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Solving Fuzzy Linear Fractional Programming Problem using LU Decomposition Method

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Abstract. This paper proposes a new approach to solve fuzzy linear fractional programming problem (FLFPP). In this paper, the FLFPP is converted into crisp linear fractional programming problem (LFPP) using ranking method. The converted LFPP is then solved by LU Decomposition method. An illustrated example shows the simplicity of the proposed approach.

Keywords: Linear fractional programming, LU decomposition, triangular fuzzy number.

AMS Mathematics Subject Classification (2010): 90C32, 03E72

1. Introduction

Linear fractional programming problem is mainly used in decision making process in which the objective function is a fraction of two linear functions. Fractional programming problems are used in many fields such as production planning, financial and corporate planning, health care and hospital planning, etc. Many researchers found various techniques to solve linear fractional programming problems.

Charnes and Cooper [1] transformed the linear fractional programming problem into the linear programming problem by adding a new constraint and a new variable and then the optimum solution is obtained by simplex method. Chinchole and Bhadane [2] proposed LU factorization method to solve linear programming problem. Jayanth Karthik and Chandrasekaran [3] solved fully fuzzy linear systems with trapezoidal fuzzy number matrices by partitioning the block matrices. Pandian and Kavitha [4] proposed a new method namely, parallel moving method to find the optimal solution to the fuzzy assignment problem. Radhakrishnan, et.al. [5] found the positive solution of the fully fuzzy system of linear equations using Cramer's rule along with Dodgson's consideration. Jain [6] proposed a Modified Gauss elimination technique to solve a separable nonlinear programming problem. Sharma and Bansal [7] used branch and bound method to find the integer solution of fractional programming problems. Swarup [8] proposed a simplex algorithm to solve linear fractional programming problem. Tantawy [9] presented an iterative method based on the conjugate gradient projection method for solving LFPP. LU Decomposition is just a compact and relatively numerically stable method to solve a system of linear equations. For large n, the computational time for LU Decomposition is proportional to $\frac{4n^3}{3}$, while for Gaussian Elimination, the computational time is proportional to $\frac{n^4}{3}$. So for large n, the ratio of the computational time for Gaussian elimination to computational time for LU Decomposition is $\frac{n^4}{3} / \frac{4n^3}{3} = \frac{n}{4}$. As an example, for the coefficient matrix of order 2000×2000, computational time by Gaussian Elimination would take n/4=2000/4=500 times the time it would take for LU Decomposition.

This paper is outlined as follows. Section 2 gives the preliminaries of the fuzzy number. In Section 3, mathematical formulation of the FLFPP is given. Section 4 describes the proposed method and Section 5 explains how the proposed method is applied to the LFPP. In Section 6, Yager's ranking method [10] is given and Section 7 gives the illustrated example and Section 8 concludes the paper.

2. Preliminaries

In this section, some basic definitions relating to fuzzy sets and triangular fuzzy numbers are given.

Definition 2.1. A fuzzy set A is called normal if there is at least one element $x \in X$ such that $\mu_A(x) = 1$.

Definition 2.2. A fuzzy set A is called convex if for any $x, y \in X$ and any $\lambda \in [0,1]$, $\mu_A(\lambda x + (1-\lambda)y) \geq \min\{\mu_A(x), \mu_A(y)\}$.

Definition 2.3. The α-level cut of a fuzzy set A is defined by $A_\alpha = \{x \in X / \mu_A(x) \geq \alpha\}$

Definition 2.4. A fuzzy set is a fuzzy number if it satisfies the conditions of normality and convexity.

Definition 2.5. Triangular fuzzy number

A triangular fuzzy number $\tilde{a} = (a_1, a_2, a_3)$ with membership function $\mu(x)$ given by

$$\mu(x) = \begin{cases} \dfrac{x-a_1}{a_2-a_1}, & for\ a_1 \leq x \leq a_2 \\ \dfrac{a_3-x}{a_3-a_2}, & for\ a_2 \leq x \leq a_3 \\ 0, & elsewhere \end{cases}$$

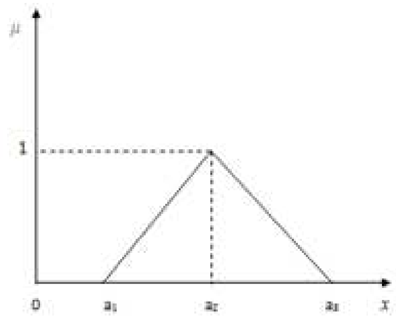

Figure 1: Triangular fuzzy number

Mathematical Formulation of FLFPP

Maximize $\tilde{z} = \dfrac{\tilde{c}^T x + \tilde{\alpha}}{\tilde{d}^T x + \tilde{\beta}}$

Subject to

$\tilde{A}x \le \tilde{b}$

$x \ge 0$

where x is an n-dimensional vector of decision variables, and $\tilde{c}, \tilde{d}$ are $n \times 1$ vectors, $\tilde{A}$ is an $m \times n$ constraint fuzzy matrix, $\tilde{b}$ is an m- dimensional fuzzy vector, $\tilde{\alpha}$ and $\tilde{\beta}$ are scalars.

