Rainbow Connection in Brick Product of Odd Cycle Graphs

K.Srinivasa Rao¹ and R.Murali²

¹Department of Mathematics, Shri Pillappa College of Engineering, Bengaluru. E-mail: srinivas.dbpur@gmail.com
²Department of Mathematics, Dr. Ambedkar Institute of Technology, Bengaluru. E-mail: murualir2968@gmail.com

Received 17 July 2016; accepted 27 July 2016

Abstract. Let G be a nontrivial connected graph on which is defined a coloring $c : E(G) \rightarrow \{1, 2, 3, \ldots, k\}, k \in N$, of the edges of G, where adjacent edges may be colored the same. A path in G is called a rainbow path if no two edges of it are colored the same. G is rainbow connected if G contains a rainbow $u \rightarrow v$ path for every two vertices $u$ and $v$ in it. The minimum $k$ for which there exists such a $k$-edge coloring is called the rainbow connection number of G, denoted by $rc(G)$. In this paper, we determine $rc(G)$ of brick product graphs associated with odd cycles.

Keywords: diameter, edge-coloring, rainbow path, rainbow connection number, brick product.

AMS Mathematics Subject Classification (2010): 05C15

1. Introduction

Let $G$ be a nontrivial connected graph with an edge coloring $c : E(G) \rightarrow \{1, 2, 3, \ldots, k\}, k \in N$, where adjacent edges may be colored the same. A path in $G$ is called a rainbow path if no two edges of it are colored the same. An edge colored graph $G$ is said to be rainbow connected if for any two vertices in $G$, there is a rainbow path in $G$ connecting them. Clearly, if a graph is rainbow connected, it must be connected. Conversely, any connected graph has a trivial edge coloring that makes it rainbow connected, i.e., a coloring such that each edge has a distinct color. The minimum $k$ for which there exist a rainbow $k$-coloring of $G$ is called the rainbow connection number of $G$, denoted by $rc(G)$.

Chartrand et.al. introduced the concept of the rainbow connection number and determined $rc(G)$ for some classes of graphs like the cycle graph, the wheel graph etc. in [1]. In [2] and [3] Srinivasa Rao and Murali determined $rc(G)$ and the strong rainbow connection number $src(G)$ of some classes of graphs like the stacked book graph, the grid graph, the prism graph etc. and discussed the critical property of these graphs with
The rainbow connection number of the fan graph, the sun graph and the gear graph has been determined in [6]. Other results on the rainbow connection number of a graph can be found in [6,7,8]. In [8], Nabila et al. determined the rainbow connection number of Origami graphs and Pizza graphs. An overview about rainbow connection number can be found in a book of Li and Sun in [4] and a survey by Li et al. in [5]. In [11] authors studied distance pattern edge coloring of a graph.

The brick product of even cycles was introduced in a paper by Alspach et al. in [9] and it was proved that these graphs exhibit hamiltonian laceability properties. Using this concept Shenoy and Murali in [10] defined the brick product related to an odd cycle. In this paper we determine the rainbow connection number $rc(G)$ of brick product graphs associated with odd cycles.

**Definition 1.1. (Brick product of odd cycles)** Let $m, n$ and $r$ be positive integers. Let $C_{2n+1} = v_0, v_1, ..., v_{2n}, v_{2n+1} = v_0$ denote a cycle of order $2n+1$ ($n > 1$). The $(m,r)$-brick product of $C_{2n+1}$, denoted by $C(2n+1,m,r)$ is defined as follows:

For $m = 1$, we require that $1 < r < 2n$. Then $C(2n+1,m,r)$ is obtained from $C_{2n+1}$ by adding chords $v_k$ to $v_{k+r}$, $0 \leq k \leq 2n$ where the computation is performed under modulo $2n+1$.

