On Edge Trimagic Labeling of Umbrella, Dumb Bell and Circular Ladder Graphs

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Received 2 January 2017; accepted 10 February 2017

Abstract. An edge trimagic total labeling of a graph $G = (V, E)$ with $p$ vertices and $q$ edges is a bijection $f: V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p+q\}$ such that for each edge $uv \in E(G)$, the value of $f(u) + f(uv) + f(v)$ is either $k_1$, $k_2$, or $k_3$. In this paper, we prove that the edge trimagic total labeling of Umbrella, Dumb bell and Circular ladder graphs.

Keywords: Graph labeling, Bijective function, Umbrella, Dumb bell, Circular ladder, Edge Trimagic.

AMS Mathematics Subject Classification (2010): 05C72

1. Introduction

Graph labeling was first introduced in the mid sixties. A labeling of a graph is a map of integers to vertices or sometimes edges in a graph based upon certain criteria. In this paper the domain will be the set of all vertices and edges and such labeling are called total labeling \cite{12}. Graph labeling are of many types such as graceful, harmonious, elegant, cordial magic, antimagic, bimagic, etc. This paper is an attempt to study of edge Trimagic total labeling. Harary \cite{4} is referred to know about the notations in graph theory.

Magic labeling was introduced by Sedlacek \cite{11}. Kotzing and Rosa \cite{9}, defined edge magic of a graph $G$ with a bijection $f: V \cup E \rightarrow \{1, 2, \ldots, p+q\}$ such that, for each edge $uv \in E(G)$, $f(u) + f(uv) + f(v)$ is a magic constant. In \cite{3} shows the cycle $C_n$ with $P_3$ chords are edge magic total labeling. Edge bimagic labeling of graphs was introduced by Babujee \cite{2} in 2004, defined by a graph $G$ with a bijection $f: V \cup E \rightarrow \{1, 2, \ldots, p + q\}$ such that for each edge $uv \in E(G)$, the value of $f(u) + f(uv) + f(v)$ is either $k_1$ or $k_2$. Magic and bimagic labeling for disconnected graphs are showed in \cite{1}.

In 2013, Jayasekaran et al. \cite{6} introduced the edge trimagic total labeling of graphs. An edge trimagic total labeling of a (p, q) graph $G$ is a bijection $f: V \cup E \rightarrow \{1, 2, \ldots, p+q\}$ such that for each edge $uv \in E(G)$, the value of $f(u) + f(uv) + f(v)$ is equal to any of the distinct constants $k_1$ or $k_2$ or $k_3$. An edge trimagic total labeling is called a super edge
tragic total labeling of a graph G, if the vertices are labeled with the smallest possible integers i.e. 1, 2, ..., p. In [7], edge tragic labeling of digraphs were discussed.

The graph $F_n = P_n + K_1$ is called a fan [10] where $P_n : u_1u_2...u_n$ be a path and $V(K_1) = u$. The umbrella [10] $U_{n,m}, m > 1$ is obtained from a fan $F_n$ by passing the end vertex of the path $P_n : v_1v_2...v_m$ to the vertex of $K_1$ of the fan $F_n$. The graph obtained by joining two disjoint cycles $u_1u_2...u_n$ and $v_1v_2...v_m$ with an edge $u_1v_1$ is called dumbbell [10] graph $Db_n$. A circular ladder [5] $CL(n)$ is the union of an outer cycle $C_0 : u_1u_2...u_nu_1$ and an inner cycle $C_1 : v_1v_2...v_nv_1$ with additional edges $(u_iv_i), i = 1, 2, 3, ..., n$ called spokes.

For more references, we use dynamic survey of graph labeling by Gallian [8]. In this paper, we prove that the graphs such as umbrella, dumb bell and circular ladder are edge tragic and super edge tragic labeling.

2. Main results

Theorem 2.1. The Umbrella $U_{n,m}$ is an edge tragic total labeling for all n.

Proof: Let $V = \{u_i, v_j / 1 \leq i \leq n, 1 \leq j \leq m\}$ be the vertex set and $E = \{u_iu_{i+1}, v_jv_{j+1} / 1 \leq i \leq n-1, 1 \leq j \leq m-1\} \cup \{v_1u_i / 1 \leq i \leq n\}$ be the edge set of the graph $U_{n,m}$. Then $U_{n,m}$ has $n+m$ vertices and $2n+m-2$ edges.

Case 1. Both n and m are odd

Define a bijection $f: V \cup E \rightarrow \{1, 2, ..., 3n+2m-2\}$ such that

$$f(u_i) = \begin{cases} 
 m + \frac{i+1}{2}, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\
 m + \frac{n+i+1}{2}, & 1 \leq i \leq n \text{ and } i \text{ is even} 
\end{cases}$$

$$f(v_j) = \begin{cases} 
 \frac{j+1}{2}, & 1 \leq j \leq m \text{ and } j \text{ is odd} \\
 m + \frac{j+1}{2}, & 1 \leq j \leq m \text{ and } j \text{ is even} 
\end{cases}$$

$f(u_iu_{i+1}) = 3n+m-i, 1 \leq i \leq n-1; f(v_jv_{j+1}) = 3n+2m-j-1, 1 \leq j \leq m-1$ and

$$f(v_1u_i) = \begin{cases} 
 \frac{2n + m + 1}{2} - \frac{i+1}{2}, & i \text{ is odd} \\
 \frac{n + m + \frac{n-i+1}{2}}, & i \text{ is even} 
\end{cases}$$

To prove this labeling is an edge tragic total labeling.

