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Inverse and Disjoint Secure Total Domination in Graphs

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Abstract. Let $G = (V, E)$ be a graph. Let D be a minimum secure total dominating set of G. If $V - D$ contains a secure total dominating set D' of G, then D' is called an inverse secure total dominating set with respect to D. The inverse secure total domination number $\gamma_{st}^{-1}(G)$ of G is the minimum cardinality of an inverse secure total dominating set of G. The disjoint secure total domination number $\gamma_{st}\gamma_{st}(G)$ of G is the minimum cardinality of the union of two disjoint secure total dominating sets in G. In this paper, we initiate a study of these two parameters and establish some results on these new parameters.

Keywords: Inverse domination number, inverse secure total domination number, disjoint secure total domination number

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1. Introduction

All graphs considered here are finite, undirected without isolated vertices, loops and multiple edges. For all further notation and terminology we refer the reader to [1].

Let $G = (V, E)$ be a graph with p vertices and q edges. A set $D \subseteq V$ is a dominating set if every vertex in $V - D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. Recently many domination parameters are given in the books by Kulli in [2,3,4]. Let D be a minimum dominating set of G. If $V - D$ contains a dominating set D' of G, then D' is called an inverse dominating set of G with respect to D. The inverse domination number $\gamma^{-1}(G)$ of G is the minimum cardinality of an inverse dominating set of G. This concept was introduced by Kulli and Sigarkanti in [5]. Many other inverse domination parameters in domination theory were studied, for example, in [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19].

A dominating set D in G is called a secure dominating set in G if for every vertex u in $V - D$, there exists v in D adjacent to u such that $(D - \{v\}) \cup \{u\}$ is a dominating set. The secure domination number $\gamma_s(G)$ of G is the minimum cardinality of a secure dominating set of G. This was introduced by Cockayne et al. in [20]. Let D be a minimum secure dominating set of G. If $V - D$ contains a secure dominating set D' of G, then D' is called an inverse secure dominating set with respect to D. The inverse secure domination number $\gamma_s^{-1}(G)$ of G is the minimum cardinality of an inverse secure dominating set of G. This was found in the paper of Enriquez et al. in [21] and was studied by Kulli in [22]. A γ_s^{-1}-set is a minimum inverse secure dominating set. Similarly other sets can be expected.

A set $D \subseteq V$ is a total dominating set of G if every vertex in V is adjacent to some vertex in D. The total domination number $\gamma_t(G)$ of G is the minimum cardinality of a total dominating set of G.

Note that every graph without isolated vertices has a total dominating set. Thus we consider only graphs with no isolated vertices. Let $\Delta(G)$ denote the maximum degree and $\lceil x \rceil$ the least integer greater than or equal to x. The complement of G is denoted by $\bar{G}$.

The disjoint domination number $\gamma\gamma(G)$ of G is the minimum cardinality of the union of two disjoint dominating sets in G. This was introduced by Hedetniemi et al. in [23]. Many other disjoint domination numbers were studied, for example, in [7, 8, 9, 14, 23, 25].

The disjoint secure domination number $\gamma_s\gamma_s(G)$ of G is the minimum cardinality of the union of two disjoint secure dominating sets in G. This concept was introduced by Kulli in [22].

In this paper, we introduce the inverse secure total domination number and the disjoint secure total domination number and study their graph theoretical properties.

2. Inverse secure total domination

We introduce the concept of inverse secure total domination as follows:

Definition 1. Let D be a minimum secure total dominating set of a graph $G=(V, E)$. If $V - D$ contains a secure total dominating set D' of G, then D' is called an inverse secure total dominating set with respect to D. The inverse secure total domination number $\gamma_{st}^{-1}(G)$ of G is the minimum cardinality of an inverse secure total dominating set of G.

Definition 2. The upper inverse secure total domination number $\Gamma_{st}^{-1}(G)$ of G is the maximum cardinality of an inverse secure total dominating set of G.

A γ_{st}^{-1}-set is a minimum inverse secure total dominating set.

Example 3. Let K_4 be the complete graph. Then $\gamma_{st}(K_4) = 2$ and $\gamma_{st}^{-1}(K_4)=2$.

Remark 4. Not all graphs have an inverse secure total dominating set. For example, the cycle C_4 has a secure total dominating set, but no inverse secure total dominating set.

Proposition 5. For any cycle C_p, $\gamma_{st}^{-1}(C_p)$ does not exist.

Proposition 6. For any tree T, $\gamma_{st}^{-1}(T)$ does not exist.

Theorem 7. Let D be a γ_{st}-set of a connected graph G. If a γ_{st}^{-1}-set exists, then G has at least 4 vertices.

Proof: Let D be a γ_{st}^{-1}-set of G. Since G has no isolated vertices, $\gamma_{st}(G)=|D|\geq 2$. If a γ_{st}^{-1}-set exists, then $V-D$ contains a secure total dominating set with respect to D. Thus $|V-D|\geq 2$. Hence G has at least 4 vertices.

