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Variation of Graceful Labeling on Disjoint Union of two Subdivided Shell Graphs

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Abstract. A shell graph is the join of a path P_k of 'k' vertices and K_1. A *subdivided shell graph* can be constructed by subdividing the edges in the path of the shell graph. In this paper we prove that the disjoint union of two subdivided shell graphs is odd graceful and also one modulo three graceful.

Keywords: Shell graph, subdivided shell graph, odd graceful labeling, one modulo three graceful labeling

AMS Mathematics Subject Classification (2010): 05C78

1. Introduction

A graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions. In 1967 Rosa[10] introduced the labeling method called *β-valuation* as a tool for decomposing the complete graph into isomorphic sub-graphs. Later on, this β-valuation was renamed as graceful labeling by Golomb [9]. A *graceful labeling* of a graph G with 'q' edges and vertex set V is an injection f :V(G) → {0,1,2,….*q*} with the property that the resulting edge labels are also distinct, where an edge incident with vertices *u* and *v* is assigned the label |f(*u*) – f(*v*)|. A graph which admits a graceful labeling is called a *graceful graph*. A variation of graceful labeling is *odd-graceful labeling.* This was introduced by Gnanajothi [8] in the year 1991. She defined a graph G with q edges to be odd-graceful if there is an injection f :V(G) → {0, 1, 2, . . . , 2q−1} such that, when each edge xy is assigned the label |f(x)−f(y)|, the resulting edge labels are {1, 3, 5, . . . , 2q−1}.

She proved many graphs as odd-graceful: paths P_n, C_n if and only if n is even, $K_{m,n}$, combs $P_n \Theta K_1$, books, crowns $C_n \Theta K_1$ if and only if n is even, the one-point union of copies of C_4, $C_n \times K_2$ if and only if n is even, caterpillars, rooted trees of height 2. Eldergill [5] generalized Gnanajothi's result on stars. Barrientos [2] has proved the following graphs are odd-graceful: every forest whose components are caterpillars, every tree with diameter at most five and all disjoint unions of caterpillars. Seoud, Diab, and Elsakhawi [12] have shown that a connected complete r-partite graph is odd-graceful if and only if r = 2.

Gao [7] has proved the following graphs are odd-graceful: the union of any number of Paths, the union of any number of stars, the union of any number of stars and paths, $C_m \cup P_n$, $C_m \cup C_n$, and the union of any number of cycles each of which has order divisible by 4. Acharya, Germina, Princy, and Rao [1] prove that every bipartite graph G can be embedded in an odd-graceful graph H. In [3] Chawathe and Krishna extend the definition of odd-gracefulness to countable infinite graphs and show that all countable infinite bipartite graphs that are connected and locally finite have odd-graceful labelings. Another variation of graceful labeling is one modulo three graceful labeling.

Sekar [11] defines the one modulo three graceful labeling as an injective function g: V(G) → { 0,1, 3, 4, 7, . . . , (3q-3), (3q-2)} if the edge labels induced by labeling each edge uv with $|g(u) – g(v)|$ is {1, 4, 7, . . . , (3q-2)} . He proves that the following graphs are one modulo three graceful. The paths, cycles C_n when $n \equiv 0 \pmod 4$, the complete bipartite graphs, caterpillars, stars, lobsters, banana trees, rooted trees of height 2, ladders are one modulo three graceful. He conjectured that every one modulo three graceful graph is graceful. For an exhaustive survey, refer to the dynamic survey by Gallian [6].

Deb and Limaye [4] have defined a *shell graph* as a cycle C_n with *(n -3)* chords sharing a common end point called the *apex*. It is the join of K_1 and a path. Shell graphs are denoted as C(n, n- 3). A *subdivided shell graph* is a shell graph in which the edges in the path of the shell are sub divided. In this paper we prove that the disjoint union of two sub divided shell graphs is odd graceful and one modulo three graceful.

2. Main result

In this section we prove two theorems on the disjoint union of two subdivided shell graphs**.**

Theorem 1. The disjoint union of two subdivided shell graphs is odd graceful.

