Annals of Pure and Applied Mathematics
Vol. 12, No. 1, 2016, 59-68
ISSN: 2279-087X (P), 2279-0888(online)
Published on 28 July 2016
www.researchmathsci.org

Annals of
Pure and Applied Mathematics

Rainbow Connection in Brick Product of Odd Cycle Graphs

***K.Srinivasa Rao*[1] and *R.Murali*[2]**

[1]Department of Mathematics, Shri Pillappa College of Engineering
Bengaluru. E-mail: srinivas.dbpur@gmail.com
[2]Department of Mathematics, Dr. Ambedkar Institute of Technology
Bengaluru. E-mail: murualir2968@gmail.com

Received 17 July 2016; accepted 27 July 2016

Abstract. Let G be a nontrivial connected graph on which is defined a coloring $c : E(G) \to \{1,2,3,....,k\}, k \in N$, of the edges of G, where adjacent edges may be colored the same. A path in G is called a rainbow path if no two edges of it are colored the same. G is rainbow connected if G contains a rainbow $u - v$ path for every two vertices u and v in it. The minimum k for which there exists such a *k-edge* coloring is called the rainbow connection number of G, denoted by $rc(G)$. In this paper, we determine $rc(G)$ of brick product graphs associated with odd cycles.

Keywords: diameter, edge-coloring, rainbow path, rainbow connection number, brick product.

AMS Mathematics Subject Classification (2010): 05C15

1. Introduction

Let G be a nontrivial connected graph with an edge coloring

$$c : E(G) \to \{1, 2, 3, ..., k\},\ k \in N,$$

where adjacent edges may be colored the same. A path in G is called a **rainbow path** if no two edges of it are colored the same. An edge colored graph G is said to be **rainbow connected** if for any two vertices in G, there is a rainbow path in G connecting them. Clearly, if a graph is rainbow connected, it must be connected. Conversely, any connected graph has a trivial edge coloring that makes it rainbow connected, i.e., a coloring such that each edge has a distinct color. The minimum k for which there exist a rainbow k-coloring of G is called the **rainbow connection number** of G, denoted by $rc(G)$.

Chartrand et.al. introduced the concept of the rainbow connection number and determined $rc(G)$ for some classes of graphs like the cycle graph, the wheel graph etc. in [1]. In [2] and [3] Srinivasa Rao and Murali, determined $rc(G)$ and the strong rainbow connection number $src(G)$ of some classes of graphs like the stacked book graph, the grid graph, the prism graph etc. and discussed the critical property of these graphs with respect to rainbow coloring. The rainbow connection number of the fan graph, the sun graph and the gear graph has been determined in [6]. Other results on the rainbow connection number of a graph can be found in [6,7,8]. In [8], Nabila et al. determined the rainbow connection number of Origami graphs and Pizza graphs. An overview about rainbow connection number can be found in a book of Li and Sun in [4] and a survey by Li et al. in [5]. In [11] authors studied distance pattern edge coloring of a graph.

The brick product of even cycles was introduced in a paper by Alspach et al. in [9] and it was proved that these graphs exhibit hamiltonian laceability properties. Using this concept Shenoy and Murali in [10] defined the brick product related to an odd cycle. In this paper we determine the rainbow connection number $rc(G)$ of brick product graphs associated with odd cycles.

Definition 1.1. (Brick product of odd cycles) Let m, n *and* r be positive integers. Let $C_{2n+1} = v_0, v_1, \ldots\ldots\ldots\ldots v_{2n}, v_{2n+1} = v_0$ denote a cycle of order $2n+1$ $(n>1)$. The (m,r)-brick product of C_{2n+1}, denoted by $C(2n+1, m, r)$ is defined as follows:

For $m=1$, we require that $1<r<2n$. Then $C(2n+1, m, r)$ is obtained from C_{2n+1} by adding chords v_k *to* v_{k+r}, $0 \le k \le 2n$ where the computation is performed under modulo $2n+1$.

