

It is given that H and K is characterstic subgroups of G, therefore $f(h) \in H$ and $f(k) \in K$.

∴ $f(x) \in HK$

Here $HK = H \times K$ [Because H ⊲G, K⊲G and $H \cap K = \{e\}$]

∴ $H \times K$ is characterstic subgroup of G.

Converse :- Let $h(\neq e) \in H$, then $h = he \in H \times K$.

$\therefore\ f(h) \in H \times K \quad \forall f \in Aut(G)$ [Because $H \times K$ is characterstic subgroup of G].
Therefore f(h) is uniquly expressed as product of elements of H and K, then $f(h) = f(h)e$.
If possible $f(h) \in K \Rightarrow |f(h)|\,|\,|K|$ (1)
But $|h|\,|\,|H|$ and $|f(h)| = |h| \Rightarrow |f(h)|\,|\,|H|$ (2)
From (1) and (2), we have
$|f(h)|\,|(|H|, |K|) \Rightarrow |f(h)|\,|1 \Rightarrow f(h) = e \Rightarrow h = e$. This contradiction shows that $f(h) \in H$.
Hence H is characterstic subgroup of G.
Similarly, K is characterstic subgroup of G.

If we denote NC(G) the number of characterstic subgroups of the group G, then by use of theorem 1.1 we have, $NC(Z_n \times Z_n) = \prod_{i=1}^{r} NC(Z_{p_i{}^{\alpha_i}} \times Z_{p_i{}^{\alpha_i}})$ where n $= p_1{}^{\alpha_1} p_2{}^{\alpha_2} p_3{}^{\alpha_3} \ldots \ldots p_r{}^{\alpha_r}$. Now our problem is reduced to find number of characteristic subgroups of a finite abelian of type $Z_{p^\alpha} \times Z_{p^\alpha}$.

2. Partition

Firstly we partition the set S (non-trivial cyclic subgroups of $Z_{p^m} \times Z_{p^n}$ $(1 \leq m \leq n)$) into (p+1) partitions.

Two cyclic subgroups H and K in S are equivalent, denoted by $H \sim K$, if and only if $H \cap K$ contains a subgroup of order p (clearly such subgroup is unique and cyclic)

Lemma 2.1. The relation $\sim$ between elements of the S is an equivalence relation on S.
Proof: Reflexive. Since H is a non-trivial cyclic subgroup of $Z_{p^{\alpha_1}} \times Z_{p^{\alpha_2}}$, then H contains a subgroup of order p. Hence $H \cap H = H$ contains a subgroup of order p, then $H \sim H$.
Symmetric. If $H \sim K$, then $H \cap K$ contains a subgroup of order p, since $H \cap K = K \cap H$. We deduce that $K \cap H$ contains a subgroup of order p and consequently $K \sim H$.
Transitive. If $H \sim K$ and $K \sim L$, then $H \cap K$ and $K \cap L$ contains a subgroup of order p. By using result "every cyclic subgroup of order $p^\alpha (\alpha \geq 1)$ has unique subgroup of order p", hence H and L contains same cyclic subgroup of order p which is contained by K. Therefore $H \cap L$ contains a subgroup of order p and consequently $H \sim L$.

Hence relation $\sim$ is called equivalence relation.

Theorem 2.2. An equivalence relation $\sim$ on a non-empty set S partitions the set S into the disjoint union of distinct equivalence class.

Here group $Z_{p^m} \times Z_{p^n}$ has only p+1 cyclic subgroups of order p , using above theorem we can partition set S into p+1 distinct equivalence class and these partition are as follows:

(a) $[\langle (0, p^{n-1}) \rangle] = \{H \in S | H \sim \langle (0, p^{n-1}) \rangle\}$ and denoted by class-0
(b) $[\langle (p^{m-1}, ip^{n-1}) \rangle] = \{H \in S | H \sim \langle (p^{m-1}, ip^{n-1}) \rangle\}$ and denoted by class-i where $(1 \leq i \leq p)$.

3. Main theorem

Theorem 3.1. Prove that there is exactly one characteristic subgroup of order p in group $Z_{p^m} \times Z_{p^n}$ where $m < n$ i.e., $\langle (0, p^{n-1}) \rangle$ which belong to class-0.

Proof: From [2], we know that there are exactly p+1 subgroups of order p in group $Z_{p^m} \times Z_{p^n}$ and they are given below:-

(i) $\langle(0, p^{n-1})\rangle$ from class-0

(ii) $\langle(p^{m-1}, ip^{n-1})\rangle$ from class-i where $1 \le i \le p$

Firstly, we prove that $\langle(0, p^{n-1})\rangle$ is a characteristic subgroup of group $Z_{p^m} \times Z_{p^n}$.
In group $Z_{p^m} \times Z_{p^n}$, order of element (0,1) is p^n and therefore in any automorphism (0,1) is transferred to element of group $Z_{p^m} \times Z_{p^n}$ which has order p^n, they are written as (j , k) where (k,p)=1.
Let x be any element of subgroup $\langle(0, p^{n-1})\rangle$, then $x = (0, rp^{n-1})$.
$\therefore f(x) = f(0, rp^{n-1}) = rp^{n-1}f(0,1) = rp^{n-1}(j,k) = (rjp^{n-1}, rkp^{n-1})$
Here m<n, then $p^m | p^{n-1}$
Hence $f(x) = (0, rkp^{n-1}) \in \langle(0, p^{n-1})\rangle$
Therefore, subgroup $\langle(0, p^{n-1})\rangle$ is a characteristic subgroup of group $Z_{p^m} \times Z_{p^n}$.