For $m > 1$, $C(2n+1,m,r)$ is obtained by first taking the disjoint union of $m$ copies $C_{2n+1}$ namely $C_{2n+1}(1), C_{2n+1}(2), C_{2n+1}(3), ..., C_{2n+1}(m)$ where for each $i = 1, 2, ..., m$, $C_{2n+1}(i) = v_i, v_{i+1}, v_{i+2}, ..., v_{i+2n}, v_i$. Further:

**Case (i):** If $m$ is odd and $1 < r < 2n$ where $r$ is defined as $r = \lfloor (2n+1)k \rfloor + 2k$, $k \geq 0$, an edge is drawn to join $v_{ik}$ to $v_{i+r+k}$ for both odd or both even $1 \leq i \leq (m-1)$, $0 \leq k \leq 2n$ whereas for each odd $1 \leq i \leq (m-1)$ and even $0 \leq k \leq 2n$ an edge is drawn to join $v_{ik}$ to $v_{m(k+2)}$.

Finally an edge is drawn to join $v_{i(2n)}$ to $v_{m(2n+r)}$.

**Case (ii):** If $m$ is even and $1 < r < 2n$ where $r$ is defined as $r = \lfloor (2n+1)k \rfloor + 3k$, $k \geq 0$, an edge is drawn to join $v_{ik}$ to $v_{i+r+k}$ for both odd or both even $1 \leq i \leq (m-1)$, $0 \leq k \leq 2n$ whereas for each odd $1 \leq i \leq (m-1)$ and even $0 \leq k \leq 2n$ an edge is drawn to join $v_{ik}$ to $v_{m(k+2)}$.

Finally an edge is drawn to join $v_{i(2n)}$ to $v_{m(2n+r)}$.

The Brick products $C(13,1,2)$ is shown in figure 1.

In the next section, we determine the values of $rc(G)$ for the brick product graph $C(2n+1,m,r)$ for $m = 1$, $r = 2$, $n \geq 2$ and for $m = 1$, $r = 3$, $n \geq 3$. 

Rainbow Connection in Brick Product of Odd Cycle Graphs

Figure 1: The brick product $C(I3,I,2)$

2. Main results

Theorem 2.1. Let $G = C(2n+1, m, r)$. Then for $m = 1$, $r = 2$ and $n \geq 2$,

$$rc(G) = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor & \text{for } 2 \leq n \leq 3 \\ \left\lfloor \frac{n}{2} \right\rfloor + 1 & \text{for } n \geq 4 \end{cases}$$

Proof: We consider the vertex set of $G$ as $V(G) = \{v_0, v_1, \ldots, v_{2n}, v_{2n+1}\}$ and the edge set of $G$ as $E(G) = \{e_i : 1 \leq i \leq 2n+1\} \cup \{e'_i : 1 \leq i \leq 2n+1\}$, where $e_i$ is the cycle edge $(v_{i-1}, v_i)$ and $e'_i$ is the brick edge $(v_{i}, v_{k+r})$, $k = 0, 1, 2, \ldots, 2n-1$. Here $k + r$ is computed modulo $2n+1$.

We prove this theorem in different cases as follows.

Case 1: $2 \leq n \leq 3$.

We have two sub cases.

Subcase 1: $n = 2$.

$G = C(5,1,2) \cong K_5$. For complete graph $K_n$, $rc(K_n) = 1$. Hence $rc(G) = 1$.

Subcase 2: $n = 3$

Since $diam(G) = 2$, it follows that $rc(G) \geq 2$. It remains to show that $rc(G) \leq 2$. Define $c : E(G) \rightarrow \{1,2\}$ and consider the assignment of colors to the edges of $G$ as

$$c(e) = \begin{cases} 1 & \text{if } e = v_0v_1 = v_1v_2 = v_2v_3 = v_3v_4 = v_4v_5 = v_5v_0 \\ 2 & \text{if } e = v_2v_3 = v_3v_4 = v_4v_5 = v_5v_6 = v_6v_1 \end{cases}$$

Then, for any two vertices $x, y \in V(G)$, the above assignment gives a rainbow $x - y$ path in $G$. Hence $rc(G) \leq 2$. This proves $rc(G) = 2$. (An illustration for the assignment of colors in $C(7,1,2)$ is provided in figure 2).
Case 2: $n \geq 4$

We prove this case in two subcases.