For $m>1$, $C(2n+1, m, r)$ is obtained by first taking the disjoint union of m copies C_{2n+1} namely $C_{2n+1}(1), C_{2n+1}(2), C_{2n+1}(3), \ldots\ldots C_{2n+1}(m)$ where for each $i = 1, 2, \ldots\ldots, m, C_{2n+1}(i) = v_{i1}, v_{i2}, v_{i3}, \ldots\ldots\ldots\ldots, v_{i(2n)}, v_{i0}$. Further:

Case (i): If *m* is odd and $1<r<2n$ where r is defined as $r = [(2n+1)k]+2, k \ge 0$, an edge is drawn to join v_{ik} *to* $v_{(i+1)k}$ for both odd or both even $1 \le i \le (m-1), 0 \le k \le 2n$ whereas for each odd $1 \le i \le (m-1)$ and even $0 \le k \le 2n$ an edge is drawn to join v_{ik} *to* $v_{m(k+2)}$.

Finally an edge is drawn to join $v_{i(2n)}$ *to* $v_{m(2n+r)}$.

Case(ii): If *m* is even and $1<r<2n$ where r is defined as $r = [(2n+1)k]+3, k \ge 0$, an edge is drawn to join v_{ik} *to* $v_{(i+1)k}$ for both odd or both even $1 \le i \le (m-1), 0 \le k \le 2n$ whereas for each odd $1 \le i \le (m-1)$ and even $0 \le k \le 2n$ an edge is drawn to join v_{ik} *to* $v_{m(k+2)}$.

Finally an edge is drawn to join $v_{i(2n)}$ *to* $v_{m(2n+r)}$.

The Brick products $C(13,1,2)$ is shown in figure 1.

In the next section, we determine the values of $rc(G)$ for the brick product graph $C(2n+1, m, r)$ for $m=1$, $r=2$, $n \ge 2$ and for $m=1$, $r=3$, $n \ge 3$.

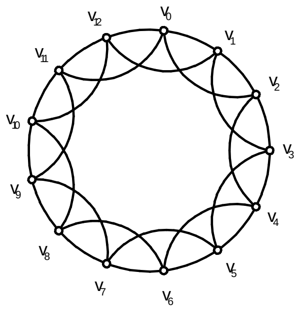

Figure 1: The brick product $C(13,1,2)$

2. Main results

Theorem 2.1. Let $G = C(2n+1, m, r)$. Then for $m = 1,\ r = 2$ and $n \geq 2$,

$$rc(G) = \begin{cases} \left\lceil \dfrac{n}{2} \right\rceil & for \quad 2 \leq n \leq 3 \\ \left\lfloor \dfrac{n}{2} \right\rfloor + 1 & for \quad n \geq 4 \end{cases}$$

Proof: We consider the vertex set of G as $V(G) = \{v_0, v_1, ...v_{2n}, v_{2n+1}\}$ and the edge set of G as $E(G) = \{e_i : 1 \leq i \leq 2n+1\} \cup \{e_i' : 1 \leq i \leq 2n+1\}$, where e_i is the ***cycle edge*** (v_{i-1}, v_i) and e_i' is the ***brick edge*** (v_k, v_{k+r}) , $k = 0,1,2,......,2n-1$. Here $k+r$ is computed modulo $2n+1$.

We prove this theorem in different cases as follows.

Case 1: $2 \leq n \leq 3$.

We have two sub cases.

Subcase 1: $n = 2$,

$G = C(5,1,2) \cong K_5$. For complete graph K_n, $rc(K_n) = 1$. Hence $rc(G) = 1$.

Subcase 2: $n = 3$

Since $diam(G) = 2$, it follows that $rc(G) \geq 2$. It remains to show that $rc(G) \leq 2$. Define $c : E(G) \to \{1,2\}$ and consider the assignment of colors to the edges of G as

$$c(e) = \begin{cases} 1 \quad if \quad e = v_0v_1 = v_1v_2 = v_4v_5 = v_5v_6 = v_0v_2 == v_1v_3 = v_4v_6 = v_5v_0 \\ 2 \quad if \quad e = v_2v_3 = v_3v_4 = v_6v_0 = v_2v_4 = v_3v_5 = v_6v_1 \end{cases}$$

Then, for any two vertices $x, y \in V(G)$, the above assignment gives a rainbow $x-y$ path in G. Hence $rc(G) \leq 2$. This proves $rc(G) = 2$. (An illustration for the assignment of colors in $C(7,1,2)$ is provided in figure 2).