Consider the edges $v_ju_i, 1 \leq i \leq n$.

For odd i, $f(v_i) + f(v_iu_i) + f(u_i) = 1 + 2n + m + 1 \cdot \frac{i+1}{2} + m + \frac{i+1}{2} = 2n + 2m + 2 = \lambda_i$.

For even i, $f(v_i) + f(v_iu_i) + f(u_i) = 1 + n + m + \frac{n-i+1}{2} + m + \frac{n+i+1}{2} = 2n + 2m + 2 = \lambda_i$.

Consider the edges $u_iu_{i+1}, 1 \leq i \leq n-1$. 

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For odd $i$, $f(u_i)+f(u_{ui})=m+\frac{i+1}{2}+3n+m-i+m+\frac{n+i+2}{2}=\frac{7n+6m+3}{2}=\lambda_2$.

For even $i$, $f(u_i)+f(u_{ui})=m+\frac{n+i+1}{2}+3n+m-i+m+\frac{i+2}{2}=\frac{7n+6m+3}{2}=\lambda_2$.

Consider the edges $v_jv_{j+1}$, $1 \leq j \leq m-1$.

For odd $j$, $f(v_j)+f(v_{j+1})+f(v_{j+1})=\frac{j+1}{2}+3n+2m-j+1+\frac{m+j+2}{2}=\frac{6n+5m+1}{2}=\lambda_3$.

For even $j$, $f(v_j)+f(v_{j+1})+f(v_{j+1})=\frac{m+j+1}{2}+3n+2m-j+1+\frac{j+2}{2}=\frac{6n+5m+1}{2}=\lambda_3$.

Hence for edge $uv \in E$, $f(u)+f(uv)+f(v)$ yields any one of the magic constants $\lambda_1=2n+2m+2$, $\lambda_2=\frac{7n+6m+3}{2}$ and $\lambda_3=\frac{6n+5m+1}{2}$. Therefore the umbrella $U_{n,m}$ is an edge trimagic for both odd $n$ and $m$.

Case 2. $n$ is odd and $m$ is even

Define a bijection $f : V \cup E \rightarrow \{1, 2, \ldots, 3n+2m-2\}$ such that

\[
\begin{align*}
  f(u_i) &= \begin{cases} 
  \frac{m+i+1}{2}, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\
  \frac{m+n+i+1}{2}, & 1 \leq i \leq n \text{ and } i \text{ is even}
  \end{cases} \\
  f(v_j) &= \begin{cases} 
  \frac{j+1}{2}, & 1 \leq j \leq m \text{ and } j \text{ is odd} \\
  \frac{m+j}{2}, & 1 \leq j \leq m \text{ and } j \text{ is even}
  \end{cases}
\end{align*}
\]

$f(u_{ui}) = 3n+m-i$, $1 \leq i \leq n-1$; $f(v_{j+1}) = 3n+2m-j+1$, $1 \leq j \leq m-1$ and

\[
\begin{align*}
  f(v_1u_i) &= \begin{cases} 
  2n+m+1+\frac{i+1}{2}, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\
  n+m+\frac{n+i+1}{2}, & 1 \leq i \leq n \text{ and } i \text{ is even}
  \end{cases}
\end{align*}
\]

To prove this labeling is an edge trimagic total labeling.

Consider the edges $v_ju_i$, $1 \leq i \leq n$.

For odd $i$, $f(v_j)+f(v_jui)+f(u_i) = 1+2n+m+1+\frac{i+1}{2}+m+\frac{i+1}{2}=2n+2m+2=\lambda_1$.

For even $i$, $f(v_j)+f(v_jui)+f(u_i) = 1+n+m+\frac{n-i+1}{2}+m+\frac{n+i+1}{2}=2n+2m+2=\lambda_1$.

Consider the edges $u_{ui}$, $1 \leq i \leq n-1$.

For odd $i$, $f(u_i)+f(u_{ui})=m+\frac{i+1}{2}+3n+m-i+m+\frac{n+i+2}{2}=\frac{7n+6m+3}{2}=\lambda_2$. 

For even $i$, $f(u_i)+f(u_{ui})=m+\frac{n+i+1}{2}+3n+m-i+m+\frac{n+i+2}{2}=\frac{7n+6m+3}{2}=\lambda_2$. 

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For even $i$, $f(u_i) + f(u_iu_{i+1}) + f(u_{i+1}) = m + \frac{n + i + 1}{2} + 3n + m - i + m + \frac{i + 2}{2} = \frac{7n + 6m + 3}{2} = \lambda_2$.

