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A Study on Ramsey Numbers and its Bounds

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Abstract. In general, Ramsey theory deals with the guaranteed occurrence of specific structures in some part of a large arbitrary structure which has been partitioned into finitely many parts. The integers R(p, q) are known as classical Ramsey numbers. The Ramsey number R(p ,q) is the minimum number n such that any graph on n vertices contains either an independent set of size s or a clique of size t. In this paper we were discuss about the examples of Ramsey numbers and their bounds.

Keywords: Ramsey theory, Ramsey numbers, Bounds of Ramsey numbers, 2-coloring

AMS Mathematics Subject Classification (2010): 05C55

1. Introduction

Ramsey theory got its start and its name when Frank Ramsey [7] published his paper “On a Problem of Formal Logic" in 1930. Ramsey Theory studies the conditions of when a combinatorial object necessarily contains some smaller given objects. The role of Ramsey numbers is to quantify some of the general existential theorems in Ramsey Theory.

Ramsey numbers one studies partitions of the edges of the complete graph, under the condition that each of the parts avoids some pre specified arbitrary graph, in contrast to classical Ramsey numbers when the avoided graphs are complete. The theorem was proved in passing, as a means to a result about logic, but it turned out to be one of the first combinatorial results that widely attracted the attention of mathematicians.

Ramsey theory to be applied in many fields like, constructive methods, computer algorithms, random graphs and the probabilistic method

More generally, we consider the following setting. We color the edges of K_n (a complete graph on *n* vertices) with a certain number of colors and we ask whether there is a complete sub graph (a *clique*) of a certain size such that all its edges have the same color.

We shall see that this is always true for a sufficiently large *n*. Note that the question about friendships corresponds to a coloring of K_6 with 2 colors, “friendly" and “unfriendly". Equivalently, we start with an arbitrary graph and we want to find either a clique or the complement of a clique, which is called an *independent set*. This leads to the definition of *Ramsey numbers*.

Van der Waerden's Theorem was proved in 1927, a year earlier than Ramsey's. Van der Waerden proved that in any finite coloring of the natural numbers there must exist, some monochromatic arithmetic progression with k terms. Finally in 1974 Hindman's Theorem, the most recent theorem proved. Hindman's Theorem states that, for every finite coloring of the natural numbers there exists some infinite subset S N such that all the finite sums of the elements of S are monochromatic.

2. Definitions

Definition 2.1. A clique is a complete sub graph, an independent set is an empty sub graph.

Definition.2.2. R(s, t) is the minimum number s such that any graph on n vertices contains a clique of order s or an independent set of order t.
Ex: R(3, 3) = 6.

Definition 2.3. A clique of size t is a set of t vertices such that all pairs among them are edges. An independent set of size s is a set of s vertices such that there is no edge between them. Ramsey's theorem states that for any large enough graph, there is an independent set of size *s* or a clique of size *t*. The smallest number of vertices required to achieve this is called a *Ramsey number*.

Definition 2.4. The Ramsey number R(s, t) is the minimum number s such that any graph on n vertices contains either an independent set of size s or a clique of size t. The Ramsey number $R_k(s_1, s_2,\ldots s_k)$ is the minimum number s such that any coloring of the edges of K_n with k colors contains a clique of size s_i in color i, for some i.

Definition 2.5. A sub graph H of G is monochromatic if all its edges receive the same color.

Definition 2.6. The integers R (p, q) are known as classical Ramsey numbers.

Definition 2.7. Given a graph G, a k-coloring of the vertices of G is a partition of V (G) into k sets $C_{1,}C_{2,\ldots}C_k$ such that for all i, no pair of vertices from C_i are adjacent. If such a partition exists, G is said to be k-colorable.

Definition 2.8. Given a graph G, a k-coloring of the edges of G is any assignment of one of k colors to each of the edges of G.

In our discussion of Ramsey theory, we will deal primarily with 2-colorings of the edges of graphs. By convention, the colors referred to are typically red and blue. Figure 1 shows an example of a graph in and several 2-colorings of its edges.

Definition 2.9. A graph is r-coloured if we colour each edge of the graph with one of r colours.

Definition 2.10. The Ramsey Number, $R_r(s)$, is the order of the smallest complete graph which, when r-coloured, must contain monochromatic Ks.

