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Secure Edge Domination in Graphs

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Abstract. Let $G = (V, E)$ be a graph without isolated vertices. A secure edge dominating set of G is an edge dominating set $F \subseteq E$ with the property that for each $e \in E - F$, there exists $f \in F$ adjacent to e such that $(F - \{f\}) \cup \{e\}$ is an edge dominating set. The secure edge domination number $\gamma'_s(G)$ of G is the minimum cardinality of a secure edge dominating set of G. In this paper, we initiate a study of the secure edge domination number and establish some results on this new parameter.

Keywords: Edge dominating set, secure edge dominating set, secure edge domination number.

AMS Mathematics Subject Classification (2010): 05C69

1. Introduction

All graphs considered here are finite, undirected without isolated vertices, loops and multiple edges. Let $G = (V, E)$ be a graph with $|V| = p$, vertices and $|E| = q$ edges. For definitions and notations, the reader may refer to [1].

A set $D \subseteq V$ is a dominating set if every vertex not in D is adjacent to at least one vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. Recently many domination parameters are given in the books by Kulli in [2, 3, 4]. A set F of edges in a graph G is an edge dominating set if every edge e in $E - F$ is adjacent to at least one edge in F. The edge domination number $\gamma'(G)$ of G is the minimum cardinality of an edge dominating set of G. A secure dominating set of G is a dominating set $D \subseteq V$ with the property that for each $u \in V - D$, there exists $v \in D$ adjacent to u such that $(D - \{v\}) \cup \{u\}$ is a dominating set. The secure domination number $\gamma'_s(G)$ of G is the minimum cardinality of a secure dominating set. The concept of secure domination was introduced by Cockayne et al. in [5]. Many other domination parameters were studied, for example, in [6, 7, 8, 9, 10, 11, 12, 13, 14].

The degree of a vertex u is denoted by deg(u) and the degree of an edge uv is defined as deg (u) + deg$(v) - 2$. Let Δ' denote the maximum degree among the edges of G. Let $\lceil x \rceil$ denote the least integer greater than or equal to x. In this paper, we introduce the secure edge domination number of a graph.

2. Secure edge domination

We introduce the concept of secure edge domination in graphs.

Definition 1. A secure edge dominating set of G is an edge dominating set $F \subseteq E$ with the property that for each $e \in E - F$, there exists $f \in F$ adjacent to e such that $(F - \{f\}) \cup \{e\}$ is an edge dominating set. The secure edge dominating number $\gamma'_s(G)$ of G is the minimum cardinality of a secure edge dominating set of G.

Note that $\gamma'_s(G)$ is defined only if G has no isolated vertices. A γ'_s-set is a minimum secure edge dominating set.

Proposition 2. Let G be a graph without isolated vertices. Then

$$\gamma'(G) \le \gamma'_s(G) \qquad (1)$$

and this bound is sharp.

Proof: Every secure edge dominating set of G is an edge dominating set. Thus (1) holds.

The graphs $K_{1,p}$, $p \ge 2$, achieve this bound.

Proposition 3. Let G be a connected graph with $p \ge 2$ vertices. Then

$$1 \le \gamma'_s(G).$$

This bound is sharp. For example, $\gamma'_s(K_{1,2}) = 1$.

We determine $\gamma'_s(G)$ for some standard graphs.

Proposition 4. For a path P_{p+1} with $p \ge 1$ vertices,

$$\gamma'_s(P_{p+1}) = \left\lceil \frac{3p}{7} \right\rceil$$

Proposition 5. For a cycle C_p with $p \ge 3$ vertices,

$$\gamma'_s(C_p) = \left\lceil \frac{3p}{7} \right\rceil.$$

Proposition 6. For a complete bipartite graph $K_{m,n}$, $2 \le m \le$ n,

$$\gamma'_s(K_{m,n}) = m.$$

Proposition 7. For a star $K_{1,p}$, $p \ge 2$,

$$\gamma'_s(K_{1,p}) = 1.$$

The double star $S_{m,n}$ is the graph obtained from joining centers of two stars $K_{1,m}$ and $K_{1,n}$ with an edge.

Proposition 8. For a double star $S_{m,n}$, $1 \le m \le n$,

$$\gamma'_s(S_{m,n}) = 2.$$

Suppose D_1 and D_2 are γ'_s-sets of G_1 and G_2 respectively. Then $D = D_1 \cup D_2$ is a γ'_s-set of $G_1 \cup G_2$. In view of this fact, we have the following proposition.

Proposition 9. For any two graphs G_1 and G_2,

$$\gamma'_s(G_1 \cup G_2) = \gamma'_s(G_1) + \gamma'_s(G_2).$$

Theorem 10. Let G be a connected graph with $p \geq 3$ vertices. Then $\gamma'_s(G) = 1$ if and only if $G = K_{1, p-1}$ or K_3.