Subcase 1: Let $n$ be even. Since $\text{diam}(G) = \frac{n}{2}$, it follows that $rc(G) \geq \frac{n}{2}$. But, if we assign $\frac{n}{2}$ - colors to the edges of $G$ as in case 1, we fail to obtain a rainbow path between the vertices $v_i$ to $v_{n-i}$ $\forall n$. (This is illustrated in figure 3 for the graph $C(9,1,2)$).

Accordingly, we construct an edge coloring $c : E(G) \rightarrow \left\{1,2,\ldots,\frac{n}{2}+1\right\}$ and assign the colors to the edges of $G$ as

$$c(e_i) = c(e'_i) = \begin{cases} \left\lfloor \frac{i}{2} \right\rfloor & \text{if} \quad 1 \leq i \leq n \\ \left\lfloor \frac{i-n}{2} \right\rfloor & \text{if} \quad n+1 \leq i \leq 2n \\ \frac{n}{2}+1 & \text{if} \quad \left\lceil \frac{3n}{2} \right\rceil + 2 \leq i \leq 2n \end{cases}$$

From the above assignment it is clear that for any 2 vertices $x, y \in V(G)$, there exists a rainbow $x - y$ path.

Subcase 2: Let $n$ be odd.
Rainbow Connection in Brick Product of Odd Cycle Graphs

Since $diam(G) = \left\lfloor \frac{n}{2} \right\rfloor$, it follows that $rc(G) \geq \left\lfloor \frac{n}{2} \right\rfloor$. It remains to show that $rc(G) \leq \left\lceil \frac{n}{2} \right\rceil$. Define $c : E(G - e) \rightarrow \left\{ 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \right\}$ and consider the assignment of colors to the edges of $G$ as,

$$c(e_i) = c(e'_i) = \begin{cases} \left\lceil \frac{i}{2} \right\rceil & \text{if } 1 \leq i \leq n + 1 \\ \left\lfloor \frac{i - (n + 1)}{2} \right\rfloor & \text{if } n + 2 \leq i \leq 2n \\ \left\lceil \frac{n}{2} \right\rceil & \text{if } i = 2n + 1 \end{cases}$$

It is easy to verify that for any two vertices $x, y \in V(G)$, the above assignment gives a rainbow $x - y$ path in $G$. Hence $rc(G) \leq \left\lceil \frac{n}{2} \right\rceil$.

Combining both the sub cases, we have $rc(G) \leq \left\lfloor \frac{n}{2} \right\rfloor + 1, \ \forall n \ n \geq 4$.

This proves $rc(G) = \left\lfloor \frac{n}{2} \right\rfloor + 1, \ \forall n \ n \geq 4$.

(An illustration for the assignment of colors in $C(9,1,2)$ is provided in figure 4).

**Figure 4:** Assignment of colors in $C(9,1,2)$

**Theorem 2.2.** Let $G = C(2n + 1, m, r)$. Then for $m = 1$, $r = 3$ and $n \geq 3$,

$$rc(G) = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor & \text{for } 3 \leq n \leq 5 \\ \left\lfloor \frac{n}{2} \right\rfloor + 1 & \text{for } n \geq 6 \end{cases}$$
K.Srinivasa Rao and R.Murali

**Proof:** We consider the vertex set \( V(G) \) and the edge set \( E(G) \) defined in Theorem 2.1. We prove this result in different cases as follows.

**Case 1:** \( 3 \leq n \leq 5 \)

Since \( \text{diam}(G) = \left\lceil \frac{n}{2} \right\rceil \), it follows that \( rc(G) \geq \left\lceil \frac{n}{2} \right\rceil \). It remains to show that \( rc(G) \leq \left\lfloor \frac{n}{2} \right\rfloor \). We have following subcases.