Consider the edges $v_jv_{j+1}$, $1 \leq j \leq m-1$.

For odd $j$, $f(v_j) + f(v_jv_{j+1}) + f(v_{j+1}) = \frac{j + 1}{2} + 3n + 2m - j - 1 + \frac{m + j + 1}{2} = \frac{6n + 5m}{2} = \lambda_1$.

For even $j$, $f(v_j) + f(v_jv_{j+1}) + f(v_{j+1}) = \frac{m + j}{2} + 3n + 2m - j - 1 + \frac{j + 2}{2} = \frac{6n + 5m}{2} = \lambda_1$.

Hence for edge $uv \in E$, $f(u) + f(uv) + f(v)$ yields any one of the magic constants $\lambda_1 = 2n+2m+2$, $\lambda_2 = \frac{7n + 6m + 3}{2}$ and $\lambda_3 = \frac{6n + 5m}{2}$. Therefore the umbrella $U_{n,m}$ is an edge trimagic for $n$ is odd and $m$ is even.

Case 3. $n$ is even and $m$ is odd

Define a bijection $f: V \cup E \to \{1, 2, \ldots, 3n+2m-2\}$ such that

- $f(u_i) = \begin{cases} m + \frac{n + i + 1}{2}, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ m + \frac{n + i}{2}, & 1 \leq i \leq n \text{ and } i \text{ is even} \end{cases}$

- $f(v_j) = \begin{cases} \frac{j + 1}{2}, & 1 \leq j \leq m \text{ and } j \text{ is odd} \\ \frac{m + j + 1}{2}, & 1 \leq j \leq m \text{ and } j \text{ is even} \end{cases}$

- $f(u_iu_{i+1}) = 3n+m-i$, $1 \leq i \leq n-1$; $f(v_jv_{j+1}) = 3n+2m-j-1$, $1 \leq j \leq m-1$ and

- $f(v_1u_i) = \begin{cases} 2n + m + 1 - \frac{i + 1}{2}, & i \text{ is odd} \\ n + m + \frac{n - i + 2}{2}, & i \text{ is even} \end{cases}$

To prove this labeling is an edge trimagic total labeling.

Consider the edges $v_ju_i$, $1 \leq i \leq n$.

For odd $i$, $f(v_i) + f(v_iu_i) + f(u_i) = 1 + 2n + m + 1 - \frac{i + 1}{2} + m + \frac{i + 1}{2} = 2n + 2m + 2 = \lambda_1$.

For even $i$, $f(v_i) + f(v_iu_i) + f(u_i) = 1 + n + m + \frac{n - i + 2}{2} + m + \frac{n + i}{2} = 2n + 2m + 2 = \lambda_1$.

Consider the edges $u_iu_{i+1}$, $1 \leq i \leq n-1$.

For odd $i$, $f(u_i) + f(u_iu_{i+1}) + f(u_{i+1}) = m + \frac{i + 1}{2} + 3n + m - i + m + \frac{n + i + 1}{2} = \frac{7n + 6m + 2}{2} = \lambda_2$. 

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For even \( i \), \( f(u_i) + f(u_iu_{i+1}) + f(u_{i+1}) = m + \frac{n + i}{2} + 3n + m - i + m + \frac{i + 2}{2} = \frac{7n + 6m + 2}{2} = \lambda_2. \)

Consider the edges \( v_jv_{j+1}, 1 \leq j \leq m-1. \)

For odd \( j \), \( f(v_j) + f(v_jv_{j+1}) + f(v_{j+1}) = \frac{j + 1}{2} + 3n + 2m - j - 1 + \frac{j + 2}{2} = \frac{6n + 5m + 1}{2} = \lambda_3. \)

For even \( j \), \( f(v_j) + f(v_jv_{j+1}) + f(v_{j+1}) = \frac{m + j + 1}{2} + 3n + 2m - j - 1 + \frac{j + 2}{2} = \frac{6n + 5m + 1}{2} = \lambda_3. \)

Hence for edge \( uv \in E \), \( f(u) + f(uv) + f(v) \) yields any one of the magic constants \( \lambda_1 = 2n+2m+2, \lambda_2 = \frac{7n + 6m + 2}{2} \) and \( \lambda_3 = \frac{6n + 5m + 1}{2} \). Therefore the umbrella \( U_{n,m} \) is an edge trimagic for \( n \) is even and \( m \) is odd.

Case 4. Both \( n \) and \( m \) are even

Define a bijection \( f : V \cup E \rightarrow \{1, 2, \ldots, 3n+2m-2\} \) such that

\[
\begin{align*}
    f(u_i) &= \begin{cases} 
        m + \frac{i + 1}{2}, & \text{if } 1 \leq i \leq n \text{ and } i \text{ is odd} \\
        m + \frac{n + i}{2}, & \text{if } 1 \leq i \leq n \text{ and } i \text{ is even}
    \end{cases} \\
    f(v_j) &= \begin{cases} 
        \frac{j + 1}{2}, & \text{if } 1 \leq j \leq m \text{ and } j \text{ is odd} \\
        \frac{m + j}{2}, & \text{if } 1 \leq j \leq m \text{ and } j \text{ is even}
    \end{cases}
\end{align*}
\]

To prove this labeling is an edge trimagic total labeling.

