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New Inversion Formulas for the L_2-Transform with Applications

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Abstract. In this article, the author established a simple formula to invert the L_2 -transform of a function. We give also an application for solution to systems of non–homogeneous singular integral equations. Constructive examples involving this transform are also provided.

Keywords: L_2-transform, Laplace transform, Efros's theorem, singular integral equations

AMS Mathematics Subject Classification (2010): 44A10, 44A35

1. Introduction

The main goal of the transform method is to transform a given problem into one that is easier to solve. Knowledge of the properties and use of the classical integral transforms such as Laplace transforms, are just as important today as they have been for the last century or so.

The Fourier and Laplace transforms are by far the most prominent in application. Many other Transforms have been developed, but most have limited applicability. This work also contains some discussion of the other integral transforms that have been used recently successfully in the solution of certain boundary value problems and in other applications.

The Laplace-type integral transform called L_2 - transform was introduced by Yurekli and Brown in [1], where the L_2 - transform is defined as

$$L_2\{f(t);s\} = \int_0^\infty t\exp(-s^2t^2)f(t)dt. \tag{1.1}$$

Although the L2 - transform is not nearly as versatile in applications as are the Fourier and Laplace transform, there are some areas of application where it can be a useful tool. In particular, it is useful in the calculation of certain integrals, and in solving some special systems of singular integral equations.

If we make a change of variable on the right-hand side of the above integral (1.1), we get,

$$L_2\{f(t);s\}=\frac{1}{2}\int_0^{\infty}e^{-ts^2}f(\sqrt{t})dt. \tag{1.2}$$

We have the following relationship between the Laplace - transform and the L_2 - transform

$$L_2\{f(t);s\}=\frac{1}{2}L\{f(\sqrt{t});s^2\}. \tag{1.3}$$

In this paper, we study two new complex inversion formulae for L2- transform.

We state the inversion formulas for the above mentioned transform and develop some of the more important techniques associated with its definition as an integral.We do not attempt to present the basic theorems in their most general forms. Proofs of the lemmas are provided, when feasible.

For further information, readers are advised to refer to the references [7,8,9].

2. Inversion formula for L_2 -transform

Lemma 2.1. Let $L_2\{f(x),s\}=F(s)$ then under certain conditions mentioned in [2] one has the following

$$L_2^{-1}\{F(s);x\}=-\frac{2}{\pi}\int_0^{+\infty}\operatorname{Im}\left\{\left[F(s)\right]_{s\to\sqrt{t}\,e^{i\frac{\pi}{2}}}\right\}e^{-x^2t}dt.$$

Proof: For the detailed proof, see [2],[3].

Example 2.1. The following relations hold true.

1 - $L_2^{-1}\{\frac{\sqrt{\pi}}{4s^3};x\}=x$.

2 - $L_2^{-1}\left\{\frac{\sqrt{\pi}}{2s}e^{-2as};x\right\}=\frac{1}{x}e^{-\frac{a^2}{x^2}}$

Proof:

$$1 - L_2^{-1}\{\frac{\sqrt{\pi}}{4s^3};x\}=-\frac{2}{\pi}\int_0^{+\infty}\operatorname{Im}\left\{\left(\frac{\sqrt{\pi}}{4s^3}\right)_{s\to\sqrt{t}e^{i\frac{\pi}{2}}}\right\}e^{-x^2t}dt, \tag{2.1}$$

$$\left(\frac{\sqrt{\pi}}{4s^3}\right)_{s\to\sqrt{t}e^{i\frac{\pi}{2}}}=\frac{\sqrt{\pi}}{4t^{\frac{3}{2}}e^{\frac{3\pi i}{2}}}=\frac{\sqrt{\pi}}{4t^{\frac{3}{2}}(-i)}$$

$$\Rightarrow \operatorname{Im}\left\{\left(\frac{\sqrt{\pi}}{4s^3}\right)_{s\to\sqrt{t}e^{i\frac{\pi}{2}}}\right\}=\frac{\sqrt{\pi}}{4}t^{-\frac{3}{2}}. \tag{2.2}$$