Proof: Let G_1 and G_2 be two subdivided shell graphs of any order. Let G be the disjoint union of G_1 and G_2. The apex of G_1 is denoted as u_0 and the remaining vertices in G_1 from bottom to top are denoted as u_1 , u_2, ... , u_m . The apex of G_2 is denoted as v_0 and the other vertices from bottom to top are denoted as $v_1, v_2, . . . , v_l$. Let e_1, e_2, . . . , $e_{(m+1)/2}$ be the edges $u_0 u_1$, $u_0 u_3$, . . . , $u_0 u_m$. Let $e_{(m+3)/2}$, $e_{(m+5)/2}$, . . . , $e_{(3m-1)/2}$ be the edges $u_1 u_2$, $u_2 u_3$, . . . , $u_{m-1} u_m$ respectively. Let $e_{(3m+1)/2}$, $e_{(3m+3)/2}$, . . . , $e_{(3m+l)/2}$ be the edges $v_0 v_1$, $v_0 v_3$, . . . , $v_0 v_l$ respectively. Let $e_{(l+3m+2)/2}$, $e_{(l+3m+4)/2}$, . . . , $e_{(3l+3m-2)/2}$ be the edges $v_1 v_2$, $v_2 v_3$, . . . , $v_{l-1} v_l$ respectively. G has n = m+ l + 2 vertices and q = (3m + 3l – 2) /2 edges. See Figure 1.

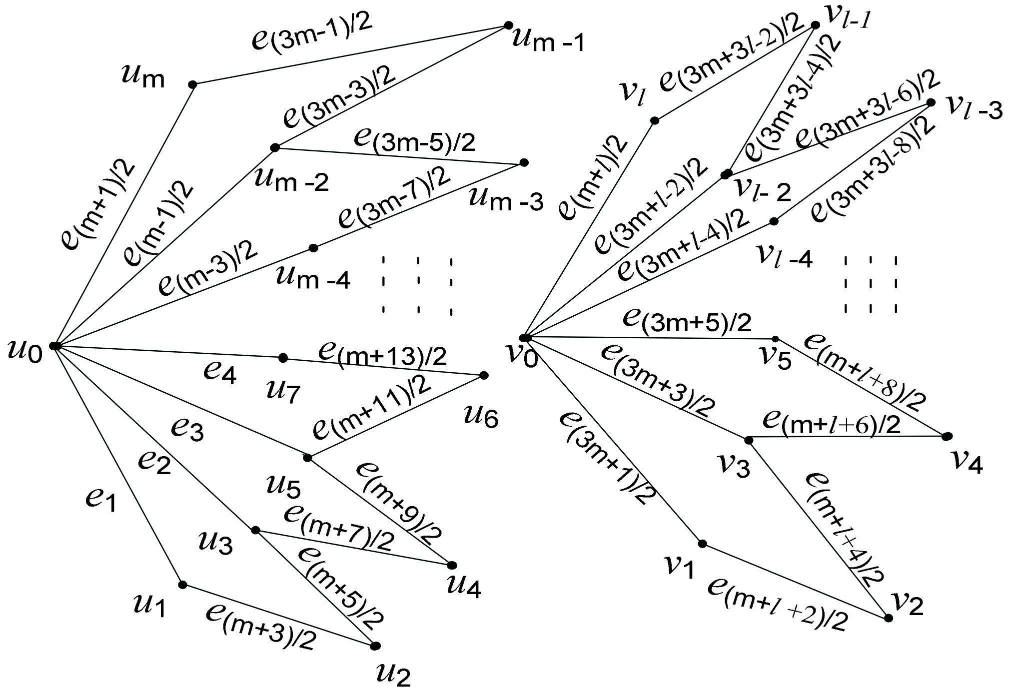

Figure 1: Disjoint union of two subdivided shell graphs

We define the vertex labels of G follows:-

$f(u_0) = 0.$ (1)

$f(u_{2i-1}) = 3l + 3m - 2i - 1,$ for $1 \leq i \leq \frac{m+1}{2}$ (2)

$f(u_{2i}) = 2l + m + 2i - 3,$ for $1 \leq i \leq \frac{m-1}{2}$ (3)

$f(v_0) = l + 1,$ (4)

$f(v_{2i-1}) = l - 2i + 2,$ for $1 \leq i \leq \frac{l+1}{2}$ (5)

$f(v_{2i}) = 2l + 2m + 2i - 2,$ for $1 \leq i \leq \frac{l-1}{2}$ (6)

From the above definitions (1) to (6) we can see that the vertex labels are distinct. No two of the vertex labels are equal. Suppose if $f(u_{2i-1}) = f(u_{2i})$ for any value of ‘i’, then we would get $l \leq -2$ which is absurd. The edge labelings are computed as follows:

$|f(u_0) - f(u_{2i-1})| = |3l + 3m - 2i - 1|$, for $1 \le i \le \frac{m+1}{2}$ (7)

$|f(u_{2i-1}) - f(u_{2i})| = |l + 2m - 4i + 2|$, for $1 \le i \le \frac{m-1}{2}$ (8)

$|f(u_{2i}) - f(u_{2i+1})| = |l + 2m - 4i|$, for $1 \le i \le \frac{m-1}{2}$ (9)

$|f(v_0) - f(v_{2i-1})| = |2i - 1|$, for $1 \le i \le \frac{m-1}{2}$ (10)

$|f(v_{2i-1}) - f(v_{2i})| = |l + 2m + 4i - 4|$, for $1 \le i \le \frac{l-1}{2}$ (11)

$|f(v_{2i}) - f(v_{2i+1})| = |l + 2m + 4i - 2|$, for $1 \le i \le \frac{l-1}{2}$ (12)

From the above computations (7) to (12) we can see that the edge labels are distinct.