3. LU decomposition method

Given a system of 'n' linear equations with n unknowns. We write this system as $AY = B$ where $A = \lfloor a_{ij} \rfloor_{n \times n}, Y = \lfloor y_j \rfloor_{n \times 1}$ $B = [b_i]_{n \times 1}$

(i) Write $A = LU$, where L is the unit lower triangular matrix and U is the upper triangular matrix. From this equation, we find L and U.

(ii) Now the system of equations becomes $LUY = B$.

(iii) Let $UY = W$. Now, we solve $LW = B$ for W

(iv) Using W, we solve $UY = W$ for Y. This will give the solution for the system $AY = B$.

4. Application of LU decomposition method to solve linear fractional programming problem

Consider the following Linear Fractional Programming Problem.

$$Maximize\, z = \frac{c_1 x_1 + c_2 x_2 + ... + c_{n-2} x_{n-2} + \alpha}{d_1 x_1 + d_2 x_2 + ... + d_{n-2} x_{n-2} + \beta}$$

Subject to

$$a_{31} x_1 + a_{32} x_2 + ... + a_{3,n-2} x_{n-2} \le b_3$$

$$a_{41} x_1 + a_{42} x_2 + ... + a_{4,n-2} x_{n-2} \le b_4$$

....

....

$$a_{n1} x_1 + a_{n2} x_2 + ... + a_{n,n-2} x_{n-2} \le b_n$$

$$x_1, x_2, ..., x_{n-2} \ge 0 \qquad \cdots\cdots\cdots (1)$$

Let us convert this linear fractional programming problem into linear programming problem using Charnes and Cooper method as below.

Let $t = \dfrac{1}{d_1x_1 + d_2x_2 + ... + d_{n-2}x_{n-2} + \beta}$ and $y_i = tx_i$, for $i = 1,2,3.........,n-2$

Now, we write the problem (1) as the following LPP.

$$
\begin{aligned}
& \textit{Maximize } z = c_1y_1 + c_2y_2 + ... + c_{n-2}y_{n-2} + \alpha t \\
& \textit{Subject to} \\
& d_1y_1 + d_2y_2 + ... + d_{n-2}y_{n-2} + \beta t = 1 \\
& a_{31}y_1 + a_{32}y_2 + ... + a_{3,n-2}y_{n-2} \le b_3t \\
& a_{41}y_1 + a_{42}y_2 + ... + a_{4,n-2}y_{n-2} \le b_4t \\
& .... \\
& .... \\
& a_{n1}y_1 + a_{n2}y_2 + ... + a_{n,n-2}y_{n-2} \le b_nt \\
& y_1, y_2, ..., y_{n-2}, t \ge 0
\end{aligned}
\qquad ,,, ,,, ,,, (2)
$$

We write this LPP in the form of less than or equality constraints.

$$
\begin{aligned}
& -c_1y_1 - c_2y_2 - .... - c_{n-2}y_{n-2} - \alpha t + z \le 0 \\
& d_1y_1 + d_2y_2 + ... + d_{n-2}y_{n-2} + \beta t = 1 \\
& a_{31}y_1 + a_{32}y_2 + ... + a_{3,n-2}y_{n-2} - b_3t \le 0 \\
& a_{41}y_1 + a_{42}y_2 + ... + a_{4,n-2}y_{n-2} - b_4t \le 0 \\
& .... \\
& .... \\
& a_{n1}y_1 + a_{n2}y_2 + ... + a_{n,n-2}y_{n-2} - b_nt \le 0 \\
& -y_1, -y_2, ..., -y_{n-2}, -t \le 0
\end{aligned}
\qquad \cdots \cdots \cdots (3)
$$

Now the system of equations can be considered as $AY = B$ where

$$
A = \begin{bmatrix}
-c_1 & -c_2 & ... & -c_{n-2} & -\alpha & 1 \\
d_1 & d_2 & ... & d_{n-2} & \beta & 0 \\
a_{31} & a_{32} & ... & a_{3,n-2} & -b_3 & 0 \\
a_{41} & a_{42} & ... & a_{4,n-2} & -b_4 & 0 \\
... & ... & ... & ... & ... & ... \\
a_{n1} & a_{n2} & ... & a_{n,n-2} & -b_n & 0
\end{bmatrix}
$$

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \cdots \\ y_{n-2} \\ t \\ z \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \ldots \\ 0 \end{bmatrix}.$$

Yager's ranking method

Given a convex triangular fuzzy number $\tilde{C} = (a,b,c)$, the α-cut of the fuzzy number $\tilde{C}$ is given by $\left(C_\alpha{}^L, C_\alpha{}^U\right) = \left((b-a)\alpha + a, c-(c-b)\alpha\right)$

The Yager's Ranking index [10] is defined by

$$R\left(\tilde{C}\right) = \int_0^1 0.5\left(C_\alpha{}^L + C_\alpha{}^U\right) d\alpha,$$

where $\left(C_\alpha{}^L + C_\alpha{}^U\right)$ is a α-level cut of fuzzy number $\tilde{C}$.