Setting the relation (2.2) in (2.1) we get

$$L_2^{-1}\{\frac{\sqrt{\pi}}{4s^3};x\}=-\frac{2}{\pi}\int_0^{+\infty}\frac{\sqrt{\pi}}{4}t^{-\frac{3}{2}}e^{-x^2t}dt=-\frac{1}{2\sqrt{\pi}}\int_0^{+\infty}t^{-\frac{3}{2}}e^{-x^2t}dt.$$

The above integral on the right-hand side may be evaluated by changing the variable of the integration from t to u where, $t=\frac{u}{x^2}$

$$L_2^{-1}\{\frac{\sqrt{\pi}}{4s^3};x\}=-\frac{x}{2\sqrt{\pi}}\int_0^{+\infty}u^{-\frac{3}{2}}\,e^{-u}du=-\frac{x}{2\sqrt{\pi}}\Gamma(-\frac{1}{2})=-\frac{x}{2\sqrt{\pi}}\left(-2\sqrt{\pi}\right)$$

$$\Rightarrow \qquad L_2^{-1}\{\frac{\sqrt{\pi}}{4s^3};x\}=x.$$

2- $$L_2^{-1}\left\{\frac{\sqrt{\pi}}{2s}e^{-2as};x\right\}=-\frac{2}{\pi}\int_0^{+\infty}\operatorname{Im}\left(\frac{\sqrt{\pi}}{2s}e^{-2as}\right)_{s\to\sqrt{t}e^{i\frac{\pi}{2}}}e^{-x^2t}dt\ , \tag{2.3}$$

$$\left(\frac{\sqrt{\pi}}{2s}e^{-2as}\right)_{s\to\sqrt{t}e^{i\frac{\pi}{2}}}=\frac{\sqrt{\pi}}{2\sqrt{t}e^{\frac{i\pi}{2}}}e^{-2a\sqrt{t}e^{\frac{i\pi}{2}}}=\frac{\sqrt{\pi}}{2i\sqrt{t}}e^{-i2a\sqrt{t}},$$

$$=\frac{-i\sqrt{\pi}}{2\sqrt{t}}\left(\cos(2a\sqrt{t})-i\sin(2a\sqrt{t})\right),$$

$$\Rightarrow \quad \operatorname{Im}\left\{\left(\frac{\sqrt{\pi}}{2s}e^{-2as}\right)_{s\to\sqrt{t}e^{i\frac{\pi}{2}}}\right\}=\frac{-\sqrt{\pi}}{2\sqrt{t}}\cos(2a\sqrt{t}).$$

Setting the above relation in (2.3), we get

$$L_2^{-1}\left\{\frac{\sqrt{\pi}}{2s}e^{-2as};x\right\}=-\frac{2}{\pi}\int_0^{+\infty}\frac{-\sqrt{\pi}}{2\sqrt{t}}\cos(2a\sqrt{t})\ e^{-x^2t}dt,$$

the integral on the right-hand side may be evaluated by changing the variable of the integration from t to u where, $\sqrt{t}=u$

$$L_2^{-1}\left\{\frac{\sqrt{\pi}}{2s}e^{-2as};x\right\}=\frac{2}{\sqrt{\pi}}\int_0^{+\infty}e^{-x^2u^2}\cos(2au)\ du=\frac{2}{\sqrt{\pi}}\times\frac{1}{x}\times\frac{\sqrt{\pi}}{2}e^{-\frac{a^2}{x^2}}.$$

In order to calculate the above integral, we use Leibnitze rule to obtain

$$L_2^{-1}\left\{\frac{\sqrt{\pi}}{2s}e^{-2as};x\right\}=\frac{1}{x}e^{-\frac{a^2}{x^2}}.$$

3. The Post–Widder inversion formula for L_2-transform

Lemma 3.1. Let $L_2\{f(t);s\}=F(s)$ and $F(\sqrt{s})=G(s)$, then one has the following

$$f(t)=\left[\lim_{n\to\infty}\frac{2(-1)^n}{n!}\left(\frac{n}{t}\right)^{n+1}G^{(n)}\left(\frac{n}{t}\right)\right]_{t\to t^2} \tag{3.1}$$

Proof: We have the following result of Post –Widder.