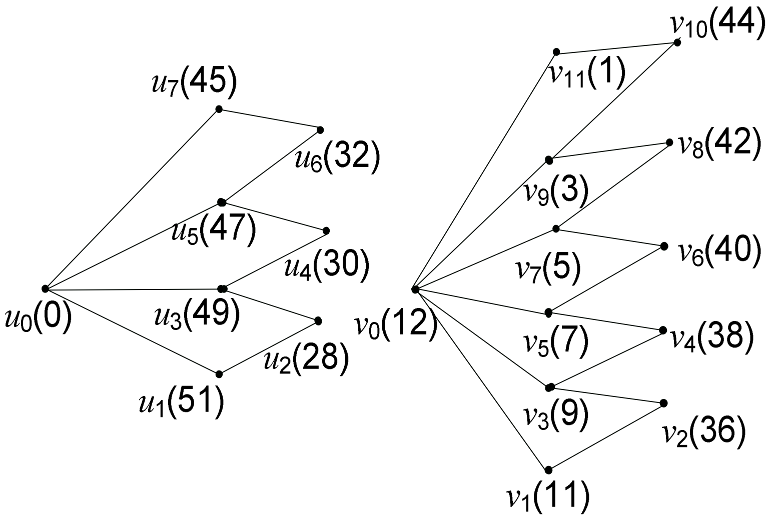

Figure 2: Odd-graceful Disjoint union of two subdivided shell graphs when m =7, l = 11, n= 20, q = 26.

Let $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$ and $\mathcal{E}_4$ be the sets of the edges $\{e_1, e_2, \ldots, e_{(m+1)/2}\}$, $\{e_{(m+3)/2}, e_{(m+5)/2}, \ldots, e_{(3m-1)/2}\}$, $\{e_{(3m+1)/2}, e_{(3m+3)/2}, \ldots, e_{(l+3m)/2}\}$ and $\{e_{(l+3m+3)/2}, e_{(l+3m+5)/2}, \ldots, e_{(3l+3m-2)/2}\}$ respectively. If E_1, E_2, E_3 and E_4 denote the set of labels of the edges given in $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$ and $\mathcal{E}_4$ respectively then we have ,

$E_1 = \{(2q-1), (2q-3), \ldots, (2q-m)\}$, (13)

$E_2 = \{l + 2m - 2, l + 2m - 4, \ldots, l + 4, l + 2\}$, (14)

$E_3 = \{1, 3, 5, \ldots, l\}$, (15)

$E_4 = \{l + 2m, l + 2m + 2, \ldots, (2q - m - 2)\}$ (16)

$E_3 \cup E_2 \cup E_4 \cup E_1 = \{ 1, 3, 5, \ldots, (2q-1)\}.$ (17)

Equation (17) shows that all the edge labels are odd. Hence the disjoint union of two subdivided shell graphs is odd graceful. An illustration is given in the Figure 2.□

Theorem 2. The disjoint union of two subdivided shell graphs is one modulo three graceful.

Proof: Let G be the disjoint union of two subdivided shell graphs G_1 and G_2. As in Theorem 1 we denote the apex of G_1 as u_0 and the remaining vertices in G_1 from bottom to top as $u_1, u_2, \ldots, u_m$. The apex of G_2 is denoted as v_0 and the other vertices from bottom to top are denoted $v_1, v_2, \ldots, v_l$. We prove the theorem when $m \le l$. G has $n = m + l + 2$ vertices and $q = \frac{3m + 3l - 2}{2}$ edges.

We define the vertex labels of G follows:-

$$f(v_0) = 3q - 8, \quad (17)$$

$$f(u_{2i-1}) = \frac{3m - 6i + 3}{2}, \quad \text{for } 1 \le i \le \frac{m+1}{2} \quad (18)$$

$$f(u_{2i}) = \frac{9l + 3m + 6i - 10}{2}, \quad \text{for } 1 \le i \le \frac{m-1}{2} \quad (19)$$

$$f(v_0) = 3q - 8, \quad (20)$$

$$f(v_{2i-1}) = \frac{9m + 3l - 6i - 12}{2}, \quad \text{for } 1 \le i \le \frac{l+1}{2} \quad (21)$$

$$f(v_{2i}) = \frac{9m + 3l + 6i - 22}{2}, \quad \text{for } 1 \le i \le \frac{l-1}{2} \quad (22)$$

