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Fixed Point Theorems for Kannan Contractions and Weakly Contractive Mappings on a Modular Metric Space Endowed with a Graph

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Abstract. The notion of a modular metric spaces were introduced by Chistyakov [5, 6]. Abdou and Khamsi [1] gave the analog of Banach contraction principle in modular metric spaces. More recently, Alfuraidan [3] gave generalization of Banach contraction principle on a modular metric space endowed with a graph which is the modular metric version of Jachymski [8] fixed point results.

In this paper, we generalize and prove some fixed point results for Kannan contraction and weakly contractive mappings in a modular metric space endowed with a graph. The result of this paper is new and improving the previously known result in modular metric spaces endowed with a graph.

Keywords: Modular metric spaces, common fixed point, connected graph, Banach contraction, Kannan contraction.

AMS Mathematics Subject Classification (2010): 47H09, 46B20, 47H10, 47E10

1. Introduction

The existence of fixed points for single valued mappings in partially ordered metric spaces was initially considered by Ran and Reurings [15]. Fixed point theorems for monotone single valued mappings in a metric space endowed with a partial ordering have been widely investigated. Recently, many results appeared giving sufficient condition for *f to* be a PO if (X , d) is endowed with a partial ordering $\preccurlyeq$. These results are the hybrid of two fundamental and useful theorems in fixed point theory, Banach Contraction Principle and the Knaster-Tarski theorem (see[7]). Jachymski [8] obtain some useful result for mappings defined on a complete metric spaces endowed with a graph instead of partial ordering. Bojor [4] proved fixed point result for Kannan mappings in metric spaces endowed with a graph. Samreen and Kamran [16] proved fixed point theorems for weakly contractive mappings on a metric space endowed with a graph. After that many researchers have investigated in this direction by weakly contractive condition and analyzing connectivity condition of graph.

The notion of modular spaces was introduce by Nakano [13] and was intensively develop by Koshi and Shimogaki [11], Yamamuro [17] and by Musielak and Orlicz [12]. Recently, Aghanians and Nourozi [2] discuss the existence and uniqueness of the fixed point for Banach and Kannan contraction defined on modular spaces endowed with a graph.

The notion of a modular metric spaces was introduced by Chistyakov [5,6]. Further Abdou and Khamsi [1] gave the analog of Banach contraction principle in modular metric spaces. More recently, Alfuraidan [3] gave generalization of Banach contraction principle on a modular metric space endowed with a graph which is the modular metric version of Jachymski [8] fixed point results for mappings on a metric space with a graph.

Ran and Reurings [15] proved the following fixed points result.

Theorem 1.1. [15] Let (X,≼) be a partially ordered set such that every pair $x, y \epsilon X$ has an upper and lower bound. Let d be a metric on X such that (X, d) is a complete metric space. Let $f: X \to X$ be a continuous monotone (either order preserving or order reversing) mapping. Suppose that the following condition hold:

(1) There exist a $k\epsilon$ (0,1) with
$$d\big(f(x), f(y)\big) \preccurlyeq kd(x,y), \qquad for\ all\ x \succcurlyeq y.$$
(2) There exist an $x_0 \epsilon X$ with $x_0 \preccurlyeq f(x_0)$ or $x_0 \succcurlyeq f(x_0)$.

Then f is a Picard operator (PO), that is, f has a unique fixed point $x_* \epsilon X$ and for each $x\epsilon X, \lim_{n\to\infty} f^n x = x_*$.

Nieto et al. in [14], proved the following fixed point theorem.