If the integral $\int_0^{+\infty} \exp(-st) f(t) dt = F(s)$ converges for $\operatorname{Re} s > \gamma$ then ,

$$f(t) = \lim_{n\to+\infty}\{\frac{(-1)^n}{n!}(\frac{n}{t})^{n+1} F^{(n)}(\frac{n}{t})\}.$$

Provided that the limit exists. By using the relationship (1.3) and setting $F(\sqrt{s}) = G(s)$,

We get finally, $$f(t) = \left[\lim_{n\to\infty} \frac{2(-1)^n}{n!}\left(\frac{n}{t}\right)^{n+1} G^{(n)}\left(\frac{n}{t}\right)\right]_{t\to t^2}.$$

For the detailed proof, see [3,4].

Example 3.1. Show that : $L_2^{-1}\left\{\ln\sqrt{\frac{s^2+b^2}{s^2+a^2}}\ ;t\right\} = \frac{e^{-a^2t^2} - e^{-b^2t^2}}{t^2}$.

Solution:

$$F(s) = \ln\sqrt{\frac{s^2+b^2}{s^2+a^2}}$$

$$\Rightarrow \quad G(s) = F(\sqrt{s}) = \ln\sqrt{\frac{s+b^2}{s+a^2}} = \frac{1}{2}\left[\ln(s+b^2) - \ln(s+a^2)\right] ,$$

$$G^{(n)}(s) = \frac{(-1)^n}{2}(n-1)!\left[\frac{1}{(s+b^2)^n} - \frac{1}{(s+b^2)^n}\right]. \tag{3.2}$$

Setting the relation (3.2) in (3.1) we get

$$f(t) = \left[\lim_{n\to\infty} \frac{2(-1)^n}{n!}\left(\frac{n}{t}\right)^{n+1} \frac{(-1)^{n-1}}{2}(n-1)!\left[\frac{1}{(\frac{n}{t}+b^2)^n} - \frac{1}{(\frac{n}{t}+b^2)^n}\right]\right]_{t\to t^2} ,$$

$$f(t) = \left[\frac{-1}{t}\left(\lim_{n\to\infty}(1+\frac{tb^2}{n})^{-n} - \lim_{n\to\infty}(1+\frac{ta^2}{n})^{-n}\right)\right]_{t\to t^2} ,$$

$$f(t) = \left[\frac{-e^{-b^2t} + e^{-a^2t}}{t}\right]_{t\to t^2} = \frac{e^{-a^2t^2} - e^{-b^2t^2}}{t^2}.$$

Lemma 3.1. (Generalized product Theorem, Efros's Theorem).
Let $L_2\{f(t);s\} = F(s)$ and assuming $\Phi(s), q(s)$ be analytic and such that,
$L_2\{\phi(\tau,t)\ ;s\} = \Phi(s)\,\tau\, e^{-\tau^2 q^2(s)}$, then one has:

$$L_2\{\int_0^\infty f(\tau)\phi(\tau,t)\,d\tau\;;s\} = F(q(s))\Phi(s)\ .$$

Proof: Application of L_2 - transform followed by changing the order of integration will do the job. For detailed proof, see[5].