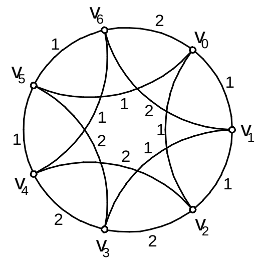

Figure 2: Assignment of colors in $C(7,1,2)$

Case 2: $n \geq 4$

We prove this case in two subcases.

Subcase 1: Let *n* be even. Since $diam(G) = \frac{n}{2}$, it follows that $rc(G) \geq \frac{n}{2}$. But, if we assign $\frac{n}{2}$ - colors to the edges of G as in case 1, we fail to obtain a rainbow path between the vertices v_1 *to* $v_{n-1}\ \forall n.$ (This is illustrated in figure 3 for the graph $C(9,1,2)$).

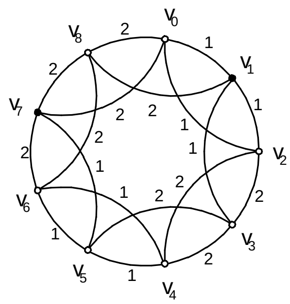

Figure 3: Assignment of colors in $C(9,1,2)$

Accordingly, we construct an edge coloring $c : E(G) \rightarrow \left\{1,2,......,\frac{n}{2}+1\right\}$ and assign the colors to the edges of G as

$$c(e_i) = c(e_i') = \begin{cases} \left\lceil \frac{i}{2} \right\rceil & if \quad 1 \leq i \leq n \\ \left\lceil \frac{i-n}{2} \right\rceil & if \quad n+1 \leq i \leq 2n \\ \frac{n}{2}+1 & if \quad \left\lceil \frac{3n}{2} \right\rceil + 2 \leq i \leq 2n \end{cases}$$

From the above assignment it is clear that for any 2 vertices $x, y \in V(G)$, there exists a rainbow $x - y$ path.

Subcase 2: Let *n* be odd.

Since $diam(G) = \left\lceil \frac{n}{2} \right\rceil$, it follows that $rc(G) \geq \left\lceil \frac{n}{2} \right\rceil$. It remains to show that $rc(G) \leq \left\lceil \frac{n}{2} \right\rceil$. Define $c : E(G-e) \to \left\{1,2,......,\left\lceil \frac{n}{2} \right\rceil\right\}$ and consider the assignment of colors to the edges of G as,

$$c(e_i) = c(e_i') = \begin{cases} \left\lceil \frac{i}{2} \right\rceil & if \quad 1 \leq i \leq n+1 \\ \left\lceil \frac{i-(n+1)}{2} \right\rceil & if \quad n+2 \leq i \leq 2n \\ \left\lceil \frac{n}{2} \right\rceil & if \quad i = 2n+1 \end{cases}$$

It is easy to verify that for any two vertices $x, y \in V(G)$, the above assignment gives a rainbow $x-y$ path in G. Hence $rc(G) \leq \left\lceil \frac{n}{2} \right\rceil$.

Combining both the sub cases, we have $rc(G) \leq \left\lfloor \frac{n}{2} \right\rfloor + 1,\ \forall n\ n \geq 4$.

This proves $rc(G) = \left\lfloor \frac{n}{2} \right\rfloor + 1, \quad \forall n\ n \geq 4.$

(An illustration for the assignment of colors in $C(9,1,2)$ is provided in figure 4).

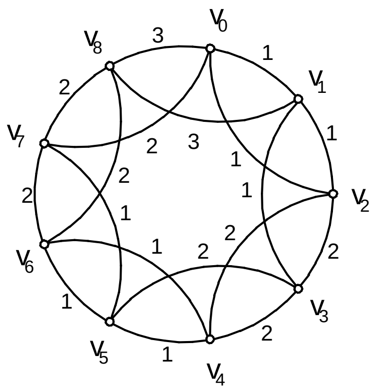

Figure 4: Assignment of colors in $C(9,1,2)$

Theorem 2.2. Let $G = C(2n+1, m, r)$. Then for $m = 1$, $r = 3$ and $n \geq 3$,

$$rc(G) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil & for \quad 3 \leq n \leq 5 \\ \left\lceil \frac{n}{2} \right\rceil + 1 & for \quad n \geq 6 \end{cases}.$$

Proof: We consider the vertex set $V(G)$ and the edge set $E(G)$ defined in **Theorem 2.1**. We prove this result in different cases as follows.