Consider the edges \( u_iu_{i+1}, 1 \leq i \leq n. \)

For odd \( i \), \( f(v_i) + f(v_iu_i) + f(u_i) = 1 + 2n + m + 1 - \frac{i + 1}{2} + m + \frac{i + 1}{2} = 2n + 2m + 2 = \lambda_1. \)

For even \( i \), \( f(v_i) + f(v_iu_i) + f(u_i) = 1 + 2n + m + 1 - \frac{n + i}{2} + m + \frac{n + i}{2} = 2n + 2m + 2 = \lambda_1. \)

Consider the edges \( u_iu_{i+1}, 1 \leq i \leq n-1. \)

For odd \( i \), \( f(u_i) + f(u_iu_{i+1}) + f(u_{i+1}) = m + \frac{i + 1}{2} + 3n + m - i + m + \frac{n + i + 1}{2} = \frac{7n + 6m + 2}{2} = \lambda_2. \)
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For even \( i \), \( f(u_i) + f(u_{i+1}) + f(u_{i+2}) = m + \frac{n + i}{2} + 3n + m - i + m + \frac{i + 2}{2} = \frac{7n + 6m + 2}{2} = \lambda_2. \)

Consider the edges \( v_jv_{j+1}, 1 \leq j \leq m-1. \)

For odd \( j \), \( f(v_j) + f(v_jv_{j+1}) + f(v_{j+1}) = \frac{j + 1}{2} + 3n + 2m - j - 1 + \frac{m + j + 1}{2} = \frac{6n + 5m}{2} = \lambda_3. \)

For even \( j \), \( f(v_j) + f(v_jv_{j+1}) + f(v_{j+1}) = \frac{m + j}{2} + 3n + 2m - j - 1 + \frac{j + 2}{2} = \frac{6n + 5m}{2} = \lambda_3. \)

Hence for edge \( uv \in E, f(u) + f(uv) + f(v) \) yields any one of the magic constants \( \lambda_1 = 2n + 2m + 2, \lambda_2 = \frac{7n + 6m + 2}{2} \) and \( \lambda_3 = \frac{6n + 5m}{2} \). Therefore the umbrella \( U_{n,m} \) is an edge trimagic for both even \( n \) and \( m \).

**Corollary 2.2.** The umbrella \( U_{n,m} \) is a super edge trimagic total labeling for all \( n \).

**Proof:** We proved that the umbrella \( U_{n,m} \) is an edge trimagic total graph for all \( n \) with \( n+m \) vertices. The labeling given in Theorem 2.1 is as follows:

For odd \( n \) and odd \( m \),

\[
f(u_i) = \begin{cases} 
m + \frac{i + 1}{2}, & 1 \leq i \leq n \text{ and } i \text{ odd} \\
m + \frac{n + i + 1}{2}, & 1 \leq i \leq n \text{ and } i \text{ even} 
\end{cases}
\]

\[
f(v_j) = \begin{cases} 
\frac{j + 1}{2}, & 1 \leq j \leq m \text{ and } j \text{ is odd} \\
\frac{m + j + 1}{2}, & 1 \leq j \leq m \text{ and } j \text{ is even} 
\end{cases}
\]

For odd \( n \) and even \( m \),

\[
f(u_i) = \begin{cases} 
m + \frac{i + 1}{2}, & 1 \leq i \leq n \text{ and } i \text{ odd} \\
m + \frac{n + i + 1}{2}, & 1 \leq i \leq n \text{ and } i \text{ even} 
\end{cases}
\]

\[
f(v_j) = \begin{cases} 
\frac{j + 1}{2}, & 1 \leq j \leq m \text{ and } j \text{ is odd} \\
\frac{m + j}{2}, & 1 \leq j \leq m \text{ and } j \text{ is even} 
\end{cases}
\]

For even \( n \) and odd \( m \),

\[
f(u_i) = \begin{cases} 
m + \frac{i + 1}{2}, & 1 \leq i \leq n \text{ and } i \text{ odd} \\
m + \frac{n + i}{2}, & 1 \leq i \leq n \text{ and } i \text{ is even} 
\end{cases}
\]

\[
f(v_j) = \begin{cases} 
\frac{j + 1}{2}, & 1 \leq j \leq m \text{ and } j \text{ is odd} \\
\frac{m + j}{2}, & 1 \leq j \leq m \text{ and } j \text{ is even} 
\end{cases}
\]
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\[ f(v_j) = \begin{cases} 
\frac{j + 1}{2}, & \text{if } 1 \leq j \leq m \text{ and } j \text{ is odd} \\
\frac{m + j + 1}{2}, & \text{if } 1 \leq j \leq m \text{ and } j \text{ is even}
\end{cases} \]