4. Solving systems of singular integral equations with trigonometric kernels using the L_2-transform

$$\begin{cases} f_1(x) = g_1(x) + \lambda_1 \int_0^{+\infty} k_1(x,t)\ f_2(t)\ dt \\ f_2(x) = g_2(x) + \lambda_2 \int_0^{+\infty} k_2(x,t)\ f_1(t)\ dt \end{cases} \tag{4.1}$$

Note: Let $L_2\{f_1(x);s\} = F_1(s)$, $L_2\{f_2(x);s\} = F_2(s)$, $L_2\{g_1(x);s\} = G_1(s)$, $L_2\{g_2(x);s\} = G_2(s)$ and assuming $\Phi_1(s), q_1(s), \Phi_2(s), q_2(s)$ are analytic and such that, $L_2\{K_1(x,t)\;;s\} = \Phi_1(s)\,x e^{-x^2 q_1^2(s)}$, $L_2\{K_2(x,t)\;;s\} = \Phi_2(s)\,x e^{-x^2 q_2^2(s)}$, using the L_2-transform of the above systems of integral equations (4.1), leads to the following relations,

$$\begin{cases} F_1(s) = G_1(s) + \lambda_1\Phi_1(s)\cdot F_2(q_1(s)) \\ F_2(s) = G_2(s) + \lambda_2\Phi_2(s)\cdot F_1(q_2(s)) \end{cases}. \tag{4.2}$$

In case of trigonometric kernel, for example, $k_1(x,t) = k_2(x,t) = \sin xt$ we have

$$L_2\{\sin(xt);s\} = x\frac{\sqrt{\pi}}{4s^3}e^{-\frac{x^2}{4s^2}} \quad \Rightarrow \quad \Phi(s) = \frac{\sqrt{\pi}}{4s^3} \quad , \quad q(s) = \frac{1}{2s}$$

$$\begin{cases} F_1(s) = G_1(s) + \lambda_1\Phi(s)\cdot F_2(\frac{1}{2s}) \\ F_2(s) = G_2(s) + \lambda_2\Phi(s)\cdot F_1(\frac{1}{2s}) \end{cases} \tag{4.3}$$

Now, in relation (4.3) we replace s by $\frac{1}{2s}$, we obtain

$$\begin{cases} F_1(\frac{1}{2s}) = G_1(\frac{1}{2s}) + \lambda_1\Phi(\frac{1}{2s})\cdot F_2(s) \\ F_2(\frac{1}{2s}) = G_2(\frac{1}{2s}) + \lambda_2\Phi(\frac{1}{2s})\cdot F_1(s) \end{cases}. \tag{4.4}$$

Combination of (4.3) and (4.4) and then calculations of $F_1(s), F_2(s)$ leads to the following

$$F_1(s)=\frac{G_1(s)+\lambda_1\,\Phi(s)\,G_2(\frac{1}{2s})}{1-\lambda_1\lambda_2\,\Phi(s)\,\Phi(\frac{1}{2s})}\ , \tag{4.5}$$

$$F_2(s)=\frac{G_2(s)+\lambda_2\,\Phi(s)\,G_1(\frac{1}{2s})}{1-\lambda_1\lambda_2\,\Phi(s)\,\Phi(\frac{1}{2s})}\ . \tag{4.6}$$

Taking the inverse L_2-transform of relations (4.5) and (4.6) we obtain finally $f_1(x)$, $f_2(x)$.

Example 4.1. Let us solve the following systems of singular integral equations.

$$\begin{cases} f_1(x)=\dfrac{e^{-\frac{a^2}{x^2}}}{x}+\lambda_1\displaystyle\int_0^{+\infty}\sin xt\ f_2(t)\ dt \\ f_2(x)=\dfrac{1}{x}+\lambda_2\displaystyle\int_0^{+\infty}\sin xt\ f_1(t)\ dt \end{cases}$$