Theorem 1.2. [14] Let (X,d) be a complete metric spaces endowed with a partial ordering≼.Let $f: X \to X$ be an order preserving mapping such that there exists a $k \epsilon [0,1)$ with
$$d\big(f(x), f(y)\big) \preccurlyeq kd(x,y), \qquad for\ all\ x \succcurlyeq y.$$
Assume that one of the following conditions holds:

(1) f is continuous and there exists an $x_0 \epsilon X$ with $x_0 \preccurlyeq f(x_0)$ or $x_0 \succcurlyeq fx_0$;
(2) (X,d, ≼) is such that for any non decreasing $(x_n)_{n\epsilon N}$, if $x_n \to x$, then $x_n \preccurlyeq x$ for $n\epsilon N$, and there exist an $x_0 \epsilon X$ with $x_0 \preccurlyeq f(x_0)$;
(3) (X,d, ≼) is such that for any non-decreasing$(x_n)_{n\epsilon N}$, if $x_n \to x$, then $x_n \succcurlyeq x$ for $n\epsilon N$, and there exist an $x_0 \epsilon X$ with $x_0 \succcurlyeq f(x_0)$;

then f has a fixed point. Moreover, if (X, ≼) is such that every pair of elements of X has an upper or a lower bound, then f is a PO.

Jachymski [9] obtained the contraction principle for mappings on a metric spaces endowed with a graph.

Theorem 1.3. [9] Let (X,d) be a complete metric spaceand let the triplet(X,d,G) have the following property:

(P) for any sequence $(x_n)_{n\epsilon N}$ in X as $n \to \infty$ and $(x_n, x_{n+1}) \epsilon E(G)$, then $(x_n, x) \epsilon E(G)$, for all n. Let $f: X \to X$ be a G-contraction. Then the following statements hold:

(1) $F_f \neq \emptyset$ if and only if $X_f \neq \emptyset$;
(2) if $X_f \neq \emptyset$ and G s weakly connected, then f is a Picard operator, i.e. $F_f = \{x^*\}$and sequence $\{f^n(x)\} \to x^*$ as $n \to \infty$, for all $x \epsilon X$;
(3) for any $x \epsilon X_f$, $f|_{[x]_{\tilde{G}}}$ is a Picard operator ;
(4) if $X_f \subseteq E(G)$, then f is a weakly Picard operator, i.e.,$F_f \neq \emptyset$ and, for each $x \epsilon X$, we have a sequence $\{f^n(x)\} \to x^*(x) \epsilon F_f$ as $n \to \infty$.

Bojor [4] proved fixed points of Kannan mappings in metric spaces endowed with a graph.

Theorem 1.4. [4] Let (X, d) be a complete metric space endowed with a graph G and $T: X \to X$ be a G-Kannan mapping. We suppose that:

(i) G is weakly T−connected;
(ii) for any $(x_n)_{n \in N}$in X, if $x_n \to x$ and $(x_n, x_{n+1}) \epsilon E(G)$for $n \in N$ then there is a subsequence $(x_{k_n})_{n \in N}$ with $(x_{k_n}, x) \in E(G)$for $n \in N$.

Then T is a PO.

Samreen and Kamran [16] proved fixed point theorem for weakly contractive mappings on a metric space endowed with a graph.

Theorem 1.5. [16] Let (X, d) be a completed metric space endowed with a graph G and f be a weakly G- contractive mapping from X into X. Suppose that the following condition holds.

(i) G satisfies property (p'),
(ii) there exist some $x_0 \epsilon X_f := \{x \epsilon X : (x, fx) \epsilon E(G)\}$.

Then $f|_{[x_0]_{\tilde{G}}}$ has a unique fixed point $\xi \epsilon [x_0]_{\tilde{G}}$ and $f^n y \to \xi$ for any $y \epsilon [x_0]_{\tilde{G}}$.

Aghaninas and Nourouzi [2] proved Banach and Kannan contraction in modular spaces with a graph.

Theorem 1.6. [2] Let X be a ρ-complete modular space endowed with a graph G and the triple (X, ρ, G). Moreover, this fixed point is unique if $k < \frac{1}{2}$ and Xsatisfies the following condition For all$x, y \epsilon X$, there exists a $z \epsilon X$ such that $(x, z), (y, z) \epsilon E(\tilde{G})$. Then a Kannan $\tilde{G} - \rho$ contraction $f: X \to X$ has a fixed point ifand only if $C_f \neq \emptyset$.

