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Dot-Line Signed Graphs

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Abstract. A *signed hypergraph* is an ordered triple $S=(X,\mathrm{E},\sigma)$ where $H=(X,\mathrm{E})$ is a hypergraph, called the *underlying hypergraph* of S, and $\sigma:\mathrm{E}\to\{-1,+1\}$ is a function called the *signature* of S. Every signed hypergraph $S=(X,\mathrm{E},\sigma)$ can be associated with a signing of its vertices by the function μ_σ, called the *canonical marking* of S, defined by the rule

$$\mu_\sigma(x)=\prod_{E_j\in\mathrm{E}_x}\sigma(E_j),$$

where E_x denotes the set of all edges of S that contain the vertex x. A signed hypergraph $S=(X,\mathrm{E},\sigma)$ together with its canonical marking μ_σ is often denoted S_μ for convenience. Hence, given a canonically marked signed hypergraph S_μ its *signed intersection graph*, denoted $\Omega(S_\mu)$ has E for its vertex set, edges defined by the rule

$$E_iE_j\in E(\Omega(S_\mu))\Leftrightarrow E_i\cap E_j\neq\varnothing$$

and its signature σ_Ω defined by

$$\sigma_\Omega(E_iE_j)=\prod_{x\in E_i\cap E_j}\mu_\sigma(x),\ \forall E_iE_j\in E(\Omega(S_\mu)).$$

The main objective of this paper is to introduce signed graphs S that are representable as signed intersection graphs $L_\bullet(S)$ of the set of edges of S, or the so called *dot-line sigraph* of S.

Keywords: Signed hypergraph, canonical marking, maxclique, edge clique cover, signed intersection graph, dot-line sigraph

AMS Mathematics Subject Classification (2010): 05C22, 05C65

1. Introduction

A *signed hypergraph* is an ordered triple $S=(X,\mathrm{E},\sigma)$ where $H=(X,\mathrm{E})$ is a hypergraph, called the *underlying hypergraph* of S and denoted S^u, and $\sigma:\mathrm{E}\to\{-1,+1\}$ is a function called the *signature* of S; this generalizes the notion of a *signed graph* (or, `sigraph' in short, as in [10], Ch.8), when H is taken to be a simple graph without isolated vertices. We shall denote by S (respectively, Ξ) the set of all signed graphs (hypergraphs) and by $\Psi(G)$ (respectively, $\Psi(H)$) the set of all signed graphs (signed hypergraphs) whose underlying graph (underlying hypergraph) is G (respectively, H). Note that our definition of signed hypergraphs is different from the one given by C.-J. Shi [21, 22, 23]. For the study on hypergraph the reader is referred to [9].

Further, as in [3], every signed hypergraph $S = (X, \mathsf{E}, \sigma)$ can be associated with a signing of its vertices by the function μ_σ, called the *canonical marking* of S, defined by the rule

$$\mu_\sigma(x) = \prod_{E_j \in \mathsf{E}_x} \sigma(E_j),$$

where E_x denotes the set of all edges of S that contain the vertex x. A signed hypergraph $S = (X, \mathsf{E}, \sigma)$ together with its canonical marking μ_σ is often denoted S_μ for convenience. Hence, given a canonically marked signed hypergraph S_μ its *signed intersection graph*, denoted $\Omega(S_\mu)$ has E for its vertex set, edges defined by the rule

$$E_i E_j \in E(\Omega(S_\mu)) \Leftrightarrow E_i \cap E_j \neq \varnothing$$

and its signature σ_Ω defined by

$$\sigma_\Omega(E_i E_j) = \prod_{x \in E_i \cap E_j} \mu_\sigma(x), \forall E_i E_j \in E(\Omega(S_\mu)).$$

As in [3], a *clique* in a signed graph S is a set of vertices any two of which are adjacent in S and an edge-clique cover (ECC) of an isolate-free signed graph S is a family Q of cliques $Q_1, Q_2, \ldots, Q_k$ in S such that

$$\bigcup_{i=1}^{k} E(\langle Q_i \rangle) = E(S),$$

where $\langle Q \rangle$ denotes the subgraph of S induced by the set $Q \subseteq V(S)$; thus, an ECC of S is a hypergraph with V as its vertex set and in which each edge is a clique of S. Also, recall that S is a *maxclique sigraph* if there exists a signed graph $H = (U, F, \xi)$ and an ECC Q of H such that (i) $\mathsf{Q} =: \mathsf{K}_S$, the set of maximal cliques (or *maxcliques*) of S and (ii) $S \cong \Omega_{\mu_\xi}(\mathsf{K}_S) =: \mathsf{K}(S)$, where μ_ξ is the canonical marking of $H = (V(S), \mathsf{K}_S)$. If, in particular, every clique in Q has exactly two vertices then $\mathsf{Q} = E(S)$, the set of edges of S and the signed intersection graph of this particular family of cliques in S is denoted $L_\bullet(S)$ and is called the *dot-line sigraph* of S. The following facts, noted in [3], are fundamental.