For even \( n \) and \( m \),

\[ f(u_i) = \begin{cases} 
\frac{m + i + 1}{2}, & \text{if } 1 \leq i \leq n \text{ and } i \text{ is odd} \\
\frac{m + n + i}{2}, & \text{if } 1 \leq i \leq n \text{ and } i \text{ is even}
\end{cases} \]

\[ f(v_j) = \begin{cases} 
\frac{j + 1}{2}, & \text{if } 1 \leq j \leq m \text{ and } j \text{ is odd} \\
\frac{m + j}{2}, & \text{if } 1 \leq j \leq m \text{ and } j \text{ is even}
\end{cases} \]

Hence the \( n+m \) vertices get labels 1, 2, ..., \( n+m \). Therefore, the umbrella \( U_{n,m} \) is a super edge trimagic total labeling graph for all \( n \).

**Example 2.3.** An edge trimagic total labeling of the Umbrella \( U_{5,3} \), \( U_{5,4} \), \( U_{6,5} \) and \( U_{4,6} \) are given in Figure 1, Figure 2, Figure 3 and Figure 4 respectively.

**Figure 1:** \( U_{5,3} \) with \( \lambda_1 = 18 \), \( \lambda_2 = 28 \) and \( \lambda_3 = 23 \)

**Figure 2:** \( U_{5,4} \) with \( \lambda_1 = 20 \), \( \lambda_2 = 31 \) and \( \lambda_3 = 25 \)
Theorem 2.4. The Dumbbell $D_b_n$ is an edge trimagic total labeling for all $n$.

Proof: Let $V = \{u_i, v_i / 1 \leq i \leq n\}$ be the vertex set and $E=\{u_iu_{i+1}, v_iv_{i+1} / 1 \leq i \leq n-1\} \cup \{u_1u_n, v_1v_n\} \cup \{u_1v_1\}$ be the edge set of the graph $D_b_n$. Then $D_b_n$ has $2n$ vertices and $2n+1$ edges.

Case 1. $n$ is even

Define a bijection $f: V \cup E \rightarrow \{1, 2, \ldots, 4n+1\}$ such that $f(u_i) = 2i-1$, $1 \leq i \leq n$; $f(v_i) = 2i$, $1 \leq i \leq n$;

$$f(u_iu_{i+1}) = \begin{cases} 4n - 4i + 3, & 1 \leq i \leq \frac{n}{2} \\ 6n - 4i - \frac{n}{2} + 1, & \frac{n}{2} + 1 \leq i \leq n - 1 \end{cases}$$
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\[
f(v_{i+1}) = \begin{cases} 
4n - 4i + 1, & 1 \leq i \leq \frac{n}{2} \\
6n - 4i - 2, & \frac{n}{2} + 1 \leq i \leq n - 1
\end{cases}
\]

\[f(u_1 v_1) = 4n + 1; \ f(u_1 u_n) = 4n; \ f(v_1 v_n) = 4n - 2.\]

To prove this labeling is an edge trimagic total labeling.

For the edges \(u_i u_{i+1}\), 
\[
f(u_i) + f(u_i u_{i+1}) + f(u_{i+1}) = 2i - 1 + 4n - 4i + 2i + 1 = 4n + 3 = \lambda_1.
\]

For the edges \(v_i v_{i+1}\), 
\[
f(v_i) + f(v_i v_{i+1}) + f(v_{i+1}) = 2i + 4n - 4i + 1 + 2i + 2 = 4n + 3 = \lambda_1.
\]

For the edge \(u_1 v_1\), 
\[
f(u_1) + f(u_1 v_1) + f(v_1) = 1 + 4n + 1 + 2 = 4n + 4 = \lambda_2.
\]

For the edges \(u_i u_{i+1}\), 
\[
f(u_i) + f(u_i u_{i+1}) + f(u_{i+1}) = 2i - 1 + 6n - 4i + 2i + 1 = 6n = \lambda_3.
\]

For the edges \(v_i v_{i+1}\), 
\[
f(v_i) + f(v_i v_{i+1}) + f(v_{i+1}) = 2i + 6n - 4i - 2 + 2i + 2 = 6n = \lambda_3.
\]

For the edge \(u_1 u_n\), 
\[
f(u_1) + f(u_1 u_n) + f(u_n) = 1 + 4n + 2n - 1 = 6n = \lambda_3.
\]

For the edge \(v_1 v_n\), 
\[
f(v_1) + f(v_1 v_n) + f(v_n) = 2 + 4n - 2 + 2n = 6n = \lambda_3.
\]

Hence for each edge \(uv \in E\), \(f(u) + f(uv) + f(v)\) yields any one of the magic constants \(\lambda_1 = 4n + 3; \ \lambda_2 = 4n + 4\) and \(\lambda_3 = 6n\). Therefore the Dumbbell graph \(D_{2n}\) is an edge trimagic for even \(n\).