From the above definition we can see that all the vertices have been given labels and they are distinct. We compute the edge labels now.

$$| f(u_0) - f(u_{2i-1})| = \left| \frac{9l + 6m + 6i - 13}{2} \right|, \quad \text{for } 1 \le i \le \frac{m+1}{2} \quad (23)$$

$$| f(u_{2i-1}) - f(u_{2i})| = \left| \frac{9l + 12i - 13}{2} \right|, \quad \text{for } 1 \le i \le \frac{m-1}{2} \quad (24)$$

$$| f(u_{2i}) - f(u_{2i+1})| = \left| \frac{9l + 12i - 7}{2} \right|, \quad \text{for } 1 \le i \le \frac{m-1}{2} \quad (25)$$

$$| f(v_0) - f(v_{2i-1})| = \left| \frac{6l + 6i - 10}{2} \right|, \quad \text{for } 1 \le i \le \frac{l+1}{2} \quad (26)$$

$$| f(v_{2i-1}) - f(v_{2i})| = |6i-5|, \quad \text{for } 1 \le i \le \frac{l-1}{2} \quad (27)$$

$$| f(v_{2i}) - f(v_{2i+1})| = |6i-2|, \quad \text{for } 1 \le i \le \frac{l-1}{2} \quad (28)$$

These computations show that the edge labels are distinct.

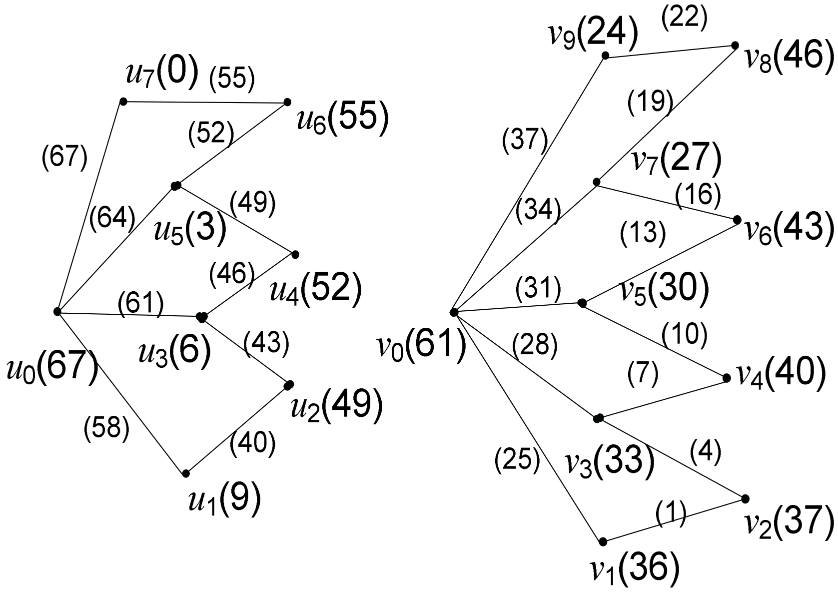

Figure 3: One modulo three-graceful disjoint union of two subdivided shell graphs when m =7, l = 9, n=18, q = 23.

Let E_1, E_2, . . . , E_6 denote the sets of edge labels given in equations (23) to (28) respectively. Then

E_1 = { (6m+9l -7)/2, (6m+9l -1)/2, . . . ,(3q-2)}, (29)

E_2 = { (9l -1)/2, (9l+11)/2, . . . , (6m+9l -19)/2}, (30)

E_3 = { (9l +5)/2, (9l+17)/2, . . . , (6m+9l -13)/2}, (31)

E_4 = { (6l -4)/2, (6l+2)/2, . . . , (9l -7)/2}, (32)

E_5 = {1, 7, 13, . . ., (3l-8)}, (33)

E_6 = {4, 10, 16, . . ., (3l-5)}, (34)

$(E_5 \cup E_6) \cup E_4 \cup (E_3 \cup E_2) \cup E_1$ = { 1, 4, 7, . . . , (3q- 5) , (3q- 2)}. If any of the edge label is 3n or 3n-1, n ≥ 1, then we would get a contradiction to the fact that *'l' is an integer.* Thus it satisfies the hypothesis of one modulo three graceful labeling. Hence the disjoint union of two subdivided shell graphs is one modulo three graceful. An illustration is given in the Figure 3. □

3. Conclusion

In this paper, we have applied two labelings to the disjoint union of two subdivided shell graphs to show that they are odd graceful and one modulo three graceful. One can also prove that the above graph satisfies other variation of graceful labeling.

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