Observation 1.1. *[3] In any canonically marked signed graph S_μ there are an even number of negative vertices.*

Observation 1.2. *[3] If S is a signed graph on a triangle-free graph then* $\mathrm{K}(S) \cong L_{\bullet}(S)$

Observation 1.3. *[3] In general, for any isolate-free signed graph* $S = (G, \sigma)$, $(L_{\bullet}(S))^{u} \cong L(G)$*, where* $L(G)$ *denotes the standard `line graph' of* G *(cf.: [16], Chapter 8).*

There are two other notions of a `signed line graph' of a given signed graph $S = (G, \sigma)$ in the literature, viz., $L(S)$ and $L_{\times}(S)$, both of which have $L(G)$ as their underlying graph; only the rule to assign signs to the edges of $L(G)$ differ. Any edge ee' in $L(S)$ is negative if and only if both the edges e and e' in S are negative (cf.: [8]) and an edge ee' in $L_{\times}$ has the product $\sigma(e)\sigma(e')$ as its sign (cf.: [4]). In $L_{\bullet}(S)$, any edge ee' has the sign given by the product of the signs of the edges incident to the vertex in $e \cap e'$; in fact, this definition was suggested by the second author in November 2009, which was indeed the motivation to extend this definition to obtain the definition of the signed intersection graph $\Omega_{\mu_{\sigma}}(\mathrm{E})$ of an arbitrary ECC E of S as in [3].

The main aim of this paper is to initiate a study of dot-line sigraphs, viz., signed graphs S for which the set $\mathrm{L}_{\bullet}^{\surd}(S) := \{H \in \mathrm{S} : L_{\bullet}(H) \cong S\}$ of `$L_{\bullet}$ *-roots*' of $S$ is nonempty. In other words, $\mathrm{L}_{\bullet}^{\surd}(S)$ is the set of `solutions' H of the *signed graph equation*

$$L_{\bullet}(H) \cong S. \tag{1}$$

If $\mathrm{L}_{\bullet}^{\surd}(S) = \varnothing$ then we shall say that S is $L_{\bullet}$ *-primitive*.

2. Dot-line sigraphs

The following fact is straightforward to infer from the above definitions.

Proposition 1. [3] For any signed graph S, $S \in \mathrm{L}_{\bullet}^{\surd}(S)$ if and only if $L_{\bullet}(S) \cong S$.

Hence, determining the set $\mathrm{L}_{\bullet}^{\surd}(S)$ of $L_{\bullet}$ -roots of S is precisely the characterization problem whose solution will be attempted here.

Lemma 2. [3] If S is a solution of (1) then S is a cycle.

Proof: This follows from the fact that if S satisfies (1) then $L(S^{u}) \cong S^{u}$ and hence, by a well known result in graph theory (e.g., see [15], Theorem 8.2), S^{u} is a cycle. Hence the result follows.

The converse of Lemma 2 does not hold. A signed graph S is said to be *balanced* if every cycle in S contains an even number of negative edges (cf.: [10, 14]), and *unbalanced* otherwise. For the study on signed graphs see [6, 7, 8, 11, 13, 16, 17, 18, 19, 20, 21, 25, 26, 27, 28]

Example 1. [3] Take $G = (V, E)$ to be the 3-cycle with $V = \{a, b, c, a\}$ and S be the unbalanced triangle on G with just one negative edge, say bc. Then, in the canonical marking μ_σ of S, a is a positive vertex, b and c are negative vertices. So, $L_\bullet(S)$ is the balanced triangle in which there are two negative edges $\{ab, bc\}, \{ac, bc\}$ and one positive edge $\{ab, ac\}$. Thus, $S \not\cong L_\bullet(S)$ (i.e., `S is not isomorphic to' $L_\bullet(S)$).

Thus, we have the following specific problem.

Problem 2.1. *Precisely which signed cycles are solutions of (1)?*

A signed graph S is said to be *all-positive* (respectively, *all-negative*) if all its edges are positive (negative); further, S is said to be *homogeneous* if it is either all-positive or all-negative, and *heterogeneous* otherwise. Clearly, all-positive cycles of any lengths are solutions of (1), as shown by V.V. Menon (cf.: [15], Theorem 8.2), and no all-negative cycle is a solution of (1). So, we seek heterogeneous cycles which are solutions of (1) to settle Problem 2.1. There do exist heterogeneous signed cycles which are solutions of (1) as the following example shows.

Example 2. [3] Take $G = (V, E)$ to be the 3-cycle with $V = \{a, b, c, a\}$ and S be the balanced triangle on G in which there are two negative edges $\{ab, bc\}$ and one positive edge $\{ac\}$. Then, it is easy to verify that S is a solution of (1).

The following theorem of `Krausz-type' characterization of dot-line sigraphs will be found useful to solve Problem 2.1.