**Subcase 1:** \( n = 3 \) and \( n = 5 \)

Define \( c : E(G) \rightarrow \{1, 2, \ldots, \left\lceil \frac{n}{2} \right\rceil\} \) and consider the assignment of colors to the edges of \( G \) as,

\[
\begin{align*}
c(e) &= \begin{cases} 
  i & \text{if } 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil \\
  i - \left\lfloor \frac{n}{2} \right\rfloor & \text{if } \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n + 1 \\
  i - \left( \frac{3n + 3}{2} \right) & \text{if } \left( \frac{3n + 3}{2} \right) + 1 \leq i \leq 2n \\
  \left\lfloor \frac{n}{2} \right\rfloor & \text{if } i = 2n + 1
\end{cases}
\]

and

\[
\begin{align*}
c(e') &= \begin{cases} 
  i & \text{if } 1 \leq i \leq n - 2 \\
  i - (n - 2) & \text{if } n - 1 \leq i \leq n \\
  i - n & \text{if } n + 1 \leq i \leq 2n - 2 \\
  2n - (i - 2) & \text{if } 2n - 1 \leq i \leq 2n + 1
\end{cases}
\]

It is easy to verify that for any two vertices \( x, y \in V(G) \), the above assignment gives a rainbow \( x - y \) path in \( G \).

(An illustration for the assignment of colors in \( C(11,1,3) \) is provided in figure 5).

**Subcase 2:** \( n = 4 \)

Define \( c : E(G) \rightarrow \{l, 2\} \) and consider the assignment of colors to the edges of \( G \) as,

\[
\begin{align*}
c(e) &= \begin{cases} 
  1 & \text{if } e = v_0v_1 = v_2v_3 = v_4v_5 = v_6v_7 = v_8v_9 = v_0v_1 = v_2v_3 = v_4v_5 = v_6v_7 = v_8v_9 \\
  2 & \text{if } e = v_1v_2 = v_3v_4 = v_5v_6 = v_7v_8 = v_9v_0 = v_1v_2 = v_3v_4 = v_5v_6 = v_7v_8 = v_9v_0
\end{cases}
\]

It is easy to verify that for any two vertices \( x, y \in V(G) \), the above assignment gives a rainbow \( x - y \) path in \( G \).

(An illustration for the assignment of colors in \( C(9,1,3) \) is provided in figure 5).
Combining both the subcases, we have $rc(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$. This proves $rc(G) = \left\lceil \frac{n}{2} \right\rceil$.

**Figure 5:** Assignment of colors in $C(9,1,3)$ and $C(11,1,3)$

**Case 2:** $n \geq 6$.

We have two subcases.

**Subcase 1:** Let $n$ be even.

In this case, $\text{diam}(G) = \left\lfloor \frac{n}{2} \right\rfloor$ and hence it follows that $rc(G) \geq \left\lceil \frac{n}{2} \right\rceil$. But, if we assign $\left\lfloor \frac{n}{2} \right\rfloor$ colors to the edges of $G$ as in case 1 above, we fail to obtain a rainbow path between the vertices $v_0$ to $v_n \forall n$ (this is illustrated in figure 6 for the graph $C(13,1,3)$) and this continues for a total of up to $\left\lfloor \frac{n}{2} \right\rfloor$ colors. Hence, we need at least one more color along with $\left\lfloor \frac{n}{2} \right\rfloor$ colors. This shows that $rc(G) \geq \left\lceil \frac{n}{2} \right\rceil + 1$

In order to show that $rc(G) \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$, construct an edge coloring

$$c : E(G) \to \left\{1,2,\ldots,\left\lfloor \frac{n}{2} \right\rfloor + 1\right\}$$

to the edges of $G$ as

$$c(e_i) = \left\{ \begin{array}{cl} \frac{i}{2} & \text{if } i \text{ is even and } 2 \leq i \leq n-2 \\ \frac{i-n}{2} & \text{if } i \text{ is odd and } n+3 \leq i \leq 2n-1 \\ \frac{n+1}{2} & \text{if } i \text{ is odd and } 1 \leq i \leq n-1 \\ \frac{n}{2} + 1 & \text{if } i \text{ is even and } n+2 \leq i \leq 2n \\ \frac{n}{2} & \text{elsewhere} \end{array} \right.$$
and