From definitions, the following result is immediate.

Proposition 8. For a connected graph G with $p\geq 4$ vertices,

$$\gamma^{-1}(G)\leq \gamma_s^{-1}(G)\leq \gamma_{st}^{-1}(G).$$

The following known results are used to prove our later results.

Proposition A [24]

(1) $\gamma_{st}(K_p)=2$, if $p\geq 2$.

(2) $\gamma_{st}(K_{m,n})=4$, if $3\leq m\leq n$.

Now we obtain the exact values of $\gamma_{st}^{-1}(G)$ for some standard graphs.

Proposition 9. For a complete graph K_p, $p\geq 4$,

$$\gamma_{st}^{-1}(K_p)=2.$$

Proof: Let D be a minimum secure total dominating set of K_p. By Proposition A(1), $|D|=2$. Let $D=\{u, v\}$. Then $S=\{x, y\}$ is a γ_{st}^{-1}-set of K_p for $x, y\in V(K_p)-\{u, v\}$. Hence $\gamma_{st}^{-1}(K_p)=2$.

Proposition 10. For a complete bipartite graph $K_{m,n}$, $4\leq m\leq n$,

$$\gamma_{st}^{-1}(K_{m,n})=4.$$

Proof: Let $V(K_{m,n})=V_1\cup V_2$ where $V_1=\{u_1, u_2, ..., u_m\}$ and $V_2=\{v_1, v_2, ..., v_n\}$. By Proposition $A(2)$, $D=\{u_1, u_2, v_1, v_2\}$ is a minimum secure total dominating set of $K_{m,n}$. Then $S=\{u_3, u_4, v_3, v_4\}$ is a γ_{st}^{-1}-set of $K_{m,n}$ for $u_3, u_4, v_3, v_4\in V(K_{m,n})-\{u_1, u_2, v_1, v_2\}$. Hence $\gamma_{st}^{-1}(K_{m,n})=4$.

Proposition 11. For any graph G with a γ_{st}^{-1}-set,

$$\gamma_{st}(G)\leq \gamma_{st}^{-1}(G)$$

and this bound is sharp.

Proof: Clearly every inverse secure total dominating set of G is a secure total dominating set. Hence

$$\gamma_{st}(G)\leq \gamma_{st}^{-1}(G).$$

The complete graph K_4 and the complete bipartite graph $K_{4,4}$ achieve this bound with $\gamma_{st}(K_4)=\gamma_{st}^{-1}(K_4)=2$ and $\gamma_{st}(K_{4,4})=\gamma_{st}^{-1}(K_{4,4})=4$.

Proposition 12. For any graph G with a γ_s^{-1}-set,

$$\gamma_{st}(G)+\gamma_{st}^{-1}(G)\leq p$$

and this bound is sharp.

Proof: This follows from the definition of $\gamma_{st}^{-1}(G)$.

The graphs K_4 and $K_{4,4}$ achieve this bound.

Theorem 13. For any graph G with a γ_{st}^{-1}-set,

$$2 \le \gamma_{st}^{-1}(G) \le p - 2$$

and these bounds are sharp.

Proof: By Proposition 11, $\gamma_{st}(G) \le \gamma_{st}^{-1}(G)$ and since $2 \le \gamma_{st}(G)$,

$$2 \le \gamma_{st}^{-1}(G).$$

By Proposition 12, $\gamma_{st}^{-1}(G) \le p - \gamma_{st}(G)$ and since $2 \le \gamma_{st}(G)$,

$$\gamma_{st}^{-1}(G) \le p - 2.$$

Hence,

$$2 \le \gamma_{st}^{-1}(G) \le p - 2.$$

These bounds are sharp as can be seen with the graph K_4.

We have the following inequality chain.

Theorem 14. For any graph G with a γ_{st}^{-1}-set,

$$\gamma(G) \le \gamma_s(G) \le \gamma_{st}(G) \le \gamma_{st}^{-1}(G) \le \Gamma_{st}^{-1}(G).$$

We now obtain lower and upper bounds on $\gamma_{st}^{-1}(G)$.

Theorem 15. For any graph G with a γ_s^{-1}-set,

$$\left\lceil \frac{p}{\Delta(G)+1} \right\rceil \le \gamma_{st}^{-1}(G) \le \left\lceil \frac{p\Delta(G)}{\Delta(G)+1} \right\rceil. \qquad (1)$$

Proof: It is known that $\left\lceil \frac{p}{\Delta(G)+1} \right\rceil \le \gamma(G)$ and since $\gamma(G) \le \gamma_{st}^{-1}(G)$, we see that the lower bound in (1) holds.

By Proposition 12,

$$\gamma_{st}^{-1}(G) \le p - \gamma_{st}(G).$$

Since $\left\lceil \frac{p}{\Delta(G)+1} \right\rceil \le \gamma(G) \le \gamma_{st}(G)$ and the above inequality,

$$\gamma_{st}^{-1}(G) \le p - \left\lceil \frac{p}{\Delta(G)+1} \right\rceil.$$

Hence the upper bound in (1) holds.