Solution: In the above example, we cannot use the Laplace transform, but the L_2-transform is suitable

$$g_1(x)=\frac{e^{-\frac{a^2}{x^2}}}{x}\quad\Rightarrow\quad G_1(x)=\frac{\sqrt{\pi}}{2s}e^{-2as}$$

$$g_2(x)=\frac{1}{x}\quad\Rightarrow\quad G_2(x)=\frac{\sqrt{\pi}}{2s}.$$

We use relations (4.5) and (4.6) to obtain:

$$F_1(s)=\frac{\frac{\sqrt{\pi}}{2s}e^{-2as}+\lambda_1\frac{\sqrt{\pi}}{4s^3}\times\frac{\sqrt{\pi}}{2(\frac{1}{2s})}}{1-\lambda_1\lambda_2\frac{\sqrt{\pi}}{4s^3}\frac{\sqrt{\pi}}{4(\frac{1}{2s})^3}}=\frac{\sqrt{\pi}}{2-\pi\lambda_1\lambda_2}\left[\frac{e^{-2as}}{s}+\lambda_1\frac{\sqrt{\pi}}{2s^2}\right], \tag{4.7}$$

$$F_2(s)=\frac{\frac{\sqrt{\pi}}{2s}+\lambda_2\frac{\sqrt{\pi}}{4s^3}\times\frac{\sqrt{\pi}}{2(\frac{1}{2s})}e^{-2a(\frac{1}{2s})}}{1-\lambda_1\lambda_2\frac{\sqrt{\pi}}{4s^3}\frac{\sqrt{\pi}}{4(\frac{1}{2s})^3}}=\frac{\sqrt{\pi}}{2-\pi\lambda_1\lambda_2}\left[\frac{1}{s}+\frac{\sqrt{\pi}}{4}\lambda_2\frac{e^{-\frac{a}{s}}}{s^2}\right], \tag{4.8}$$

Taking the inverse L_2-transform of relations (4.7) and (4.8), we get

$$f_1(x) = -\frac{2}{\pi}\int_0^{+\infty} \operatorname{Im}\left(\frac{\sqrt{\pi}}{2-\pi\lambda_1\lambda_2}\left[\frac{e^{-2as}}{s}+\lambda_1\frac{\sqrt{\pi}}{2s^2}\right]\right)_{s\to\sqrt{t}e^{i\frac{\pi}{2}}} e^{-x^2t}dt$$

$$\left(\frac{\sqrt{\pi}}{2-\pi\lambda_1\lambda_2}\left[\frac{e^{-2as}}{s}+\lambda_1\frac{\sqrt{\pi}}{2s^2}\right]\right)_{s\to\sqrt{t}e^{i\frac{\pi}{2}}} = \frac{\sqrt{\pi}}{2-\pi\lambda_1\lambda_2}\left[\frac{e^{-2a\sqrt{t}\,e^{i\frac{\pi}{2}}}}{\sqrt{t}\,e^{i\frac{\pi}{2}}}+\lambda_1\frac{\sqrt{\pi}}{2\left(\sqrt{t}\,e^{i\frac{\pi}{2}}\right)^2}\right]$$

$$= \frac{\sqrt{\pi}}{2-\pi\lambda_1\lambda_2}\left[\frac{-ie^{-i2a\sqrt{t}}}{\sqrt{t}}-\lambda_1\frac{\sqrt{\pi}}{2t}\right] = \frac{\sqrt{\pi}}{2-\pi\lambda_1\lambda_2}\left[\frac{-i}{\sqrt{t}}\left(\cos(2a\sqrt{t})-i\sin(2a\sqrt{t})\right)-\lambda_1\frac{\sqrt{\pi}}{2t}\right]$$

$$\operatorname{Im}\left(\frac{\sqrt{\pi}}{2-\pi\lambda_1\lambda_2}\left[\frac{e^{-2as}}{s}+\lambda_1\frac{\sqrt{\pi}}{2s^2}\right]\right)_{s\to\sqrt{t}e^{i\frac{\pi}{2}}} = \frac{\sqrt{\pi}}{2-\pi\lambda_1\lambda_2}\left[\frac{-\cos(2a\sqrt{t})}{\sqrt{t}}\right]$$

$$f_1(x) = -\frac{2}{\pi}\int_0^{+\infty} \frac{\sqrt{\pi}}{2-\pi\lambda_1\lambda_2}\left[\frac{-\cos(2a\sqrt{t})}{\sqrt{t}}\right] e^{-x^2t}d$$

$$f_1(x) = \frac{2}{\sqrt{\pi}\,(2-\pi\lambda_1\lambda_2)}\int_0^{+\infty} \frac{\cos(2a\sqrt{t})}{\sqrt{t}}\, e^{-x^2t}dt.$$