Case 1: $3 \le n \le 5$

Since $diam(G) = \left\lceil \frac{n}{2} \right\rceil$, it follows that $rc(G) \ge \left\lceil \frac{n}{2} \right\rceil$. It remains to show that $rc(G) \le \left\lceil \frac{n}{2} \right\rceil$. We have following subcases.

Subcase 1: $n = 3$ and $n = 5$

Define $c : E(G) \to \left\{1,2,......,\left\lceil \frac{n}{2} \right\rceil\right\}$ and consider the assignment of colors to the edges of G as,

$$c(e_i) = \begin{cases} i & if \quad 1 \le i \le \left\lceil \frac{n}{2} \right\rceil \\ i - \left\lceil \frac{n}{2} \right\rceil & if \quad \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n+1 \\ i - (n+1) & if \quad n+2 \le i \le \frac{3n+3}{2} \\ i - \left(\frac{3n+3}{2}\right) & if \quad \left(\frac{3n+3}{2}\right) + 1 \le i \le 2n \\ \left\lceil \frac{n}{2} \right\rceil & if \quad i = 2n+1 \end{cases}$$

and

$$c(e_i') = \begin{cases} i & if \quad 1 \le i \le n-2 \\ i - (n-2) & if \quad n-1 \le i \le n \\ i - n & if \quad n+1 \le i \le 2n-2 \\ 2n - (i-2) & if \quad 2n-1 \le i \le 2n+1 \end{cases}$$

It is easy to verify that for any two vertices $x, y \in V(G)$, the above assignment gives a rainbow $x - y$ path in G.

(An illustration for the assignment of colors in $C(11,1,3)$ is provided in figure 5).

Subcase 2: $n = 4$

Define $c : E(G) \to \{1,2\}$ and consider the assignment of colors to the edges of G as,

$$c(e) = \begin{cases} 1 & if \quad e = v_0v_1 = v_2v_3 = v_4v_5 = v_6v_7 = v_0v_3 = v_2v_5 = v_4v_7 = v_6v_0 \\ 2 & if \quad e = v_1v_2 = v_3v_4 = v_5v_1 = v_7v_8 = v_1v_4 = v_3v_6 = v_5v_8 = v_7v_1 \end{cases}$$

It is easy to verify that for any two vertices $x, y \in V(G)$, the above assignment gives a rainbow $x - y$ path in G.

(An illustration for the assignment of colors in $C(9,1,3)$ is provided in figure 5).

Combining both the subcases, we have $rc(G) \le \left\lceil \frac{n}{2} \right\rceil$. This proves $rc(G) = \left\lceil \frac{n}{2} \right\rceil$.

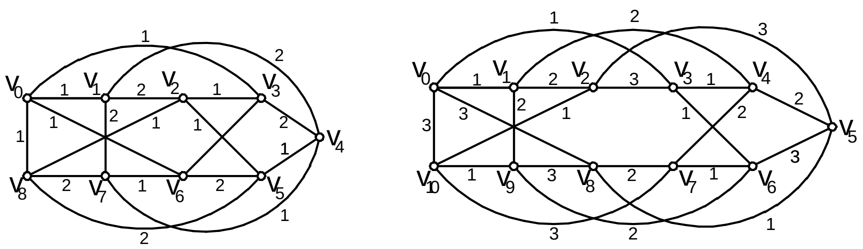

Figure 5: Assignment of colors in $C(9,1,3)$ and $C(11,1,3)$

Case 2: $n \ge 6$.

We have two subcases.