Alfuraidan [3] proved the contraction principle for mappings on a modular metric space with a graph.

Theorem 1.7. [3] Let (X, ω) be a modular metric space with a graphG_ω. Suppose that $\omega$is a convex regular modular metric which satisfies the $\Delta_2 -$ type condition. Assume that $M = V(G_\omega)$ is a nonempty $\omega -$ bounded, $\omega -$ complete subset of X_ω and the triple $(M, d_\omega^*, G_\omega)$ has property (P) Let $T: M \to M$ be G_ω-contraction map and $M_T := \{x \epsilon M; (x, Tx) \epsilon E(G_\omega)\}$.

If$(x_0, T(x_0)) \epsilon E(G_\omega)$, then the following statement holds:

(i) For any $x \epsilon M_T, T|_{[x]_{\widetilde{G_\omega}}}$has a fixed point.

(ii) If G_ω is weakly connected, then T has a fixed point in M.

(iii) If $M' := \cup\{[x]_{\widetilde{G_\omega}} : x \epsilon M_T\}$, then $T|_{M'}$ has a fixed point in M.

2. Basic definition and preliminaries

Let X be a nonempty set. Throughout this paper for a function $\omega : (0,\infty) \times X \times X \to (0,\infty)$ will be written as $\omega_\lambda(x,y) = \omega(\lambda,x,y)$ for all $\lambda > 0$ and $x, y \epsilon X$.

Definition 2.1. [5,6] Let X be a non-empty set. A function $\omega : (0,\infty) \times X \times X \to [0,\infty]$ is said to be a metric modular on X if it satisfies the following three axioms:

(i) given $x, y \epsilon X, \omega_\lambda(x,y) = 0 \; for \; all \; \lambda > 0 \; if \; and \; only \; if \; x = y$;

(ii) $\omega_\lambda(x,y) = \omega_\lambda(y,x) \; for \; all \; \lambda > 0$ and $x, y \epsilon X$;

(iii) $\omega_{\lambda+\mu}(x,y) \le \omega_\lambda(x,z) + \omega_\mu(z,y) for \; all \; \lambda, \mu > 0$ and $x, y, z \epsilon X$.

If instead of (i), we have only the condition

$$\omega_\lambda(x,x) = 0 \; for \; all \; \lambda > 0 \text{ and } x \epsilon X.$$

Then ω is said to be a (metric) pseudo modular on X. A modular ω on X is said to be regular if the following weaker version of (i) is satisfied:

$x = y$ if and only if $\omega_\lambda(x,y) = 0 \; for \; some \; \lambda > 0$.

Finally ω is said to be convex if for $\lambda, \mu > 0$ and x, y, zϵX, it satisfies the inequality

$$\omega_{\lambda+\mu}(x,y) = \frac{\lambda}{\lambda+\mu}\omega_\lambda(x,z) + \frac{\mu}{\lambda+\mu}\omega_\mu(z,y) .$$

Note that for a pseudo modularω on a set X and any $x, y \epsilon X$, the function $\lambda \to \omega_\lambda(x,y)$ is non increasing on $(0,\infty)$. Indeed, if $0 < \mu < \lambda$, then

$$\omega_\lambda(x,y) \le \omega_{\lambda-\mu}(x,x) + \omega_\mu(x,y) = \omega_\mu(x,y)$$

Definition 2.2. Let X_ω be a modular metric space.

(1) The sequence $(x_n)_{\, n\epsilon N}$ in X_ω is said to be convergent to $x \; \epsilon \; X_\omega$if

$$\omega_\lambda(x_n,x) \to 0 \; as \; n \to \infty \; for \; all \; \lambda > 0.$$

(2) The sequence $(x_n)_{\, n\epsilon N}$ in X_ω is said to be Cauchy if

$$\omega_\lambda(x_m,x_n) \to 0 \; as m,n \to \infty \; for \; all \; \lambda > 0.$$

(3) A subset C of X_ω is said to be closed if the limit of the convergent sequence of C always belong to C.