Secondly, we prove that $\langle(p^{m-1}, ip^{n-1})\rangle$ is not a characteristic subgroup of group $Z_{p^m} \times Z_{p^n}$ for $1 \le i \le p$.

In group $Z_{p^m} \times Z_{p^n}$, order of element (1,0) is p^m and therefore in any automorphism (1,0) is transferred to element of group $Z_{p^m} \times Z_{p^n}$ which has order p^m which belong to class other than-0. Take ($j \not\equiv 0$ $(modp)$. Let f_j be an Automorphism of group $Z_{p^m} \times Z_{p^n}$ such that $f_j(1,0) = (1, jp^{n-m})$ and$f_j(0,1) = (0,1)$
Then $f_j(kp^{m-1}, ikp^{n-1}) = kp^{m-1}f_j(1,0) + ikp^{n-1}f_j(0,1) = kp^{m-1}(1, jp^{n-m}) + ikp^{n-1}(0,1) = (kp^{m-1}, k(i+j)p^{n-1}) \notin \langle(p^{m-1}, ip^{n-1})\rangle \ \ \forall\, k \not\equiv 0(modp)$
Hence, subgroup $\langle(p^{m-1}, ip^{n-1})\rangle$ is a not characteristic subgroup of group $Z_{p^m} \times Z_{p^n}$.

Theorem 3.2. Prove that there is no subgroup of order p which is characteristic subgroup of group $Z_{p^n} \times Z_{p^n}$.
Proof: From [2], we know that there are exactly p+1 subgroups of order p in group $Z_{p^n} \times Z_{p^n}$ and they are given below:-

(i) $\langle(0, p^{n-1})\rangle$

(ii) $\langle(p^{n-1}, ip^{n-1})\rangle$ where $1 \le i \le p$.

Firstly, we prove that $\langle(0, p^{n-1})\rangle$ is not a characteristic subgroup of group $Z_{p^n} \times Z_{p^n}$.
Let f_0 be an Automorphism of group $Z_{p^n} \times Z_{p^n}$ such that $f_0(1,0) = (0,1)$ and$f_0(0,1) = (1,0)$.
Then
$f_0(0, kp^{n-1}) = kp^{n-1}f_0(0,1) = kp^{n-1}(1,0) = (kp^{n-1}, 0) \notin \langle(0, p^{n-1})\rangle \ \ \forall\, k \not\equiv 0(modp)$.

Secondly, we prove that $\langle(p^{n-1}, ip^{n-1})\rangle$ is not a characteristic subgroup of group $Z_{p^n} \times Z_{p^n}$ for $1 \le i \le p$.

Let f_i be an Automorphism of group $Z_{p^n} \times Z_{p^n}$ such that $f_i(1,0) = (p-i, 1)$ and$f_i(0,1) = (1,0)$

Then $f_i(kp^{n-1}, ikp^{n-1}) = kp^{n-1}f_i(1,0) + ikp^{n-1}f_i(0,1) = kp^{n-1}(p-i, 1) + ikp^{n-1}(1,0) = (kp^n, kp^{n-1}) = (0, kp^{n-1}) \notin \langle(p^{n-1}, ip^{n-1})\rangle \ \ \forall\, k \not\equiv 0(modp)$

Hence there is no subgroup of order p which is characteristic subgroup of group $Z_{p^n} \times Z_{p^n}$

Theorem 3.3. [3] Characteristic property is transitive. That is, if N is characteristic subgroup of K and K is characteristic subgroup of G, then N is characteristic subgroup of G.

Theorem 3.4. Number of characteristic subgroup of a group $Z_{p^n} \times Z_{p^n}$ are $\tau(p^n)$ and its characteristic subgroup lattice is isomorphic to subgroup lattice of group Z_{p^n}.

Proof:

Case 1: When subgroup of group $\boldsymbol{Z_{p^n} \times Z_{p^n}}$ which is isomorphic group $\boldsymbol{Z_{p^{\alpha_1}} \times Z_{p^{\alpha_2}}}$ where $\boldsymbol{1 \le \alpha_1 < \alpha_2 \le n}$

If possible there exist a characteristic subgroup H from group $Z_{p^n} \times Z_{p^n}$ which is isomorphic group $Z_{p^{\alpha_1}} \times Z_{p^{\alpha_2}}$ where $1 \le \alpha_1 < \alpha_2 \le n$

By using theorem 3.2, then there exists a characteristic subgroup K of order p from subgroup H.