Case 2. \(n\) is odd

Define a bijection \(f: V \cup E \rightarrow \{1, 2, \ldots, 4n + 1\}\) such that

\[
f(u_i) = \begin{cases} 
\frac{i + 1}{2}, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\
\frac{n + i + 1}{2}, & 1 \leq i \leq n \text{ and } i \text{ is even}
\end{cases}
\]

\[
f(v_i) = \begin{cases} 
\frac{2n + i + 1}{2}, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\
\frac{3n + i + 1}{2}, & 1 \leq i \leq n \text{ and } i \text{ is even}
\end{cases}
\]

\[
f(u_1 u_{i+1}) = 3n - i, \ 1 \leq i \leq n - 1; \ f(u_1 u_n) = 3n; \ f(v_1 v_{i+1}) = 4n - i, \ 1 \leq i \leq n - 1; \ f(v_1 v_n) = 4n \text{ and } f(u_1 v_1) = 4n + 1.
\]

To prove this labeling is an edge trimagic total labeling.

Consider the edges \(u_1 u_{i+1}, \ 1 \leq i \leq n - 1\).

For odd \(i\), \(f(u_i) + f(u_i u_{i+1}) + f(u_{i+1}) = \frac{i + 1}{2} + 3n - i + \frac{n + i + 2}{2} = \frac{7n + 3}{2} = \lambda_1.\)

For even \(i\), \(f(u_i) + f(u_i u_{i+1}) + f(u_{i+1}) = \frac{n + i + 1}{2} + 3n - i + \frac{i + 2}{2} = \frac{7n + 3}{2} = \lambda_1.\)
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For the edge $u_1u_n$, $f(u_1) + f(u_1u_n) + f(u_n) = 1 + 3n + \frac{n+1}{2} = \frac{7n+3}{2} = \lambda_1$.

For the edge $u_1v_1$, $f(u_1) + f(u_1v_1) + f(v_1) = 1 + 4n + 1 + n + 1 = 5n + 3 = \lambda_2$.

Consider the edges $v_1v_{i+1}$, $1 \leq i \leq n - 1$.

For odd $i$, $f(v_i) + f(v_{1}v_{i+1}) + f(v_{i+1}) = \frac{2n + i + 1}{2} + 4n + 1 + n + i + 2 = \frac{13n + 3}{2} = \lambda_3$.

For even $i$, $f(v_i) + f(v_{1}v_{i+1}) + f(v_{i+1}) = \frac{3n + i + 1}{2} + 4n + 1 + n + i + 2 = \frac{13n + 3}{2} = \lambda_3$.

For the edge $v_1v_n$, $f(v_1) + f(v_1v_n) + f(v_n) = n + 1 + 4n + \frac{3n + 1}{2} = \frac{13n + 3}{2} = \lambda_3$.

Hence for each edge $uv \in E$, $f(u) + f(uv) + f(v)$ yields any one of the magic constants $\lambda_1 = \frac{7n+3}{2}$, $\lambda_2 = 5n + 3$ and $\lambda_3 = \frac{13n+3}{2}$. Therefore the dumbbell graph $Db_n$ is an edge trimagic for odd $n$. From cases (1) and (2), the Dumbbell $Db_n$ is an edge Trimagic total labeling for all $n$.

**Corollary 2.5.** The Dumbbell $Db_n$ is a super edge trimagic total labeling for all $n$.

**Proof:** We proved that the Dumbbell $Db_n$ is an edge trimagic total graph for all $n$ with $2n$ vertices. The labeling given in Theorem 2.4 is as follows:

For even $n$, $f(u_i) = 2i - 1$, $1 \leq i \leq n$ and $f(v_i) = 2i$, $1 \leq i \leq n$.

For odd $n$,

$f(u_i) = \begin{cases} 
\frac{i+1}{2}, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\
\frac{n+i+1}{2}, & 1 \leq i \leq n \text{ and } i \text{ is even} 
\end{cases}$

$f(v_i) = \begin{cases} 
\frac{2n+i+1}{2}, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\
\frac{3n+i+1}{2}, & 1 \leq i \leq n \text{ and } i \text{ is even} 
\end{cases}$

Hence the $2n$ vertices get labels $1, 2, \ldots, 2n$. Therefore, the Dumbbell $Db_n$ is a super edge trimagic total labeling graph for all $n$.

**Example 2.6.** An edge Trimagic total labeling of the Dumbbell $Db_5$; and $Db_3$ are given in figure 5 and figure 6 respectively.

![Figure 5: $Db_8$ with $\lambda_1 = 35$, $\lambda_2 = 36$ and $\lambda_3 = 48$](image)

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Theorem 2.7. The circular ladder $\text{CL}(n)$ is an edge trimagic total labeling for all $n$.

Proof: Let $V = \{u_i, v_i / 1 \leq i \leq n\}$ be the vertex set and $E = \{u_iu_{i+1}, v_iv_{i+1} / 1 \leq i \leq n-1\} \cup \{u_1v_1 / 1 \leq i \leq n\} \cup \{u_1u_n, v_1v_n\}$ be the edge set of the graph $\text{CL}(n)$. Then $\text{CL}(n)$ has $2n$ vertices and $3n$ edges.