(4) A subset C of X_ω is said to be complete if any Cauchy sequence in C is a convergent sequence and its limit in C.

(5) A subset C of X_ω is said to be bounded if for all$\lambda > o$

$$\delta_\omega(C) = \sup\{\omega_\lambda(x,y); x, y \epsilon C\} < \infty.$$

We will use following notations and terminology of graph theory (see [3]) related to the rest of our result.

Let (X,ω) be a modular metric space and M be a non empty subset of X_ω. Let Δ denote the diagonal of the Cartesian product $M \times M$. Consider a directed graph G_ω such that the set $V(G_\omega)$ of its vertices coincide with M, and the set $E(G_\omega)$ of its edges contain all loops, i.e. $E(G_\omega) \supseteq \Delta$. We assume G_ω has no parallel edges (arcs), so we can identify G_ω with the pair $(V(G_\omega), E(G_\omega))$. Our graph theory notation and terminology are standard and can be found in all graph theory books, like [14]. Moreover, we may treat G_ω as a weighted graph (see [10]) by assigning to each edge the distance between its vertices.

By G^{-1} we denote the conversion of a graph G, i.e., the graph obtained from G by reversing the direction of edges. Thus we have

$$E(G^{-1}) = \{(y,x)|(x,y)\epsilon\, E(G)\}.$$

A diagraph G is called an oriented graph if whenever $(u,v)\epsilon\, E(G)$, then$(v,u) \notin E(G)$. The letter $\tilde{G}$ denotes the undirected graph obtain from G by ignoring the direction of edges.

Actually, it will be more convenient for us to treat $\tilde{G}$ as a directed graph for which the set of its edges is symmetric. Under this convention,
$E(\tilde{G}) = E(G) \cup E(G^{-1})$.

We call (V', E') a sub graph of $V' \subseteq V(G), E' \subseteq E(G)$, and for any edge $(x,y) \in E', x,y \in V'$.

If x and y are vertices in a graph G, then a (directed) path in G from x to y of length N is a sequence $(x_i)_{i=1}^{N}$ of $N+1$ vertices such that $x_0 = x, x_N = y$ and $(x_{n-1}, x_n) \in E(G)$ for $i = 1, \dots, N$. A graph G is connected if there is a directed path between any two vertices. G is a weakly connected if $\tilde{G}$ is connected. If G is such that $E(G)$ is symmetric and x is a vertex in G, then the sub graph G_x consisting of all edges and vertices which are contained in some path beginning at x is called the component of G containing x. In this case $V(G_x) = [x]_G$, where $[x]_G$ is the equivalence class of the following relation $\mathcal{R}$ defined on $V(G)$ by the rule:

$y\mathcal{R}z$ if there is a (directed) path in G from y to z.

Clearly G_x is connected.

Definition 2.3. [3] Let (X,ω) be a modular metric space and M be a non empty subset of X_ω. A mapping $T : M \to M$ is called
(i) G_ω- contraction if T preserve edges of G_ω, i.e.,

$$\forall\, x,y \,\epsilon\, M\ ((x,y)\epsilon E(G_\omega) \Longrightarrow \big(T(x),T(y)\big)\epsilon E(G_\omega)\big),$$

and if there exists a constant $\alpha\epsilon[0,1)$ such that

$$\omega_1\big(T(x),T(y)\big) \le \alpha\omega_1(x,y) \text{ for any } (x,y)\epsilon E(G_\omega).$$

(ii) $(\varepsilon,\alpha) - G_\omega$-uniformly locally contraction if T preserve edges of G_ω and there exists a Constant $\alpha \in [0,1)$ such that for any $(x,y)\epsilon E(G_\omega)$
$\omega_1(T(x),T(y)) \le \alpha\omega_1(x,y)$ whenever $\omega_1(x,y) < \varepsilon$.

Definition 2.4. [3] A point $x\,\epsilon\, M$ is called a fixed point of T whenever $x = T(x)$. The set of fixed points of T will be denoted by Fix(T).