Subcase 1: Let *n* be even

In this case, $diam(G) = \left\lfloor \frac{n}{2} \right\rfloor$ and hence it follows that $rc(G) \ge \left\lfloor \frac{n}{2} \right\rfloor$. But, if we assign $\left\lfloor \frac{n}{2} \right\rfloor$ colors to the edges of G as in case 1 above, we fail to obtain a rainbow path between the vertices v_0 *to* v_n $\forall n$ (this is illustrated in figure 6 for the graph $C(13,1,3)$) and this continues for a total of up to $\left\lceil \frac{n}{2} \right\rceil$ colors. Hence, we need at least one more color along with $\left\lceil \frac{n}{2} \right\rceil$ colors. This shows that $rc(G) \ge \left\lceil \frac{n}{2} \right\rceil + 1$

In order to show that $rc(G) \le \left\lceil \frac{n}{2} \right\rceil + 1$, construct an edge coloring

$c : E(G) \to \left\{1,2,\ldots\ldots,\left\lceil \frac{n}{2} \right\rceil + 1\right\}$ to the edges of G as

$$c(e_i) = \begin{cases} \frac{i}{2} & if \quad i \text{ is even and } 2 \le i \le n-2 \\ \left\lfloor \frac{i-n}{2} \right\rfloor & if \quad i \text{ is odd and } n+3 \le i \le 2n-1 \\ \frac{n}{2}+1 & if \quad i \text{ is odd and } 1 \le i \le n-1 \\ \frac{n}{2}+1 & if \quad i \text{ is even and } n+2 \le i \le 2n \\ \frac{n}{2} & elswhere \end{cases}$$

and

$$c(e'_i)=\begin{cases}\left\lceil \dfrac{i}{2}\right\rceil & if \quad i \text{ is odd and } 1\le i\le n-3\\ \left\lceil \dfrac{i-(n+1)}{2}\right\rceil & if \quad i \text{ is even and } n+2\le i\le 2n-2\\ \dfrac{n}{2}+1 & if \quad i \text{ is even and } 2\le i\le 2n-2\\ \dfrac{n}{2}+1 & if \quad i \text{ is odd and } n+1\le i\le 2n-3\\ \dfrac{n}{2} & elswhere\end{cases}$$

It is easy to verify that for any two vertices $x, y \in V(G)$, the above assignment gives a rainbow $x-y$ path in G. (An illustration for the assignment of colors in $C(13,1,3)$ is provided in figure 6).

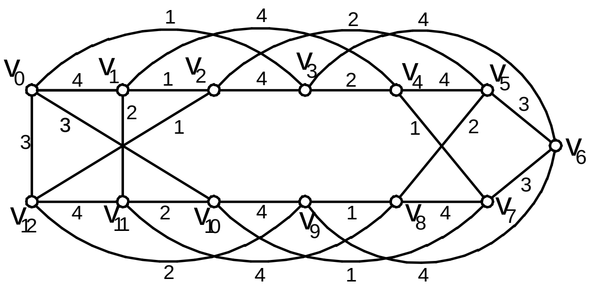

Figure 6: Assignment of colors in $C(13,1,3)$.

Subcase 2: Let *n* be odd

As in subcase 1, in this case also $diam(G)=\left\lfloor \frac{n}{2}\right\rfloor$, and if we assign $\left\lfloor \frac{n}{2}\right\rfloor$ colors to the edges of G we fail to obtain a rainbow path between the pair of vertices and this continues for a total of up to $\left\lceil \frac{n}{2}\right\rceil$ colors. Hence, here again, we need at least $\left\lceil \frac{n}{2}\right\rceil+1$ colors.

This shows that $rc(G)\ge\left\lceil \frac{n}{2}\right\rceil+1$. In order to show that $rc(G)\le\left\lceil \frac{n}{2}\right\rceil+1$, we construct an edge coloring $c: E(G)\to\left\{1,2,......,\left\lceil \frac{n}{2}\right\rceil+1\right\}$ to the edges of G as