\[
\begin{align*}
\left\lfloor \frac{i}{2} \right\rfloor & \quad \text{if } i \text{ is odd and } 1 \leq i \leq n-3 \\
\left\lfloor \frac{i-(n+1)}{2} \right\rfloor & \quad \text{if } i \text{ is even and } n+2 \leq i \leq 2n-2 \\
c(e') = & \frac{n}{2} + 1 \quad \text{if } i \text{ is even and } 2 \leq i \leq 2n-2 \\
\frac{n}{2} + 1 & \quad \text{if } i \text{ is odd and } n+1 \leq i \leq 2n-3 \\
\frac{n}{2} & \quad \text{elsewhere}
\end{align*}
\]

It is easy to verify that for any two vertices \(x, y \in V(G)\), the above assignment gives a rainbow \(x-y\) path in \(G\). (An illustration for the assignment of colors in \(C(13,1,3)\) is provided in figure 6).

**Figure 6:** Assignment of colors in \(C(13,1,3)\).

**Subcase 2:** Let \(n\) be odd

As in subcase 1, in this case also \(\text{diam}(G) = \left\lfloor \frac{n}{2} \right\rfloor\), and if we assign \(\left\lfloor \frac{n}{2} \right\rfloor\) colors to the edges of \(G\) we fail to obtain a rainbow path between the pair of vertices and this continues for a total of up to \(\left\lfloor \frac{n}{2} \right\rfloor\) colors. Hence, here again, we need at least \(\left\lfloor \frac{n}{2} \right\rfloor + 1\) colors.

This shows that \(rc(G) \geq \left\lfloor \frac{n}{2} \right\rfloor + 1\). In order to show that \(rc(G) \leq \left\lfloor \frac{n}{2} \right\rfloor + 1\), we construct an edge coloring \(c : E(G) \rightarrow \left\{1,2,\ldots, \left\lfloor \frac{n}{2} \right\rfloor + 1\right\}\) to the edges of \(G\) as
Rainbow Connection in Brick Product of Odd Cycle Graphs

\[
c(e_i) = \begin{cases} 
\frac{i}{2} & \text{if } i \text{ is even and } 2 \leq i \leq n-1 \\
\frac{i-(n+1)}{2} & \text{if } i \text{ is even and } n+3 \leq i \leq 2n \\
\frac{n}{2} + 1 & \text{if } i \text{ is odd and } 1 \leq i \leq n-2 \\
\frac{n}{2} + 1 & \text{if } i \text{ is odd and } n+2 \leq i \leq 2n-1 \\
\frac{n}{2} & \text{elsewhere}
\end{cases}
\]

and

\[
c(e'_i) = \begin{cases} 
\frac{i}{2} & \text{if } i \text{ is odd and } 1 \leq i \leq n-2 \\
\frac{i-n}{2} & \text{if } i \text{ is odd and } n+2 \leq i \leq 2n-3 \\
\frac{n}{2} + 1 & \text{if } i \text{ is even and } 2 \leq i \leq n-3 \\
\frac{n}{2} + 1 & \text{if } i \text{ is even and } n+1 \leq i \leq 2n-2 \\
\frac{n}{2} & \text{elsewhere}
\end{cases}
\]

From the above assignment it is easy to verify that for any two vertices \(x, y \in V(G)\), the above assignment gives a rainbow \(x - y\) path in \(G\).

Combining both the subcases, we have \(rc(G) \leq \left\lfloor \frac{n}{2} \right\rfloor + 1\). This proves \(rc(G) = \left\lfloor \frac{n}{2} \right\rfloor + 1\). (An illustration for the assignment of colors in \(C(15,1,3)\) is provided in figure 7).

**Figure 7:** Assignment of colors in \(C(15,1,3)\).
K.Srinivasa Rao and R.Murali

REFERENCES