Now we introduce the G_ω Kannan contraction and weakly G_ω contractive mappings in a modular metric space endowed with a graph as follows.

Definition 2.5. Let (X,ω) be a modular metric space with a graph G_ω. Amapping $T : M \to M$ is called

(1) G_ω- Kannan contraction if T preserve the edges of G_ω ,
i.e., for all $x,y\,\epsilon M$ ($(x,y)\,\epsilon\, E(G_\omega) \Longrightarrow (Tx,Ty)\epsilon E(G_\omega)$)
and if there exists positive number $k\epsilon(0,\frac{1}{2})$ such that

$\omega_\lambda(Tx,Ty) \le k(\omega_\lambda(Tx,x) + \omega_\lambda(Ty,y))$ for any $x, y \epsilon M$ with $(x,y) \epsilon E(G_\omega)$.

(2) weakly G_ω contractive if T preserve the edges of G_ω,
i.e., for all $x, y \epsilon M$ $((x,y) \epsilon E(G_\omega) \Longrightarrow (Tx,Ty) \epsilon E(G_\omega))$
and $\omega_\lambda(Tx,Ty) \le \omega_\lambda(x,y) - \psi\big(\omega_\lambda(x,y)\big)$
whenever $\psi: [0,\infty) \to [0,\infty)$ is continuous non decreasing such that ψ is positive on$(0,\infty)$ and $\psi(0) = 0$.

Our first result can be seen as an extension of Jachymski [9] fixed point theorems to modular metric spaces. As Jachymski [8] did, we introduce the following property.
We say that the triple $(M, d_\omega^*, G_\omega)$ has property (P) if
(P) For any sequence $\{x_n\}_{n\in N}$in M, if $x_n \to x$ as $n \to \infty$ and $(x_n, x_{n+1}) \epsilon E(G_\omega)$, then $(x_n, x) \epsilon E(G_\omega)$, for all n.

Note that property (P) is precisely the Nieto et al. [14] hypothesis relaxing continuity assumption as in Theorem 1.2 ((2) and (3)) rephrased in terms of edges.

Lemma 2.1. [16] Let (X, d) be a metric space and $T: X \to X$ be a weakly G-contractive map. Then for any $x \epsilon X$ and $y \epsilon [x]_{\tilde{G}}$ we have

$$\lim_{n\to\infty} d(T^n x, T^n y) = \lim_{n\to\infty} r(T^n x, T^n y) = 0.$$

Proposition 2.2. [16] Let (X, d) be a metric space and T be a weakly G-contractive mapping from X into X. Let there exist $x_0 \epsilon X$ such that $Tx_0 \epsilon [x_0]_{\tilde{G}}$ then the sequence $\{T^n x_0\}$ is Cauchy.

3. Main results

Theorem 3.1. Let (X, ω) be a modular metric space with a graph G_ω. Suppose that ω is a convex regular modular metric which satisfies the Δ_2- type condition. Assume that $M = V(G_\omega)$ is a nonempty $\omega-$ bounded, $\omega-$complete subset of X_ω and the triple $(M, d_\omega^*, G_\omega)$ has property (P). Let $T: M \to M$ be Kannan contraction map and $M_T := \{x \epsilon M; (x, Tx) \epsilon E(G_\omega)\}$.

If $(x_0, T(x_0)) \epsilon E(G_\omega)$, then the following statements hold:

(i) For any $x \epsilon M_T, T|_{[x]_{\widetilde{G_\omega}}}$, has a fixed point.

(ii) If G_ω is weakly connected, then T has a fixed point in M.

(iii) If$M' = \cup\{[x]_{\widetilde{G_\omega}}: x \epsilon M_T\}$, then $T|_{M'}$ has a fixed point in M.