Case 1. $n$ is odd

Define a bijection $f: \cup_{E} \rightarrow \{1, 2, \ldots, 5n\}$ such that

$$f(u_i) = \begin{cases} n + \frac{n + i + 2}{2}, & \text{if } 1 \leq i \leq n - 1 \text{ and } i \text{ is odd} \smallskip \end{cases}$$

$$f(v_i) = \begin{cases} \frac{i + 1}{2}, & \text{if } 1 \leq i \leq n \text{ and } i \text{ is odd} \smallskip \end{cases}$$

$$f(u_n) = n+1; f(u_{u_n+1}) = 4n-i-1, 1 \leq i \leq n-2; f(u_nu_n+1) = 4n; f(u_nu_n) = 4n-1; f(v_nv_{n+1}) = 5n-i, 1 \leq i \leq n-1; f(v_nv_n) = 5n; f(u_nv_n) = 3n-i, 1 \leq i \leq n-1 \text{ and } f(u_nv_n) = 3n.$$  

To prove this labeling is an edge trimagic total labeling.

Consider the edges $u_iu_{i+1}, 1 \leq i \leq n-2$.

For odd $i$, $f(u_i) + f(u_{u_n+1}) + f(u_{i+1}) = n + \frac{n + i + 2}{2} + 4n - i - 1 + n + \frac{i + 3}{2} = \frac{13n + 3}{2} = \lambda_j$.

For even $i$, $f(u_i) + f(u_{u_n+1}) + f(u_{i+1}) = n + \frac{i + 2}{2} + 4n - i - 1 + n + \frac{n + i + 3}{2} = \frac{13n + 3}{2} = \lambda_j$.

For the edge $u_1u_n$, $f(u_1)+f(u_{u_n})+f(u_n) = n + \frac{n + 3}{2} + 4n - 1 + n + 1 = \frac{13n + 3}{2} = \lambda_j$.

For the edge $u_{n-1}u_n$, $f(u_{u_n})+f(u_{u_n-1})+f(u_n) = n + \frac{n + 1}{2} + 4n + n + 1 = \frac{13n + 3}{2} = \lambda_j$.

Consider the edges $u_nv_i, 1 \leq i \leq n-1$.  

Figure 6: $D_{5}$ with $\lambda_1 = 19, \lambda_2 = 28$ and $\lambda_3 = 34$.  

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For odd $i$, $f(u_i) + f(u_i v_i) + f(v_i) = n + \frac{i + 2}{2} + 3n - i + \frac{i + 1}{2} = \frac{9n + 3}{2} = \lambda_2$.

For even $i$, $f(u_i) + f(u_i v_i) + f(v_i) = n + \frac{i + 2}{2} + 3n - i + \frac{n + i + 1}{2} = \frac{9n + 3}{2} = \lambda_2$.

For the edge $u_n v_n$, $f(u_n) + f(u_n v_n) + f(v_n) = n+1+3n+\frac{n + 1}{2} = \frac{9n + 3}{2} = \lambda_2$.

Consider the edges $v_i v_{i+1}$, $1 \leq i \leq n-1$.

For odd $i$, $f(v_i) + f(v_i v_{i+1}) + f(v_{i+1}) = \frac{i + 1}{2} + 5n - i + \frac{n + i + 2}{2} = \frac{11n + 3}{2} = \lambda_3$.

For even $i$, $f(v_i) + f(v_i v_{i+1}) + f(v_{i+1}) = \frac{n + i + 1}{2} + 5n - i + \frac{i + 2}{2} = \frac{11n + 3}{2} = \lambda_3$.

For the edge $v_1 u_n$, $f(v_1) + f(v_1 u_n) + f(u_n) = 1 + 5n + \frac{n + 1}{2} = \frac{11n + 3}{2} = \lambda_3$.

Hence for each edge $uv \in E$, $f(u) + f(uv) + f(v)$ yields any one of the magic constants $\lambda_1 = \frac{13n + 3}{2}$, $\lambda_2 = \frac{9n + 3}{2}$ and $\lambda_3 = \frac{11n + 3}{2}$. Therefore the Circular ladder $CL(n)$ is an edge trimagic for odd $n$.

Case 2. $n$ is even

Define a bijection $f : V \cup E \rightarrow \{1, 2, \ldots, 5n\}$ such that

$$
f(u_i) = \begin{cases} 
n + \frac{i + 1}{2}, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\
n + \frac{n + i}{2}, & 1 \leq i \leq n \text{ and } i \text{ is even} 
\end{cases}
$$

$$
f(v_i) = \begin{cases} 
\frac{i + 1}{2}, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\
\frac{n + i}{2}, & 1 \leq i \leq n \text{ and } i \text{ is even} 
\end{cases}
$$

$f(u_i u_{i+1}) = 3n-i+1$, $1 \leq i \leq n-1$; $f(u_i u_n) = 2n+1$; $f(v_i v_{i+1}) = 5n-i+1$, $1 \leq i \leq n-1$; $f(v_1 v_n) = 4n+1$; and $f(u_1 v_i) = 4n-i+1$, $1 \leq i \leq n$.