But the integral on the right-hand side may be evaluated by changing the variable of the integration as $\sqrt{t}=u$

$$f_1(x) = \frac{4}{\sqrt{\pi}\,(2-\pi\lambda_1\lambda_2)}\int_0^{+\infty} \cos(2au)\, e^{-x^2u^2}du = \frac{\pi}{2(2-\pi\lambda_1\lambda_2)}\,\frac{e^{-\frac{a^2}{x^2}}}{x}.$$

By following the same procedure, we may calculate $f_2(x)$ as follows,

$$f_2(x) = -\frac{2}{\pi}\int_0^{+\infty} \operatorname{Im}\left(\frac{\sqrt{\pi}}{2-\pi\lambda_1\lambda_2}\left[\frac{1}{s}+\frac{\sqrt{\pi}}{4}\lambda_2\frac{e^{-\frac{a}{s}}}{s^2}\right]\right)_{s\to\sqrt{t}e^{i\frac{\pi}{2}}} e^{-x^2t}dt$$

$$\left(\frac{\sqrt{\pi}}{2-\pi\lambda_1\lambda_2}\left[\frac{1}{s}+\frac{\sqrt{\pi}}{4}\lambda_2\frac{e^{-\frac{a}{s}}}{s^2}\right]\right)_{s\to\sqrt{t}\,e^{i\frac{\pi}{2}}} = \frac{\sqrt{\pi}}{2-\pi\lambda_1\lambda_2}\left[\frac{1}{\sqrt{t}\,e^{i\frac{\pi}{2}}}+\frac{\sqrt{\pi}}{4}\lambda_2\frac{e^{-\frac{a}{\sqrt{t}\,e^{i\frac{\pi}{2}}}}}{\left(\sqrt{t}\,e^{i\frac{\pi}{2}}\right)^2}\right]$$

$$=\frac{\sqrt{\pi}}{2-\pi\lambda_1\lambda_2}\left[\frac{-i}{\sqrt{t}}-\frac{\sqrt{\pi}}{4}\lambda_2\frac{e^{\frac{ai}{\sqrt{t}}}}{t}\right]=\frac{\sqrt{\pi}}{2-\pi\lambda_1\lambda_2}\left[\frac{-i}{\sqrt{t}}-\frac{\sqrt{\pi}}{4t}\lambda_2\left(\cos\frac{a}{\sqrt{t}}+i\sin\frac{a}{\sqrt{t}}\right)\right]$$

$$\operatorname{Im}\left(\frac{\sqrt{\pi}}{2-\pi\lambda_1\lambda_2}\left[\frac{1}{s}+\frac{\sqrt{\pi}}{4}\lambda_2\frac{e^{-\frac{a}{s}}}{s^2}\right]\right)_{s\to\sqrt{t}e^{i\frac{\pi}{2}}}=\frac{\sqrt{\pi}}{2-\pi\lambda_1\lambda_2}\left[\frac{-1}{\sqrt{t}}-\frac{\sqrt{\pi}}{4t}\lambda_2\sin\frac{a}{\sqrt{t}}\right]$$

$$f_2(x)=-\frac{2}{\pi}\int_0^{+\infty}\frac{\sqrt{\pi}}{2-\pi\lambda_1\lambda_2}\left[\frac{-1}{\sqrt{t}}-\frac{\sqrt{\pi}}{4t}\lambda_2\sin\frac{a}{\sqrt{t}}\right]e^{-x^2t}dt$$