Proof (i): As $(x_0, T(x_0)) \epsilon E(G_\omega)$ and $(y_0, T(y_0)) \epsilon E(G_\omega)$ then$x_0, y_0 \epsilon M_T$. Since T is a Kannan-contraction, there exists a constant $k \in (0, \frac{1}{2})$ such that $(T(x_0), T(y_0)) \epsilon E(G_\omega)$ and $\omega_1(Tx_0, Ty_0) \le k[\omega_1(x_0, Tx_0) + \omega_1(y_0, Ty_0)]$ (3.1.1)

By induction we can construct a sequence $\{x_n\}$ such that $x_{n+1} = Tx_n$ and $(x_n, x_{n+1}) \epsilon E(G_\omega)$

$$\omega_1(x_{n+1}, x_n) = \omega_1(Tx_n, Tx_{n-1})$$

$$\omega_1(x_{n+1}, x_n) \le k[\omega_1(Tx_n, x_n) + \omega_1(Tx_{n-1}, x_{n-1})]$$

$$\le k[\omega_1(x_{n+1}, x_n) + \omega_1(x_n, x_{n-1})]$$

$$(1-k)\,\omega_1(x_{n+1}, x_n) \le k\,\omega_1(x_n, x_{n-1})$$

$\omega_1(x_{n+1}, x_n) \le \frac{k}{(1-k)} \omega_1(x_n, x_{n-1})$ where $\alpha = \frac{k}{(1-k)} < 1$

$$\omega_1(x_{n+1}, x_n) \le \alpha \omega_1(x_n, x_{n-1})$$

So by induction, we construct a sequence $\{x_n\}$ such that $(x_{n+1}, x_n) \epsilon E(G_\omega)$ and $\omega_1(x_{n+1}, x_n) \leq \alpha^n \omega_1(x_0, x_1)$ for any $n \geq 1$. Since M is ω-bounded, we have

$$\omega_1(x_{n+1}, x_n) \leq \delta_\omega(M) k^n$$

for any $n \geq 1$. Then by lemma 2.2 .

$\Longrightarrow \{x_n\}$ is ω -Cauchy. Since M is $\omega -$ Complete, therefore $\{x_n\}$ is ω - convergence to some point ϵM.

By property (P), $(x_n, x) \epsilon E(G_\omega)$ for all n and hence

$$\begin{aligned} \omega_1\big(x_{n+1}, T(x)\big) &= \omega_1(Tx_n, Tx) \\ &\leq k(\omega_1(Tx_n, x_n) + \omega_1(Tx, x)) \end{aligned}$$

Taking limit $n \to \infty$ both sides we get

$$\omega_1(x, Tx) \leq k(\omega_1(x, x) + \omega_1(Tx, x))$$

i.e. $\omega_1(x, Tx) \leq k\omega_1(Tx, x))$ which is a contradiction.

Hence $\omega_1(x, Tx) = 0$.

Therefore $x = Tx$.

i.e. x is a fixed point of T.

As $(x_0, x) \epsilon E(G_\omega)$, we have $x \,\epsilon [x_0]_{\widetilde{G_\omega}}$.

Uniqueness. Let x and y be two fixed point of T.

Consider $\omega_1(x, y) = \omega_1(Tx, Ty) \leq k[\omega_1(x, Tx) + \omega_1(y, Ty)]$

$$\leq k[\omega_1(x, x) + \omega_1(y, y)]$$

This gives

$$\omega_1(x, y) = 0 \Longrightarrow x = y.$$

Hence point is unique.

(ii) Since $M_T \neq \emptyset$, there exists an $x_0 \epsilon M_T$ and since G_ω is weakly connected, then $[x_0]_{\widetilde{G_\omega}} = M$ and by M and by (i), mapping T has a fixed point.

(iii) It follows easily from (i) and (ii).

Theorem 3.2. Let (X, ω) be a modular metric space with a graph G_ω. Suppose that ω is a convex regular modular metric which satisfies the Δ_2- type condition. Assume that $M = V(G_\omega)$ is a nonempty $\omega -$ bounded, $\omega -$ complete subset of X_ω and the triple $(M, d_\omega^*, G_\omega)$ has property (P). Let $T : M \to M$ be weak contraction mapping and $M_T := \{x \,\epsilon\, M; (x, Tx) \epsilon E(G_\omega)\}$.