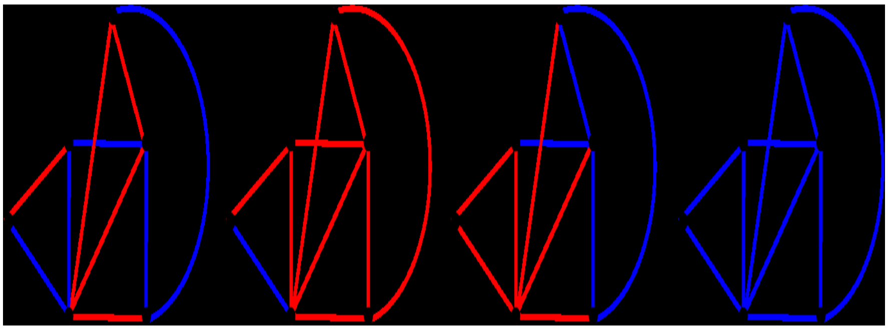

Figure 1: Four possible 2-colorings of the edges of a graph.

Definition 2.11. The (2,n)-Barbell graph is the simple graph obtained by connecting two copies of a complete graph K_n by a bridge and it is denoted by $B(K_n, K_n)$.

Definition 2.12. **[18]** The (3,n)-Barbell graph is the simple graph obtained by connecting three copies of a complete graph K_n by a bridge and it is denoted by $B(K_n, K_n, K_n)$.

Definition 2.13. A harmonious coloring of a graph G(V,E) is a line-distinguishing coloring which is also proper. The harmonious chromatic number of G (denoted by $\chi_h(G)$) is the smallest number k such that there exists a harmonious coloring of G of k colors.

3. Observation

Proposition 3.1. (Putnam 1952). Among any six people, there are three of them any two of whom are friends, or else no two of whom are friends.

Theorem 3.2. For any two natural numbers, s and t, there exists a natural number, R(s, t) = n, such that any 2-colored complete graph of order at least n, colored red and blue, must contain a monochromatic red K_s or blue K_t.

Proof: We prove that R(s, t) exists by proving it is bounded. We shall use proof by induction first assuming that R(s −1, t) and R(s, t − 1) exist. As was shown earlier R(s, 2) = R(2, s) = s and R(s, 1) = R(1, s) = 1 are trivial results.

Claim. $R(s, t) \leq R(s-1, t) + R(s, t-1)$.

We first take a 2-coloring of a complete graph with n = R(s−1, t)+R(s, t− 1) vertices. We now pick one of the vertices in Kn, say x. We then produce two sets, R_x and B_x, R_x is the set of vertices adjacent to x such that every edge connecting a vertex in R_x to x is red. Similarly B_x is the set of vertices adjacent to x such that every edge connecting a vertex in B_x to x is blue.

Since K_n is a complete graph $B_x = [n] \backslash (R_x □ \{x\})$ and so $|R_x|+|B_x| = n-1$. If $|R_x| < R(s-1, t)$ and $|B_x| < R(s-1, t)$ then since $n = R(s-1, t)+R(s, t-1)$ we must have $|R_x| + |B_x| \leq n - 2$, a contradiction. So $|B_x| \geq R(s, t-1)$ or $|R_x| \geq R(s-1, t)$.

If $|B_x| \geq R(s, t-1)$ and B_x induces a red Ks we are done. If B_x induces a blue K_{t-1} then K_n must contain a blue Kt since B_x □ {x} must induce a blue K_t. Indeed, each edge xt is blue for all t □ B_x, from the definition of Bx. So B_x □ {x} must induce a blue K_t if B_x contains a blue K_{t-1}. The case for R_x is completely symmetric, that is, if Rx induces a blue Kt we are done and if R_x induces a red K_{s-1} then K_n must contain a red K_s since R_x □ {x} must induce a red K_s.

We have shown that a 2-coloured complete graph of order $R(s-1, t) + R(s, t-1)$ must contain a red K_s or a blue K_t, proving that $R(s, t) \leq R(s-1, t) + R(s, t-1)$. This completes our induction.

Theorem 3.3. (Schur) For any k there exists n such that for any k-coloring of {1,…,n}, there exist x, y, z of the same color such that x + y = z.

Proof: Consider n = Rk(3,…, 3). Given a coloring c : [n] → [k], define an edge-coloring of K_n. The color of edge{i, j} is □ {I, j}) = c($|j-i|$). Then, there exists a monochromatic triangle with vertices i, j ,k.