5. Numerical example

Consider the following FLFPP

$$Maximize\ z = \frac{(1,2,3)x_1 + (0,1,2)x_2}{(1,3,5)x_1 + (1,1,1)x_2 + (3,6,9)}$$

Subject to

$$(1,5,9)x_1 + (2,3,4)x_2 \leq (5,6,7)$$
$$(4,5,10)x_1 + (0,1,2)x_2 \leq (4,6,8) \qquad \cdots\cdots\cdots (4)$$
$$x_1, x_2 \geq 0$$

Now, we convert the FLFPP into the following crisp LFPP using Yager's ranking method.

The α-cut of fuzzy number $(1,2,3)$ is $\left(C_\alpha{}^L, C_\alpha{}^U\right) = (\alpha+1, 3-\alpha)$

$$R(1,2,3) = \int_0^1 0.5(\alpha + 1 + 3 - \alpha) d\alpha = 2$$

Proceeding similarly, the problem (4) can be written as the following crisp LFPP

$$Maximize\ z = \frac{2x_1 + x_2}{3x_1 + x_2 + 6}$$

Subject to

$5x_1 + 3x_2 \le 6$

$7x_1 + x_2 \le 6$

$x_1, x_2 \ge 0$

Let $\dfrac{1}{3x_1 + x_2 + 6} = t$ and $tx_1 = y_1$, $tx_2 = y_2$

The given LFPP becomes a LPP as below.

$Maximize\ z = 2y_1 + y_2$

Subject to

$3y_1 + y_2 + 6t = 1$

$7y_1 + y_2 - 6t \le 0$

$5y_1 + 3y_2 - 6t \le 0$

$y_1, y_2, t \ge 0$

Solution by LU decomposition method:

We write the above LPP as

$-2y_1 - y_2 + z \le 0$

$3y_1 + y_2 + 6t = 1$

$7y_1 + y_2 - 6t \le 0$

$5y_1 + 3y_2 - 6t \le 0$

$-y_1, -y_2, -t \le 0$

We write the system as $AY = B$

$$\text{where } A = \begin{bmatrix} -2 & -1 & 0 & 1 \\ 3 & 1 & 6 & 0 \\ 7 & 1 & -6 & 0 \\ 5 & 3 & -6 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ t \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

We write $A = LU$ where L is a unit lower triangular matrix U is an upper triangular matrix.

That is, $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}$

Now, $\begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix} = \begin{bmatrix} -2 & -1 & 0 & 1 \\ 5 & 3 & -6 & 0 \\ 7 & 1 & -6 & 0 \\ 3 & 1 & 6 & 0 \end{bmatrix}$

On simplification, we get,

$$u_{11} = -2, u_{12} = -1, u_{13} = 0, u_{14} = 1,$$
$$l_{21} = -\frac{3}{2}, u_{22} = -\frac{1}{2}, u_{23} = 6, u_{24} = \frac{3}{2},$$
$$l_{31} = -\frac{7}{2}, l_{32} = -5, u_{33} = -36, u_{34} = -4,$$
$$l_{41} = -\frac{5}{2}, l_{42} = -1, l_{43} = 0, u_{44} = 4$$

Thus, $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{3}{2} & 1 & 0 & 0 \\ -\frac{7}{2} & 5 & 1 & 0 \\ -\frac{5}{2} & -1 & 0 & 1 \end{bmatrix}$, $U = \begin{bmatrix} -2 & -1 & 0 & 1 \\ 0 & -\frac{1}{2} & 6 & \frac{3}{2} \\ 0 & 0 & -36 & -4 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

Now, $LUY = B$. We write $LW = B$ where $UY = W$.

Now, we solve *LW*=*B* for *W*, where $W = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{3}{2} & 1 & 0 & 0 \\ -\frac{7}{2} & 5 & 1 & 0 \\ -\frac{5}{2} & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

On simplification, we get

$w_1 = 0, w_2 = 1, w_3 = -5, w_4 = 1$

Finally, the solution matrix $Y = \begin{bmatrix} y_1 \\ y_2 \\ t \\ z \end{bmatrix}$ is given by UY=W.

That is, $\begin{bmatrix} -2 & -1 & 0 & 1 \\ 0 & -\frac{1}{2} & 6 & \frac{3}{2} \\ 0 & 0 & -36 & -4 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ t \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -5 \\ 1 \end{bmatrix}$

On simplification we get,

$y_1 = \frac{1}{12}, y_2 = \frac{1}{12}, t = \frac{1}{9}, z = \frac{1}{4}$

Thus, the optimal solution of the LFPP is given by

$x_1 = \frac{y_1}{t} = \frac{3}{4}, x_2 = \frac{y_2}{t} = \frac{3}{4}, \ z = \frac{1}{4}$

6. Conclusion

In this paper, we proposed a new approach called LU Decomposition Method of matrices to solve fuzzy linear fractional programming problem (FLFPP). The FLFPP is converted into crisp LFPP using Yager's Ranking method and then it is converted into LPP using Charnes and Cooper method. In this approach, we get the solution directly and the time consumption is very less. A numerical example is given to show the simplicity of the proposed approach and the solution is verified by LINGO 13.0 version also.

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