If $(x_0, T(x_0)) \epsilon E(G_\omega)$, then the following statements hold:

(iv) For any $x \,\epsilon\, M_T, T|_{[x]_{\widetilde{G_\omega}}}$, has a fixed point.

(v) If G_ω is weakly connected, then T has a fixed point in M.

(vi) If $M' = \bigcup\{[x]_{\widetilde{G_\omega}} : x \,\epsilon\, M_T\}$, then $T|_{M'}$ has a fixed point in M.

Proof: As $(x_0, T(x_0)) \epsilon E(G_\omega)$ and $(y_0, T(y_0)) \epsilon E(G_\omega)$ then $x_0, y_0 \,\epsilon\, M_T$. Since T is a weak contraction, there exists a constant $k \in (0, \frac{1}{2})$ such that $(T(x_0), T(y_0)) \epsilon E(G_\omega)$ and

$$\omega_1(Tx_0, Ty_0) \leq \omega_1(x_0, y_0) - \Psi(\omega_1(x_0, y_0))$$

By induction we can construct a sequence $\{x_n\}$ such that $x_{n+1} = Tx_n$ and

$$(x_n , x_{n+1}) \epsilon E(G_\omega) \omega_1(x_{n+1}, x_n) = \omega_1(Tx_n, Tx_{n-1})$$

$$\omega_1(x_{n+1}, x_n) \leq \omega_1(x_n, x_{n-1}) - \psi(\omega_1(x_n, x_{n-1}))$$

$$\omega_1(x_{n+1}, x_n) \leq \omega_1(x_n, x_{n-1})$$

Similarly

$$\omega_1(x_{n+2}, x_{n+1}) = \omega_1(Tx_{n+1}, Tx_n)$$
$$\omega_1(x_{n+2}, x_{n+1}) \leq \omega_1(x_{n+1}, x_n) - \Psi(\omega_1(x_{n+1}, x_n))$$
$$\omega_1(x_{n+2}, x_{n+1}) \leq \omega_1(x_{n+1}, x_n)$$
$$\omega_1(x_{n+3}, x_{n+2}) = \omega_1(Tx_{n+2}, Tx_{n+1})$$
$$\omega_1(x_{n+3}, x_{n+2}) \leq \omega_1(x_{n+2}, x_{n+1}) - \Psi(\omega_1(x_{n+2}, x_{n+1}))$$
$$\omega_1(x_{n+3}, x_{n+2}) \leq \omega_1(x_{n+2}, x_{n+1})$$

Hence in general

$$\omega_1(x_{i+1}, x_n) \leq \omega_1(x_i, x_{i-1}) - \Psi(\omega_1(x_i, x_{i-1}); \qquad \forall\, i = 1,2,3 \dots n$$

Since Ψ is non decreasing and this shows that $\{x_i\}_{i=1}^{n}$ is a ω-cauchy sequence

$$\omega_1(x_{n+1}, x_n) \leq \omega_1(x_n, x_{n-1}) \leq \cdots \leq \omega_1(x_1, x_0)$$

Since $\omega_1(x_{n+1}, x_n)$ is non increasing sequence of non-negative real number bounded below by 0, thus convergent.

Taking limit as n →∞,

$\lim_{n\to\infty} \omega_1(x_{n+1}, x_n) = 0; \qquad \forall\, i = 1,2,3 \dots n.$

Uniqueness: Let x and y be two fixed point of T.

Consider $\omega_1(x, y) = \omega_1(Tx, Ty) \leq \omega_1(x, y) - \psi\omega_1(x, y)]$

This gives

$$\omega_1(x, y) = 0 \Rightarrow x = y.$$

Hence point is unique.

(ii) Since $M_T \neq \emptyset$, there exists an $x_0 \epsilon\, M_T$ and since G_ω is weakly connected, then $[x_0]_{\widetilde{G_\omega}} = M$ and by M and by (i), mapping T has a fixed point.

(iii) It follows easily from (i) and (ii).

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