Now K is characteristic subgroup of H and H is characteristic subgroup of $Z_{p^n} \times Z_{p^n}$, by use of theorem 3, we conclude that K is a characteristic subgroup of $Z_{p^n} \times Z_{p^n}$. By use of theorem 3.1, K is not a characteristic subgroup of $Z_{p^n} \times Z_{p^n}$, which contraction with fact that there exist a characteristic subgroup H from group $Z_{p^n} \times Z_{p^n}$ which is isomorphic group $Z_{p^{\alpha_1}} \times Z_{p^{\alpha_2}}$ where $1 \le \alpha_1 < \alpha_2 \le n$.

Case 2: When subgroup of group $\boldsymbol{Z_{p^n} \times Z_{p^n}}$ which is isomorphic $\boldsymbol{Z_{p^\alpha} \times Z_{p^\alpha}}$ where $\boldsymbol{0 \le \alpha \le n}$

From [2], there is exactly one subgroup from group $Z_{p^n} \times Z_{p^n}$ which is isomorphic to $Z_{p^\alpha} \times Z_{p^\alpha}$. This subgroup must be characteristic subgroup. Hence there exist one subgroup for each α, therefore total number of characteristic subgroups of group $\boldsymbol{Z_{p^n} \times Z_{p^n}}$ **are n+1 or** $\boldsymbol{\tau(p^n)}$. These subgroups are $<\left(p^{n-i},0\right),\left(0,p^{n-i}\right)>$ where $i = 0,1,2,\ldots,n$

Its characteristic subgroup lattice is as follows:-

$< (0,0) > \subseteq < (p^{n-1},0),(0,p^{n-1}) > \subseteq < (p^{n-2},0),(0,p^{n-2}) > \subseteq \ldots \subseteq < (1,0),(0,1) > = Z_{p^n} \times Z_{p^n}$

Subgroup lattice of group Z_{p^n} is as follows:-

$< 0 > \subseteq < p^{n-1} > \subseteq < p^{n-2} > \subseteq \ldots \subseteq < 1 > = Z_{p^n}$

Let as define a mapping f from a set of characteristic subgroup of group $Z_{p^n} \times Z_{p^n}$ to set of subgroups of Z_{p^n} such that $f\left(<\left(p^{n-i},0\right),\left(0,p^{n-i}\right)>\right) = < p^{n-i} >$. This mapping f also preserve subset property means $<\left(p^{n-i},0\right),\left(0,p^{n-i}\right)> \subseteq <\left(p^{n-j},0\right),\left(0,p^{n-j}\right)> \Leftrightarrow f(<\left(p^{n-i},0\right),\left(0,p^{n-i}\right)>) \subseteq f(<\left(p^{n-j},0\right),\left(0,p^{n-j}\right)>)$

Hence characteristic subgroup lattice of group $Z_{p^n} \times Z_{p^n}$ is isomorphic to subgroup lattice of group Z_{p^n}

Theorem 3.5. Number of characteristic subgroup of a group $Z_n \times Z_n$ are $\tau(n)$ and its characteristic subgroup lattice is isomorphic to subgroup lattice of group Z_n.

Proof: We know that $NC(Z_n \times Z_n) = \prod_{i=1}^{r} NC(Z_{{p_i}^{\alpha_i}} \times Z_{{p_i}^{\alpha_i}})$ where n $={p_1}^{\alpha_1}{p_2}^{\alpha_2}{p_3}^{\alpha_3} \dots \dots {p_r}^{\alpha_r}$, hence $NC(Z_n \times Z_n) = \prod_{i=1}^{r} \tau({p_i}^{\alpha_i}) = \tau(n)$.
If LC(G) for characteristic subgroup lattice of G, then LC($Z_n \times Z_n$) $\approx LC\left(Z_{{p_1}^{\alpha_1}} \times Z_{{p_1}^{\alpha_1}}\right) \times LC\left(Z_{{p_2}^{\alpha_2}} \times Z_{{p_2}^{\alpha_2}}\right) \times \dots \times LC(Z_{{p_r}^{\alpha_r}} \times Z_{{p_r}^{\alpha_r}})$ the direct product of corresponding subgroup lattices (Suzuki[5]).
From theorem 3.4, we have $LC(Z_{{p_i}^{\alpha_i}} \times Z_{{p_i}^{\alpha_i}}) \approx L(Z_{{p_i}^{\alpha_i}})$ where $L(Z_{{p_i}^{\alpha_i}})$ denotes subgroup lattice of group $Z_{{p_i}^{\alpha_i}}$.
Hence, $LC(Z_n \times Z_n) \approx L\left(Z_{{p_1}^{\alpha_1}}\right) \times L\left(Z_{{p_2}^{\alpha_2}}\right) \times \dots \times L(Z_{{p_r}^{\alpha_r}}) \approx L(Z_n)$.

4. Conclusion

In this paper, we have conclude that Number of characteristic subgroup of a group $Z_n \times Z_n$ are $\boldsymbol{\tau(n)}$ and its characteristic subgroup lattice is isomorphic to subgroup lattice of group Z_n

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