Assume i < j < k. Then,c(j-i) = c(k - j) = c(k - i). Then x = j - i, y = k - j, and z = k- i is the desired monochromatic solution. If we consider to the linear equation $c_1x_1 + c_2x_2 + \ldots + c_nx_n = 0$, the theorem holds iff some non-empty subset of the coefficients sum to 0. This is a special case of Rado's theorem.

Theorem 3.4. For all $m \geq 1$ there exists p_0 such that for all primes $p > p_0$ the congruence $x^m + y^m \equiv z^m \pmod{p}$.

Theorem 3.3. (Ramsey, Erd'os-Szekeres[12]). R(s, t) exists and $R_k(s_1,\ldots, s_k)$ in general $R(s, t) \leq R(s-1, t) + R(s, t-1)$.

Theorem 3.5. For every positive integer k, R(2, k) = k for all $k \geq 2$

Proof: Choose some positive integer $k \geq 2$. First we will show that R(2, k) > k-1 by constructing a 2-coloring on K_{k-1} that contains neither a red K_2 nor a blue K_k. The coloring in which every edge is blue satisfies these requirements. It will certainly not contain a red K_2 and cannot possibly contain a blue K_k, so R(2,k) > k -1.

Next, suppose that the edges of K_k are 2-colored in some fashion. If any of the edges are red, then K_k will contain a red K_2. If none of the edges are red, then we are left with a blue K_k. So $R(2, k) \leq k$. Thus, we can conclude that R(2, k) = k for all $k \geq 2$.

The difficulty of determining additional Ramsey numbers grows quickly as p and q increase.

Theorem 3.6. For every positive integer k, R(3, 3) = 6.

Proof: Consider any 2-coloring on K_6. Choose some vertex v from the graph. Because there are 5 edges incident to v, by the pigeon hole principle, at least three of these edges must be the same color. We will call them vx, vy, and vz, and we will suppose they are red. If at least one of xy, xz, or yz is red, then we have a red K_3. If none of these is red, then we have a blue K_3. Thus, $R(3; 3) \leq 6$. Next, consider the 2-coloring on K_5 as depicted in Figure 2. This coloring does not contain a monochromatic K_3 in either red or blue, so we know that R(3, 3) > 5. Thus, R(3, 3) = 6.

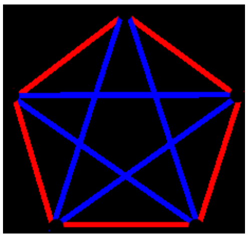

Figure 2: A 2-coloring on K_5 that contains no monochromatic K_3.

Theorem 3.7. (Ramsey 1930**[7]**) R(s, t) is finite for all s, t ≥ 2 and for s, t > 2 we have R(s, t) ≤ R(s − 1, t) + R(s, t − 1).

Proof: Consider an arbitrary vertex v of the graph K_N, where N = R(s − 1, t) + R(s, t − 1). Let c be an arbitrary coloring of K_N. Then, R(s − 1, t) + R(s, t − 1) − 1 edges arrive in v. Either R(s − 1, t) of them are red or R(s, t − 1) are blue. Without loss of generality, assume we have R(s − 1, t) vertices incident to v by means of red edges. These vertices form a $K_{R(s-1,t)}$ graph.

Thus, for each coloring, including coloring c, we either have a blue K_t or a red K_{s-1} in this $K_{R(s-1,t)}$ graph. This completes the proof, as in the latter case a red K_s is formed by adding *v* to the red K_{s-1}.

Theorem 3.8.

For all s, t ≥ 2 we have $R(s, t) \leq \binom{s+t-2}{s-1}$

Theorem 3.9. Let m and n be positive integers. Then r (χ ≥ m, χ ≥ n) = (m - 1)(n -1) + 1, where χ signifies the chromatic number of a graph.

Proof: The assertion is obviously true if one of *m* and *n* is one. Assume that *m, n* ≥ 2. Let *G* be a graph consisting of disjoint *n*-1 copies of $K_{m-1.}$ Then $\chi(G) = m - 1$ and $\chi(\overline{G}) = n - 1$, yielding that *r* (χ ≥ m, χ ≥ n) = (*m* - 1)(*n* -1) + 1.

On the other hand, if *G* is a graph of order *N* = (*m* - 1)(*n* - 1) + 1, then by the fact that $\chi(G)\alpha(G) \geq N$ we have $\alpha(G) \geq \left|\frac{N}{\chi(G)}\right| = \left|\frac{(m-1)(n-1)+1}{m-1}\right| \geq n$

Therefore $\chi(\overline{G}) \geq \omega(\overline{G}) = \alpha(G) \geq n$, proving *r* (□ ≥ m, □ ≥ n) = (*m* - 1)(*n* -1) + 1.