We establish a Nordhaus-Gaddum type result for secure total domination number.

Theorem 16. Let G be a graph with $p \ge 4$ vertices. If a γ_{st}^{-1}-set exists and G and $\overline{G}$ have no isolated vertices, then

$$4 \le \gamma_{st}^{-1}(G) + \gamma_{st}^{-1}(\overline{G}) \le 2(p-2)$$

$$4 \le \gamma_{st}^{-1}(G) + \gamma_{st}^{-1}(\overline{G}) \le (p-2)^2.$$

Proof: Since a γ_{st}^{-1}-set exists and $G, \overline{G}$ have no isolated vertices,

$$2 \le \gamma_{st}^{-1}(G) \text{ and } 2 \le \gamma_{st}^{-1}(\overline{G}).$$

Hence both lower bounds follow.
By Theorem 13, we have

$$\gamma_{st}^{-1}(G) \le p-2 \text{ and } \gamma_{st}^{-1}(\overline{G}) \le p-2.$$

Hence both upper bounds follow.

3. Disjoint secure total domination

The inverse secure total domination number inspires us to introduce the concept of disjoint secure total domination number.

Definition 17. The disjoint secure total domination number $\gamma_{st}\gamma_{st}(G)$ of a graph G is defined as follows: $\gamma_{st}\gamma_{st}(G) = \min \{|D_1|+|D_2|: D_1, D_2$ are disjoint secure total dominating sets of $G\}$. We say that two disjoint secure total dominating sets, whose union has cardinality $\gamma_{st}\gamma_{st}(G)$, is a $\gamma_{st}\gamma_{st}$-pair of G.

Remark 18. Not all graphs have a disjoint secure total domination number. For example, the cycle C_4 does not have two disjoint secure total dominating sets.

Theorem 19. For any graph G with $\gamma_{st}^{-1}(G)$,

$$2\gamma_{st}(G) \le \gamma_{st}\gamma_{st}(G) \le \gamma_{st}(G) + \gamma_{st}^{-1}(G) \le p.$$

Definition 20. A graph G is called $\gamma_{st}\gamma_{st}$-minimum if it has two disjoint γ_{st}-sets, that is, $\gamma_{st}\gamma_{st}(G) = 2\gamma_{st}(G)$.

Definition 21. A graph G is called $\gamma_{st}\gamma_{st}$-maximum if $\gamma_{st}\gamma_{st}(G) = p$.

When the disjoint secure total domination number exists, the following inequalities hold.

Proposition 22. For any graph G having two disjoint two secure total dominating sets,

$$\gamma\gamma(G) \le \gamma_s\gamma_s(G) \le \gamma_{st}\gamma_{st}(G).$$

The exact values of $\gamma_{st}\gamma_{st}(G)$ for some standard graphs are given below.

Proposition 23. For the complete graph K_p, $p \ge 4$,

$$\gamma_{st}\gamma_{st}(K_p) = 2\gamma_{st}(K_p) = 4.$$

Proof: This follows from Theorem A(1) and Proposition 9.

Proposition 24. For the complete bipartite graph $K_{m,n}$, $4 \le m \le n$,

$$\gamma_{st}\gamma_{st}(K_{m,n}) = 2\gamma_{st}(K_{m,n}) = 8.$$

Proof: This follows from Theorem A(2) and Proposition 10.

The complete graphs K_p, $p \geq 4$ and the complete bipartite graphs $K_{m,n}$, $4 \leq m \leq n$ are $\gamma_{st}\gamma_{st}$ -minimum.

The graphs K_4 and $K_{4,4}$ are $\gamma_{st}\gamma_{st}$ -maximum.

4. Some open problems

In this paper, we have introduced a new type of inverse domination, namely inverse secure total domination. Also we have introduced disjoint secure total domination. Many questions are suggested by this research, among them are the following:

Problem 1. Characterize graphs G for which $\gamma_{st}(G) = \gamma_{st}^{-1}(G)$.

Problem 2. Characterize graphs G for which $\gamma_{st}(G) + \gamma_{st}^{-1}(G) = p$.

Problem 3. Characterize graphs G for which $\gamma_s\gamma_s(G) = \gamma_{st}\gamma_{st}(G)$.

Problem 4. Characterize graphs G for which $\gamma_{st}\gamma_{st}(G) = 2\gamma_s(G)$.

Problem 5. Characterize the class of $\gamma_{st}\gamma_{st}$ -minimum graphs.

Problem 6. Characterize the class of $\gamma_{st}\gamma_{st}$ -maximum graphs.

Problem 7. Obtain bounds for $\gamma_{st}\gamma_{st}(G) + \gamma_{st}\gamma_{st}(\bar{G})$.

Problem 8. What is the complexity of the decision problem corresponding to $\gamma_{st}\gamma_{st}(G)$?

Problem 9. Is DISJOINT SECURE TOTAL DOMINATION NP-complete for a class of graphs?

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