To prove this labeling is an edge trimagic total labeling.

For the edge $v_1 u_n$, $f(v_1) + f(v_1 v_n) + f(v_n) = 1+4n+1+n = 5n+2 = \lambda_1$.

For the edge $u_1 u_n$, $f(u_1) + f(u_1 u_n) + f(u_n) = n+1+2n+1+2n = 5n+2 = \lambda_1$.

Consider the edges $u_i v_i$, $1 \leq i \leq n$.

For odd $i$, $f(u_i) + f(u_i v_i) + f(v_i) = \frac{i + 1}{2} + 4n - i + 1 + n + \frac{i + 1}{2} = 5n + 2 = \lambda_1$.

For even $i$, $f(u_i) + f(u_i v_i) + f(v_i) = \frac{n + i}{2} + 4n - i + 1 + n + \frac{n + i}{2} = 6n + 1 = \lambda_2$. 

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Consider the edges $u_i u_{i+1}$, $1 \leq i \leq n-1$.

For odd $i$, $f(u_i) + f(u_i u_{i+1}) + f(u_{i+1}) = \frac{n+i+1}{2} + 3n - i + 1 + n + \frac{n + i + 1}{2} = \frac{11n + 4}{2} = \lambda_3$.

For even $i$, $f(u_i) + f(u_i u_{i+1}) + f(u_{i+1}) = \frac{n + i + 1}{2} + 3n - i + 1 + n + \frac{i + 2}{2} = \frac{11n + 4}{2} = \lambda_3$.

Consider the edges $v_i v_{i+1}$, $1 \leq i \leq n-1$.

For odd $i$, $f(v_i) + f(v_i v_{i+1}) + f(v_{i+1}) = \frac{i + 1}{2} + 5n - i + 1 + \frac{n + i + 1}{2} = \frac{11n + 4}{2} = \lambda_3$.

For even $i$, $f(v_i) + f(v_i v_{i+1}) + f(v_{i+1}) = \frac{n + i}{2} + 5n - i + 1 + \frac{i + 2}{2} = \frac{11n + 4}{2} = \lambda_3$.

Hence for each edge $uv \in E$, $f(u) + f(uv) + f(v)$ yields any one of the magic constants $\lambda_1 = 5n+2$, $\lambda_2 = 6n+1$ and $\lambda_3 = \frac{11n + 4}{2}$. Therefore, the Circular ladder $CL(n)$ is an edge trimagic for even $n$. Hence by case 1 and case 2, the circular ladder $CL(n)$ is an edge Trimagic total labeling for all $n$.

**Corollary 2.8.** The Circular ladder $CL(n)$ is a super edge trimagic total labeling for all $n$.

**Proof:** We proved that the Circular ladder $CL(n)$ is an edge trimagic total graph for all $n$ with $2n$ vertices. The labeling given in Theorem 2.7 is as follows:

For odd $n$,

$$f(u_i) = \begin{cases} 
\frac{n + i + 2}{2}, & 1 \leq i \leq n - 1 \text{ and } i \text{ is odd} \\
\frac{i + 1}{2}, & 1 \leq i \leq n - 1 \text{ and } i \text{ is even}
\end{cases}$$

$$f(v_i) = \begin{cases} 
\frac{n + i + 1}{2}, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\
\frac{n + i + 1}{2}, & 1 \leq i \leq n \text{ and } i \text{ is even}
\end{cases}$$

For even $n$,

$$f(u_i) = \begin{cases} 
\frac{n + i + 1}{2}, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\
\frac{i + 1}{2}, & 1 \leq i \leq n \text{ and } i \text{ is even}
\end{cases}$$

$$f(v_i) = \begin{cases} 
\frac{n + i}{2}, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\
\frac{n + i}{2}, & 1 \leq i \leq n \text{ and } i \text{ is even}
\end{cases}$$
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Hence the 2n vertices get labels 1, 2, ..., 2n. Therefore, the Circular ladder CL(n) is a super edge trimagic total labeling graph for all n.

**Example 2.9.** An edge Trimagic total labeling of the Circular ladder CL(7) and CL(6) are given in figure 7 and figure 8 respectively

**Figure 7:** CL(7) with $\lambda_1 = 47$, $\lambda_2 = 33$ and $\lambda_3 = 40$

**Figure 8:** CL(6) with $\lambda_1 = 32$, $\lambda_2 = 37$ and $\lambda_3 = 35$

**3. Conclusion**

In this paper we have determined the edge trimagic total labeling of the Umbrella, Dumbbell and Circular ladder graphs. Also we have determined the above graphs are super edge Trigramic total labeling.

**Acknowledgement.** The authors are grateful to the anonymous referee for providing thoughtful comments and valuable suggestions.
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