Theorem 3.10. (Erdös,1947, Lower bounds on Ramsey numbers) For all k ≥ 3, R(k) > 2^{k-2}.

S/ t	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7	8	9	10
3	1	3	6	9	14	18	23	28	36	40-43

4	1	4	9	18	25	35-41	49-61	56-84	73-115	92-149
5	1	5	14	25	43-49	58-87	80-143	101-216	125-316	143-442
6	1	6	18	35-41	58-87	102-165	113-298	127-495	169-780	179-1171
7	1	7	23	49-61	80-143	113-298	205-540	216-1031	233-1713	298-2826
8	1	8	28	56-84	101-216	127-495	216-1031	282-1870	317-3583	317-6090
9	1	9	36	73-115	125-316	169-780	233-1713	317-3583	565-6588	580-12677
10	1	10	40-43	92-149	143-442	179-1171	289-2826	317-6090	580-12677	798-23556

Table 3.1: Known Ramsey numbers R(s ,t) and bounds.

Cycles. The initial general result for cycles, $R(C_3,C_n) = 2n - 1$ for $n \geq 4$, was obtained by Chartrand and Schuster in 1971. The complete solution of the case $R(C_n, C_m)$ was obtained soon afterwards, independently by Faudree and Schelp and Rosta **[13]**.

Theorem 3.11. (Triple odd cycles - Kohayakawa, Simonovits, Skokan, 2005, 2009) $R(C_n, C_n, C_n) = 4n - 3$ for all sufficiently large odd n.

Theorem 3.12. (Faudree, Schelp ,1974; Rosta , 1973 [20])

$R(C_n, C_m) = 2n - 1$ for $3 \leq m \leq n$, m *odd* , $(n, m) \neq (3,3)$

$n - 1 + m/2$ for $4 \leq m \leq n$, m and n *even* , $(n, m) \neq (4,4)$

$\max\{n - 1 + m/2, 2m - 1\}$ for $4 \leq m < n$, m *even* and n *odd*

Conjecture 3.12. The cycle-complete graph Ramsey number $r(C_m, K_n)$ is the smallest integer N such that for every graph G of order N contains Cm or $\alpha(G) \geq$ n. The graph $(n-1)K_{m-1}$ shows that $r(C_m, K_n) \geq (m - 1)(n - 1) + 1$.The above result was improved by Nikiforov [19] when he proved the equality for $m \geq 4n + 2$.

Erd'os gave the following conjecture: $r(C_m,K_n) = (m - 1)(n - 1) + 1$, for all $m \geq n \geq 3$ except $r(C_3,K_3) = 6$.

Theorem 3.14. The Ramsey number of cycle C_n versus W_4 is $R(C_n,W_4) = 2n - 1$, for $n \geq 5$.

Theorem 3.15. The Ramsey number cycle C_n versus $W4$ is $R(C_n, W_5) = 3n - 1$, for $n \geq 5$.

Theorem 3.16. (Surahmat, etb 2001) For all $n \geq 3$, $R(P_n,W_4) = 2n - 1$ and $R(P_n, W_5) = 3n - 2$.

Theorem 3.17. If $n \geq \frac{m(m-2)}{2}$ and $m \geq 4$ even then $R(P_n,W_m) = 2n - 1$.

Theorem 3.18. (etb, Surahmat 2001) If $n \geq \frac{m(m-3)}{2}$ and m ≥ 5 odd then $R(P_n,W_m) = 3n - 2$.

Theorem 3.19. For any complete graph K_n, $\chi_h\,[B(K_n,K_n)]=2n-1$, $n \geq 2$.

Theorem 3.20. [18] For any complete graph K_n, $\chi_h\,[B(K_n,K_n,K_n)]=3n-2$, $n\geq 2$.

From the theorems we observed that , the values for the harmonious coloring of barbell graphs for 2 copies and the Ramsey number of paths and wheels for n,m (theorem 3.17 and 3.19) the values are same with different conditions.

Similarly we observed that , the values for the harmonious coloring of barbell graphs for 3 copies and the Ramsey number of paths and wheels for n, m (theorem 3.18 and 3.20) the values are same with different conditions.

4. