Annals of Pure and Applied Mathematics
Vol. 13, No. 1, 2017, 125-130
ISSN: 2279-087X (P), 2279-0888(online)
Published on 3 March 2017
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v13n1a12

Annals of
Pure and Applied Mathematics

On Eccentricity Sum Eigenvalue and Eccentricity Sum Energy of a Graph

***D.S.Revankar*[1], *M.M.Patil*[2] and *H.S.Ramane*[3]**

[1]Department of Mathematics
KLE DR. M. S. Sheshgeri College of Engineering & Technology
Belgaum – 590008, India.
Corresponding author. E-mail: revankards@rediffmail.com
[2]Department of Mathematics, Angadi Institute of Technology & Management
Belgaum – 590009, India. E-mail: manjushamagdum@gmail.com
[3]Department of Mathematics, Karnatak University, Dharwad – 580003, India
E-mail: hsramane@yahoo.com

Received 15 February 2017; accepted 28 February 2017

Abstract. Let G be a simple graph with n vertices and m edges. For a vertex v_i its eccentricity, e_i is the largest distance from v_i to any other vertices of G. In this paper we introduce the concept of eccentricity sum matrix $ES(G)$ and eccentricity sum energy $E_{ES}(G)$ of a simple connected graph G and obtain bounds for eigenvalues of $ES(G)$ and bounds for the eccentricity sum energy $E_{ES}(G)$ of a graph G.

Keywords: Eccentricity sum matrix, Eigenvalues, Eccentricity sum Energy.

AMS Mathematics Subject Classification (2010): 05C50

1. Introduction

Let G be a simple graph with n vertices and m edges. Let the vertices of G be labeled as $v_1, v_2, \ldots, v_n$. The degree of a vertex v in a graph G, denoted by $d(v)$ is the number of edges incident to v. The distance between the vertices v_i and v_j is the length of the shortest path joining v_i and v_j in G. For a vertex v_i its eccentricity, e_i is the largest distance from v_i to any other vertices of G. The adjacency matrix $A(G)$ of a graph G is a square matrix of order n whose (i, j)-entry is equal to unity if the vertex v_i is adjacent to v_j, and is equal to zero otherwise. The eigenvalues of adjacency matrix $A(G)$ are denoted by $\lambda_1, \lambda_2, \ldots, \lambda_n$ and since they are real it can be ordered as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. [2]

The energy of a graph G is defined as [1],

$$E = E_\pi(G) = \sum_{i=1}^{n} |\lambda_i|$$

This definition of energy was motivated by large number of results for the Huckel molecular orbital total π-electron energy [1].

Motivated by previous researches on Degree Sum Energy of a Graph [3, 4], in this paper we introduce eccentricity sum matrix and eccentricity sum energy associated with a graph and study its bounds. For more results on degree sum energy see [8, 9]

Let G be a simple graph with n vertices v_1, v_2, ..., v_n and let $e_i = ecc(v_i)$ be the eccentricity of v_i, i =1, 2, … n. Then $ES(G) = [a_{ij}]$ is called the eccentricity sum matrix of a graph G where

$$a_{ij} = \begin{cases} e_i + e_j, & \text{if} \quad i \neq j \\ 0 , & \text{otherwise} \end{cases} \tag{1}$$

The characteristic polynomial of the eccentricity sum matrix is defined as,

$\phi(G: \xi) = det(\xi I - ES(G))$.

where I is the identity matrix of order n .

Since $ES(G)$ is real symmetric matrix, the roots of $\phi(G: \xi) = 0$ are real. These roots can be ordered as $\xi_1 \geq \xi_2 \geq \cdot \cdot \cdot \geq \xi_n$, where ξ_1 is largest and ξ_n is smallest eigenvalues. If G has ξ_1, ξ_2, … , ξ_n distinct eigenvalues with respective multiplicities k_1, k_2, … , k_n then the spectrum can be written as,

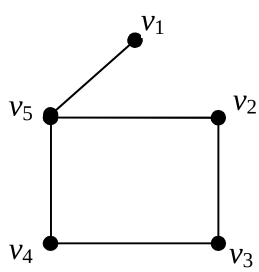
$$Spe(G) = \begin{pmatrix} \xi_1 & \xi_2 & \cdots & \xi_n \\ k_1 & k_2 & \cdots & k_n \end{pmatrix}$$

The eccentricity sum energy of a graph G is defined as,

$$E_{ES}(G) = \sum_{i=1}^{n} |\xi_i| \tag{2}$$

Example 1.

G :



$$ES(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0 & 5 & 6 & 5 & 5 \\ 5 & 0 & 5 & 4 & 4 \\ 6 & 5 & 0 & 5 & 5 \\ 5 & 4 & 5 & 0 & 4 \\ 5 & 4 & 5 & 4 & 0 \end{bmatrix} \end{array}$$

Figure 1:

$\phi(G: \xi) = (\xi + 4)^2 (\xi + 6) (\xi^2 - 14\xi - 102)$.

$\xi_1 = 19.2882$, $\xi_2 = -4$, $\xi_3 = -4$, $\xi_4 = -5.2882$, $\xi_5 = -6$.

Therefore, $E_{ES}(G) = 38.5764$

Lemma 1.1. [3] The Cauchy – Schwarz inequality states that if $(a_1, a_2,\ldots, a_p)$ and $(b_1, b_2,\ldots, b_p)$ are real p – vectors then

$$\left(\sum_{i=1}^{p} a_i b_i \right)^2 \leq \left(\sum_{i=1}^{p} a_i^2 \right) \left(\sum_{i=1}^{p} b_i^2 \right). \tag{3}$$

Lemma 1.2. [7] Let a_1, a_2, … , a_n be non negative numbers. Then

$$n\left[\frac{1}{n}\sum_{i=1}^{n}a_i-\left(\prod_{i=1}^{n}a_i\right)^{1/n}\right]\le n\sum_{i=1}^{n}a_i-\left(\sum_{i=1}^{n}\sqrt{a_i}\right)^2$$
$$\le n(n-1)\left[\frac{1}{n}\sum_{i=1}^{n}a_i-\left(\prod_{i=1}^{n}a_i\right)^{1/n}\right]$$

2. Bounds for the largest eigenvalue of eccentricity sum matrix

Since $trace(ES(G)) = 0$, the eigenvalues of $ES(G)$ satisfies the relations

$$\sum_{i=1}^{n}\xi_i = 0 \tag{4}$$

Further,

$$\sum_{i=1}^{n}\xi_i^2 = trace\,(ES(G))^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}\,a_{ji} = \sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ij})^2 = 2\sum_{i<j}\left(e_i+e_j\right)^2$$

$$\Rightarrow \quad \sum_{i=1}^{n}\xi_i^2 = 2M \quad \text{where,} \quad M = \sum_{1\le i<j\le n}\left(e_i+e_j\right)^2 \tag{5}$$

If r is the radius and D is the diameter of a graph, then $r \le e_i \le D$.
Hence, $2n(n-1)r^2 \le M \le 2n(n-1)d^2$. Equality holds if $r = e_i = D$.

Theorem 2.1. If G is a connected graph with $\mathrm{ecc}(v_i)= e_i = e$, $i = 1, 2, \ldots, n$, then G has only one positive eigenvalue equal to $2(n-1)e$.

Proof: Let G be a simple connected graph with n vertices. Let $ecc(v_i)= e_i = e$, $i = 1, 2, \ldots, n$.

Then, $$a_{ij} = \begin{cases} e_i+e_j & ,\text{if } \quad i\ne j \\ 0 & ,\text{otherwise}\end{cases} = \begin{cases} 2e & ,\text{if } \quad i\ne j \\ 0 & ,\text{otherwise}\end{cases}$$

Then the characteristic polynomial of the eccentricity sum matrix is,
$\phi(G: \xi) = det(\xi I - ES(G)) = det(\xi I - 2e\,(A(K_n)) = 0$. Where $A(K_n)$ is adjacency matrix of a complete graph K_n.

$$det(\xi I - 2e\,(A(K_n))) = (2e)^n\left|\frac{\xi}{2e}I - A(K_n)\right| = (2e)^n\left(\frac{\xi}{2e}-(n-1)\right)\left(\frac{\xi}{2e}+1\right)^{n-1}$$
$$= [\xi - 2(n-1)e]\,(\xi + 2e)^{n-1}$$

Therefore $[\xi - 2(n-1)e]\,(\xi + 2e)^{n-1} = 0$.

$$\Rightarrow \quad spect(ES(G)) = \begin{pmatrix} 2(n-1)e & -2e \\ 1 & n-1 \end{pmatrix}.$$

Hence G has only one positive eigenvalue equal to $2(n-1)e$. □

Corollary 2.2. If $G = K_n$ is a complete graph, then

$$spe(ES(K_n)) = \begin{pmatrix} 2(n-1) & -2 \\ 1 & n-1 \end{pmatrix}.$$

Corollary 2.3. If $G = K_{p,q}$ is a complete bipartite graph, then for $p, q \ne 1$

$$spe(ES(K_{p,q})) = \begin{pmatrix} 4(p+q-1) & -4 \\ 1 & p+q-1 \end{pmatrix}.$$

Corollary 2.4. If $G = C_n$ is a cycle, then

$$spe(ES(C_n)) = \begin{pmatrix} n(n-1) & -n \\ 1 & n-1 \end{pmatrix}, \quad if\ n\ is\ even,$$

and $$spect(ES(C_n)) = \begin{pmatrix} (n-1)^2 & -(n-1) \\ 1 & n-1 \end{pmatrix}, \quad if\ n\ is\ odd.$$

Corollary 2.5. If G is a star graph $S_n = K_{1,n}$, then for $n \geq 2$

$$spect(ES(S_n)) = \begin{pmatrix} -4 & 2n-2+\sqrt{4n^2+n+4} & 2n-2-\sqrt{4n^2+n+4} \\ n-1 & 1 & 1 \end{pmatrix}.$$

Theorem 2.6. If G is any graph with n vertices, then

$$\xi_1 \leq \sqrt{\frac{2M(n-1)}{n}} \tag{6}$$

Equality holds if $ecc(v_i) = e_i = e$, $i = 1, 2, \ldots, n$.

Proof: Let $a_i = 1$ and $b_i = \xi_i$ for $i = 2, 3, \ldots, n$ in Eqn. (3).

Therefore

$$\left(\sum_{i=2}^{n} \xi_i\right)^2 \leq (n-1)\left(\sum_{i=2}^{n} \xi_i^{\,2}\right). \tag{7}$$

From Eqn. (4) and (5)

$$\sum_{i=2}^{n} \xi_i = -\xi_1 \qquad \text{and} \qquad \sum_{i=2}^{n} \xi_i^2 = 2M - \xi_1^2 .$$

Therefore Eqn. (7) becomes

$$(-\xi_1)^2 \leq (n-1)\,(2M - \xi_1^{\,2})$$

which gives, $\xi_1 \leq \sqrt{\frac{2M(n-1)}{n}}$.

For equality, let $ecc(v_i) = e_i = e$, $i = 1, 2, \ldots, n$.

Therefore, $M = \sum_{1 \leq i < j \leq n} (e_i + e_j)^2 = \sum_{1 \leq i < j \leq n} 4e^2 = \binom{n}{2} 4e^2 = 2n(n-1)e^2$

Hence $\sqrt{\frac{2M(n-1)}{n}} = \sqrt{\frac{2(n-1)}{n} 2n(n-1)e^2} = 2(n-1)e$.

From Theorem 2.1, $\xi_1 = 2(n-1)e$ is the only one positive eigenvalue. Hence it is largest. Therefore equality holds. □

3. Bounds for the eccentricity sum energy of a graph

Theorem 3.1. If G is any graph with n vertices, then

$$\sqrt{2M} \leq E_{ES}(G) \leq \sqrt{2Mn}\,. \tag{8}$$

Proof: Put $a_i = 1$ and $b_i = |\xi_i|$ in Eq. (3), we get

$$\left(\sum_{i=1}^{n} |\xi_i|\right)^2 \leq n \sum_{i=1}^{n} \xi_i^{\,2}$$

from which

$$[E_{ES}(G)]^2 \le n\,(2M) \;\Rightarrow\; E_{ES}(G) \le \sqrt{2nM}. \tag{9}$$

Now, $[E_{ES}(G)]^2 = \left(\sum_{i=1}^{n}|\xi_i|\right)^2 \ge \sum_{i=1}^{n}|\xi_i|^2 = 2M$

Therefore $E_{ES}(G) \ge \sqrt{2M}$ (10)

Combining Eqs. (9) and (10) we get the result (8). □

Theorem 3.2. Let G be any graph with n vertices and let Δ be the absolute value of the determinant of the eccentricity sum matrix $ES(G)$. Then

$$\sqrt{2M + n(n-1)\Delta^{2/n}} \le E_{ES}(G) \le \sqrt{2(n-1)M + n\Delta^{2/n}} \tag{11}$$

Proof: <u>Lower bound</u>.
By the definition of the eccentricity sum energy and by Eq. (5)

$$[E_{ES}(G)]^2 = \left(\sum_{i=1}^{n}(\xi_i)\right)^2 = \sum_{i=1}^{n}(\xi_i)^2 + 2\sum_{i<j}|\xi_i||\xi_j| = 2M + \sum_{i\ne j}|\xi_i||\xi_j| \tag{12}$$

Since for nonnegative numbers the arithmetic mean is not smaller than the geometric mean,

$$\frac{1}{n(n-1)}\sum_{i\ne j}|\xi_i||\xi_j| \ge \left(\prod_{i\ne j}|\xi_i||\xi_j|\right)^{1/n(n-1)} = \left(\prod_{i=1}^{n}|\xi_i|^{2(n-1)}\right)^{1/n(n-1)} = \prod_{i=1}^{n}|\xi_i|^{2/n} = \Delta^{2/n}.$$

Therefore,

$$\sum_{i\ne j}|\xi_i||\xi_j| \ge n(n-1)\Delta^{2/n}. \tag{13}$$

Combining Eqns. (12) and (13) we get,

$[E_{ES}(G)]^2 \ge 2M + n(n-1)\Delta^{2/n}$

$$i.e. \quad E_{ES}(G) \ge \sqrt{2M + n(n-1)\Delta^{2/n}} \tag{14}$$

<u>Upper bound.</u>
Put $\sqrt{a_i} = |\xi_i|$, $i = 1, 2, \dots, n$. Then from Lemma (1.2) we obtain,

$$n\left[\frac{1}{n}\sum_{i=1}^{n}\xi_i^2 - \left(\prod_{i=1}^{n}\xi_i^2\right)^{1/n}\right] \le n\sum_{i=1}^{n}\xi_i^2 - \left(\sum_{i=1}^{n}|\xi_i|\right)^2 \le n(n-1)\left[\frac{1}{n}\sum_{i=1}^{n}\xi_i^2 - \left(\prod_{i=1}^{n}\xi_i^2\right)^{1/n}\right]$$

That is, $2M - n\Delta^{2/n} \le 2nM - [E_{ES}(G)]^2$

$$\text{Thus, } [E_{ES}(G)]^2 \le 2(n-1)M + n\Delta^{2/n} \tag{15}$$

where, $M = \sum_{1\le i<j\le n}(e_i + e_j)^2$.

Combining Equations (14) and (15) we obtain the result (11) □

Theorem 3.3. If G is a connected graph with $ecc(v_i) = e_i = e$, $i = 1, 2, \dots, n$, then $E_{ES}(G) = 4(n-1)e$.

Proof: If G is a connected graph with $ecc(v_i) = e_i = e$, $i = 1, 2, \dots, n$, then from Theorem 2.1, G has only one positive eigenvalue equal to $\xi_1 = 2(n-1)e$. Since $trace(ES(G)) = 0$, sum of the remaining eigenvalues is equal to $-2(n-1)e$.

Therefore, $E_{ES}(G) = \sum_{i=1}^{n} |\xi_i| = 4(n-1)e$.

Eccentricity sum energy of some graphs

Graph G	ξ_1	Eccentricity Sum Energy = $E_{ES}(G)$
K_n	$2(n-1)$	$4(n-1)$
$K_{m,n}$	$4(n-1)$	$8(n-1)$
C_n	$n(n-1)$, if n is even	$2n(n-1)$, if n is even
	$(n-1)^2$, if n is odd	$2(n-1)^2$, if n is odd

4. Conclusión

For a connected graph G, we have defined eccentricity sum matrix $ES(G)$ and eccentricity sum energy $E_{ES}(G)$. Shown spectrum of some standard graphs. Also obtained bounds for eigenvalues of $ES(G)$ and bounds for the eccentricity sum energy $E_{ES}(G)$ of a graph G.

REFERENCES

1. I.Gutman, The energy of a graph, *Ber. Math. Stat.*, 103 (1978) 1–22.
2. D.M.Cvetkovic, M.Doob and H.Sachs, *Spectra of Graphs*, Academic Press, New York, 1980.
3. H.S.Ramane, D.S.Revankar and J.B.Patil, Bounds for the degree sum eigenvalues and degree sum energy of a graph, *International Journal of Pure and Applied Mathematical Sciences*, 6 (2) (2013) 161-167.
4. S.R.Jog, S.P.Hande and D.S.Revankar, Degree sum polynomial of graph valued functions on regular graphs, *Int. J. Graph Theory*, 1(3) (2013) 108 – 115.
5. S.Bernard, J.M.Child, *Higher Algebra*, Macmillan India Ltd, New Delhi, 2001.
6. X.Li, Y.Shi and I.Gutman, *Graph Energy*, Springer, 2013.
7. H.Kober, On the arithmetic and geometric means and on Holder's inequality, *Proc. Amer. Math. Soc.*, 9 (1958) 452-459.
8. D.S.Revankar, M.M.Patil and S.P.Hande, Note on the bounds for the degree sum energy of a graph, degree sum energy of a common neighborhood graph and terminal distance energy of a graph, *International Journal of Mathematical Archive*, 7(6) (2016) 34-37.
9. S.M.Hosamani and H.S.Ramane, On degree sum energy of a graph, *European Journal of Pure and Applied Mathematics*, 9(3) (2016) 340-345.