Proof: Suppose $\gamma'_s(G) = 1$. If G is a connected graph with 3 vertices, then G is $K_{1, 2}$ or K_3. Clearly $\gamma'_s(K_{1, 2})=1$ and $\gamma'_s(K_3)=1$.

Suppose G is a connected graph with $p \geq 4$ vertices. Let $D=\{e\}$ be a secure edge dominating set of G. We now prove that $G = K_{1, p-1}$. On the contrary, assume $G \neq K_{1, p-1}$. We consider the following two cases.

Case 1. Let $H = K_{1, p-1}$ and let endvertices $a, b \in V(H)$. Consider the graph obtained G from H by adding the edge $f=ab \notin E(H)$, see Figure 1. It follows that the set $(D - \{e\}) \cup \{f\} = \{f\}$ is not an edge dominating set of G. This implies that D is not a secure edge dominating set, which is a contradiction. Thus $G = K_{1, p-1}$.

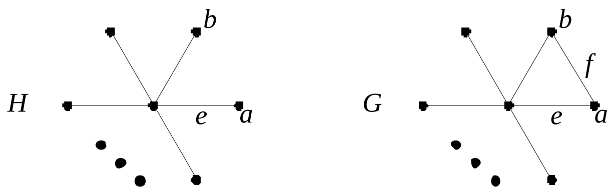

Figure 1:

Case 2: Let $H = K_{1, p-1}$ and an endvertex $a \in V(H)$. Consider the graph obtained G from H by adding the vertex $v \notin V(H)$ and the edge $f = av$, see Figure 2. It follows that the set $(D - \{e\}) \cup \{f\} = \{f\}$ is not an edge dominating set of G. This implies that D is not a secure edge dominating set, which is a contradiction. Thus
$G=K_{1, p-1}$.

Converse is obvious.

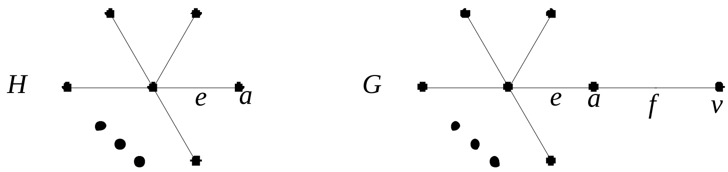

Figure 2:

The following result is immediate.

Proposition 11. If G is a connected graph and G is not a star or K_3, then

$$2 \leq \gamma'_s(G)$$

and this bound is sharp.

Proof: This result follows from Theorem 10.

The double star $S_{m, n}$ achieves this bound.

Proposition 12. For any graph G with maximum edge degree Δ',

$$\left\lceil \frac{q}{\Delta'+1} \right\rceil \leq \gamma'_s(G). \tag{2}$$

Further, equality holds if $G = P_3$.

Proof: Due to Jayaram [15], $\gamma'(G) \geq \left\lceil \frac{q}{\Delta'+1} \right\rceil$.

Also by Proposition 2, we have $\gamma'(G) \leq \gamma'_s(G)$. Thus (2) holds.

The following result gives that the value of the parameter $\gamma'_s(G)$ ranges over all positive integers.

Theorem 13. Given positive integers k and p such that $p \geq 3$ and $1 \leq k < p$, there exists a connected graph G with p vertices and $\gamma'_s(G) = k$.

Proof: We consider the following cases.

Case 1. Suppose $k = 1$. Let $G = K_3$. Clearly $|V(G)| = 3$ and $\gamma'_s(G) = 1$.

Case 2. Let $C_{2m} = \{v_1, e_1, v_2, e_2, v_3, \ldots v_{2m}, e_{2m}, v_1\}$ be an even cycle with $2m$ vertices. For each odd integer i, join the vertices v_i and v_{i+1} to a new vertex u_i to form the graph G, see Figure 3.

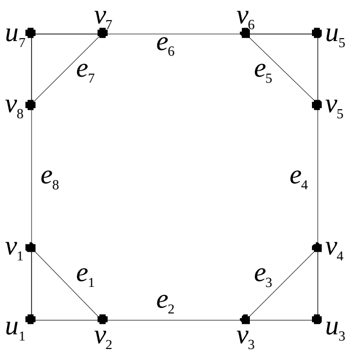

Figure 3: A graph with $\gamma'_s = 4$

It is easy to see that the set $F = \{e_1, e_3, ..., e_{2m-1}\}$ is a minimum secure edge dominating set of G. Then $|F|=m$. Hence $|V(G)| = 3m$ and $\gamma'_s(G) = m = k$.

Problem 14. Characterize graphs G for which $\gamma'(G) = \gamma'_s(G)$.

Problem 15. Characterize graphs G for which $\gamma'_s(G) = 2$.

Problem 16. Characterize graphs G for which $\gamma'_s(G) = \left\lceil \frac{q}{\Delta'+1} \right\rceil$.

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