$$c(e_i)=\begin{cases}\dfrac{i}{2} & \text{if} \quad i \text{ is even and } 2\le i\le n-1\\[2mm] \dfrac{i-(n+1)}{2} & \text{if} \quad i \text{ is even and } n+3\le i\le 2n\\[2mm] \left\lceil\dfrac{n}{2}\right\rceil+1 & \text{if} \quad i \text{ is odd and } 1\le i\le n-2\\[2mm] \left\lceil\dfrac{n}{2}\right\rceil+1 & \text{if} \quad i \text{ is odd and } n+2\le i\le 2n-1\\[2mm] \left\lceil\dfrac{n}{2}\right\rceil & \text{elswhere}\end{cases}$$

and

$$c(e_i')=\begin{cases}\left\lceil\dfrac{i}{2}\right\rceil & \text{if} \quad i \text{ is odd and } 1\le i\le n-2\\[2mm] \dfrac{i-n}{2} & \text{if} \quad i \text{ is odd and } n+2\le i\le 2n-3\\[2mm] \left\lceil\dfrac{n}{2}\right\rceil+1 & \text{if} \quad i \text{ is even and } 2\le i\le n-3\\[2mm] \left\lceil\dfrac{n}{2}\right\rceil+1 & \text{if} \quad i \text{ is even and } n+1\le i\le 2n-2\\[2mm] \left\lceil\dfrac{n}{2}\right\rceil & \text{elswhere}\end{cases}$$

From the above assignment it is easy to verify that for any two vertices $x, y\in V(G)$, the above assignment gives a rainbow $x-y$ path in G.

Combining both the subcases, we have $rc(G)\le\left\lceil\dfrac{n}{2}\right\rceil+1$. This proves $rc(G)=\left\lceil\dfrac{n}{2}\right\rceil+1$. (An illustration for the assignment of colors in $C(15,1,3)$ is provided in figure 7).

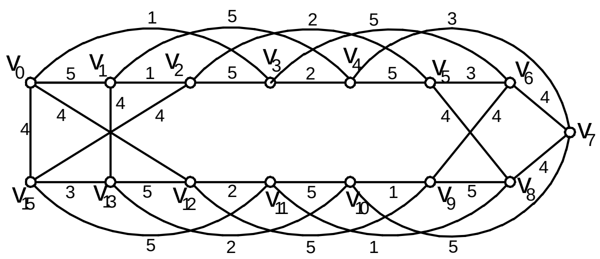

Figure 7: Assignment of colors in $C(15,1,3)$.

REFERENCES

1. G.Chartrand, G.L.Johns, K.A.McKeon and P.Zhang, Rainbow connection in graphs *Math Bohem*,133 (2008) 85-98.
2. K.Srinivasa Rao and R.Murali, Rainbow critical graphs, *Int. Journal of Comp. Application*, 4(4) (2014) 252-259.
3. K.Srinivasa Rao, R.Murali and S.K.Rajendra, Rainbow and strong rainbow criticalness of some standard graphs, *Int. Journal of Mathematics and Computer Research,* 3(1) (2015) 829-836.
4. X.Li, Y.Shi and Y.Sun, Rainbow connection of graphs: a survey, *Graphs Combin.*, 29 (1) (2013) 1-38.
5. X.Li and Y.Sun, Rainbow connection of graphs, New York, Springer-Verlag, 2012.
6. S.Syafrizal, G.H.Medika and L.Yulianti, The rainbow connection number of fan and sun, *Applied Mathematical Science*, 7 (2013) 3155-3160.
7. S.Syafrizal, R.Wijaya and Surahamat, Rainbow connection of some graphs, *Applied Mathematical Science*, 8 (2014) 4693-4696.
8. S.Nabila and A.N.M.Salmal, The rainbow connection number of origami graphs and Pizza graphs, *Procedia Computer Science,* 74 (2015) 162-167.
9. B.Alspach, B.Alspach, C.C.Chen and Kevin Mc Avaney, On a class of Hamiltonian laceable 3-regular graphs, *Discrete Mathematics*, 151 (1996) 19-38.
10. L.N.Shenoy and R.Murali, Laceability on a class of regular graphs, *International Journal of Computational Science and Mathematics*, (20) 3 (2010) 397-406.
11. P.T.Marykutty and K.A.Germina, Open distance pattern edge coloring of a graph, *Annals of Pure and Applied Mathematics,* 6(2) (2014) 191-198.