$$f_2(x)=\frac{2}{\sqrt{\pi}\left(2-\pi\lambda_1\lambda_2\right)}\int_0^{+\infty}\left(\frac{1}{\sqrt{t}}+\frac{\sqrt{\pi}}{4t}\lambda_2\sin\frac{a}{\sqrt{t}}\right)e^{-x^2t}dt=$$

$$=\frac{2}{\sqrt{\pi}\left(2-\pi\lambda_1\lambda_2\right)}\left[\int_0^{+\infty}\frac{1}{\sqrt{t}}e^{-x^2t}dt+\lambda_2\frac{\sqrt{\pi}}{4}\int_0^{+\infty}\frac{1}{t}\sin\frac{a}{\sqrt{t}}e^{-x^2t}dt\right],$$

the integral $\int_0^{+\infty}\frac{1}{\sqrt{t}}e^{-x^2t}dt$ may be evaluated by changing the variable of the integration as $\sqrt{t}=u$,

$$\int_0^{+\infty}\frac{1}{\sqrt{t}}e^{-x^2t}dt=\frac{\sqrt{\pi}}{x},$$

but , for the integral $\int_0^{+\infty}\frac{1}{t}\sin\frac{a}{\sqrt{t}}e^{-x^2t}dt$ we change the variable of the integration as $\frac{1}{\sqrt{t}}=u$, then we obtain

$$f_2(x)=\frac{2}{\sqrt{\pi}\left(2-\pi\lambda_1\lambda_2\right)}\left[\frac{\sqrt{\pi}}{x}+\lambda_2\frac{\sqrt{\pi}}{4}\int_0^{+\infty}\sin(au)\,\frac{e^{-\frac{x^2}{u^2}}}{u}du\right]$$

$$f_2(x)=\frac{2}{x\left(2-\pi\lambda_1\lambda_2\right)}+\lambda_2\frac{1}{2\left(2-\pi\lambda_1\lambda_2\right)}\int_0^{+\infty}\sin(au)\,\frac{e^{-\frac{x^2}{u^2}}}{u}du\,.$$

4.1. Matrix form of the solution to the systems of singular integral equations (The general case)

$$\begin{cases} f_1(x) = g_1(x) + \lambda_1 \int_0^{+\infty} k_1(x,t)\; f_2(t)\; dt \\ f_2(x) = g_2(x) + \lambda_2 \int_0^{+\infty} k_2(x,t)\; f_3(t)\; dt \\ \cdots\cdots \\ f_{n-1}(x) = g_{n-1}(x) + \lambda_{n-1} \int_0^{+\infty} k_{n-1}(x,t)\; f_n(t)\; dt \\ f_n(x) = g_n(x) + \lambda_n \int_0^{+\infty} k_n(x,t)\; f_1(t)\; dt \end{cases} \tag{4.9}$$

Note: Let $L_2\{ f_1(x);s\} = F_1(s)$,…, $L_2\{ f_n(x);s\} = F_n(s)$, $L_2\{g_1(x);s\} = G_1(s)$, … , $L_2\{g_n(x);s\} = G_n(s)$ and assuming $\Phi_1(s), q_1(s),\ldots,\Phi_n(s), q_n(s)$ be analytic and such that, $L_2\{ K_1(x,t)\;;s\} = \Phi_1(s)\, x e^{-x^2 q_1^2(s)}$,…, $L_2\{ K_n(x,t)\;;s\} = \Phi_n(s)\, x e^{-x^2 q_n^2(s)}$, and using the above theorem, then taking L_2-transform of the above systems of integral equations (4.9), leads to the following relations,

$$\begin{cases} F_1(s) = G_1(s) + \lambda_1 \Phi_1(s)\cdot F_2(q_1(s)) \\ F_2(s) = G_2(s) + \lambda_2 \Phi_2(s)\cdot F_3(q_2(s)) \\ \quad\vdots \qquad\quad \vdots \qquad\qquad\qquad \vdots \\ F_{n-1}(s) = G_{n-1}(s) + \lambda_{n-1} \Phi_{n-1}(s)\cdot F_n(q_{n-1}(s)) \\ F_n(s) = G_n(s) + \lambda_n \Phi_n(s)\cdot F_1(q_n(s)) \end{cases} \tag{4.10}$$