Theorem 3. A signed graph S is a dot-line sigraph if and only if it has an ECC Q such that

- every vertex of S is common to at most two cliques in Q,
- every clique in Q is homogeneous, and
- there are an even number of all-negative cliques in Q.

Proof: Suppose first that S is a dot-line sigraph. Then, there exists a signed graph $H = (U, F, \xi)$ such that $S \cong L_\bullet(H)$. Without loss of generality, we assume that H has no isolates. Then the edges in the star H_x formed by the edges incident at each vertex x of H induce a complete subgraph in S, and every edge of S lies in exactly one such subgraph. Since each edge of H belongs to the stars of exactly two vertices of H, no vertex of S lies in more than two of these complete subgraphs, establishing condition (i).

Next, the rule for assigning signs to the edges of the dot-line sigraph implies that all the edges of the clique in S formed by the vertices corresponding to the edges in H_x must have the sign $\lambda_\xi(x) = \prod_{e \in H_x} \xi(e)$, condition (ii) holds.

The condition (iii) follows from Observation 1.1 applied to H and condition (ii).

Conversely, suppose that we are given an ECC

$$\mathsf{Q} = \{Q_1, Q_2, \ldots, Q_n\}$$

of S satisfying conditions (i)-(iii) in the statement of the theorem. We then indicate the construction of a signed graph H such that $S \cong L_\bullet(H)$. The vertices of H correspond to the ECC $\mathcal{Q}$ together with the set V of vertices of S belonging to only one of the subgraphs $\langle Q_i \rangle$. Thus, $\mathcal{Q} \cup V$ is the set of vertices of H and two of these vertices are adjacent whenever they have a nonempty intersection; that is, H is the standard intersection graph $\Omega(\mathcal{Q} \cup V)$. Now, construct a marked graph on H by first assigning the common sign of the edges in each $\langle Q_i \rangle$ to the vertex Q_i in H and then by assigning to each vertex $v \in V$ the common sign of the edges of the homogeneous subgraph containing v. This construction is made possible by conditions (i) and (ii). Condition (iii) ensures that the number of negative vertices in the resulting marked graph H_λ, λ being the marking so defined, is even. Then, by a theorem of Sampathkumar [20], there exists a signing ξ of the edges of H such that λ is the canonical marking with respect to ξ. It is now amply evident that $S \cong L_\bullet(H)$.Hence the result.

Remark 2.2. It may be seen that given one ECC satisfying conditions (i)-(iii) of Theorem 3, there is a whole class of signed graphs on the graph H constructed in the proof of Theorem 3 and this class has a number of interesting properties as recently propounded by Zaslavsky [29].

3. $L_\bullet$ -periodicity of a signed graph

Next, there do exist heterogeneous signed cycles S which are $\mathcal{K}$ -periodic (or, equivalently in this particular case of cycles, \`$L_\bullet$ -periodic') with $L_\bullet$ -period exceeding 1 (cf.: [1, 2]).

Example 9. In a heterogeneously signed cycle Z, a *negative section* is defined as a maximal all-negative path (see [12]). Hence, take $G = (V, E)$ to be the 6-cycle with $V = \{a, b, c, d, e, f\}$ and S be the signed 6-cycle with one negative section of length 4. It is easy to verify that $L_\bullet(S) \ncong S$ and $L_\bullet^2(S) \cong S$.

Example 10. Take $G = (V, E)$ to be the 5-cycle with $V = \{a, b, c, d, e\}$ and S be the signed 5-cycle with just two adjacent negative edges, say ab and ae. Then, its canonical marking makes the vertices a, c, d positive and the vertices b and e negative. It is easy to verify that $L_\bullet^t(S) \ncong S$ for $t = 1,2$ and $L_\bullet^3(S) \cong S$.

Example 11. Take $G = (V, E)$ to be the 7-cycle with $V = \{a, b, c, d, e, f, g\}$ and S be the signed 7-cycle on G with one negative section of length 2 and two negative sections of length 1 each. One can then to verify that no two signed cycles in the set $\{L_\bullet^t(S) : t \in \{0,1,2,\ldots,6\}\}$ are isomorphic to each other and $L_\bullet^7(S) \cong S$, whence the $L_\bullet$ -period of S is 7.

The above three examples prompt one to raise the following problem.

Problem 3.1. *Characterize signed cycles which are $L_\bullet$ -periodic with $L_\bullet$ -period for*

$t \geq 1$.

Remark 3.2. *The $L_{\bullet}$-period t of a signed cycle Z of length n can be anywhere in the range from 1 to n. Hence, determination of exact values of t, given n, is a part of Problem 3.1.*

While the chracterization problem has been solved for line signed graph (i.e., signed graph S for which their exists a signed graph H such that $L(H) \cong S$ as in [5]), the same remains to be solved for the $\times$-line sigraph (i.e., signed graph S for which there exists a signed graph H such that $L_{\times}(H) \cong S$). Further dot-line sigraph, viz., a signed graph S for which there exists a signed graph H such that $L_{\bullet}(H) \cong S$ has been reported in [24] under a different set up.

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