In the case of trigonometric kernel, for example,
$k_1(x,t) = \cdots = k_2(x,t) = \sin xt$ we have

$$L_2\{\sin(xt);s\} = x\frac{\sqrt{\pi}}{4s^3} e^{-\frac{x^2}{4s^2}} \quad \Rightarrow \quad \Phi(s) = \frac{\sqrt{\pi}}{4s^3} \quad , \quad q(s) = \frac{1}{2s}$$

$$\begin{cases} F_1(s) = G_1(s) + \lambda_1 \Phi(s)\cdot F_2(\frac{1}{2s}) \\ F_2(s) = G_2(s) + \lambda_2 \Phi(s)\cdot F_3(\frac{1}{2s}) \\ \quad\vdots \qquad\quad \vdots \qquad\qquad\qquad \vdots \\ F_{n-1}(s) = G_{n-1}(s) + \lambda_{n-1} \Phi(s)\cdot F_n(\frac{1}{2s}) \\ F_n(s) = G_n(s) + \lambda_n \Phi(s)\cdot F_1(\frac{1}{2s}) \end{cases} \tag{4.11}$$

Now, in relation (4.11) we replace s by $\frac{1}{2s}$, to obtain

$$\begin{cases} F_1(\frac{1}{2s}) = G_1(\frac{1}{2s}) + \lambda_1\Phi(\frac{1}{2s})\cdot F_2(s) \\ F_2(\frac{1}{2s}) = G_2(\frac{1}{2s}) + \lambda_2\Phi(\frac{1}{2s})\cdot F_3(s) \\ \quad\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \\ F_{n-1}(\frac{1}{2s}) = G_{n-1}(\frac{1}{2s}) + \lambda_{n-1}\Phi(\frac{1}{2s})\cdot F_n(s) \\ F_n(\frac{1}{2s}) = G_n(\frac{1}{2s}) + \lambda_n\Phi(\frac{1}{2s})\cdot F_1(s) \end{cases} \tag{4.12}$$

Combination of (4.11) and (4.12) and calculation of $F_1(s),\cdots,F_n(s)$ leads to the following matrix equation $AX = b$, where matrix A and b and X are as following with $\det A \neq 0$.

$$A = \begin{bmatrix} 1 & 0 & -\lambda_1\lambda_2\Phi(s)\Phi(\frac{1}{2s}) & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \cdots & 1 & 0 & -\lambda_{n-2}\lambda_{n-1}\Phi(s)\Phi(\frac{1}{2s}) \\ -\lambda_{n-1}\lambda_n\Phi(s)\Phi(\frac{1}{2s}) & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & -\lambda_n\lambda_1\Phi(s)\Phi(\frac{1}{2s}) & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} F_1(s) \\ F_2(s) \\ \vdots \\ F_{n-1}(s) \\ F_n(s) \end{bmatrix} \quad , \quad b = \begin{bmatrix} G_1(s) + \lambda_1\Phi(s)G_2(\frac{1}{2s}) \\ G_2(s) + \lambda_2\Phi(s)G_3(\frac{1}{2s}) \\ \vdots \\ G_{n-1}(s) + \lambda_{n-1}\Phi(s)G_n(\frac{1}{2s}) \\ G_n(s) + \lambda_n\Phi(s)G_1(\frac{1}{2s}) \end{bmatrix}$$

6. Conclusion

In this article, the author derived simple inversion formula for the above- mentioned L_2-transform. We give also an application for the solution of the systems of non - homogeneous singular integral equations. They conclude by remarking that many identities involving various integral transforms can be obtained and some other type of singular integral equations with other type of kernels (Bessel's -kernels) can be solved by applying the results considered here.

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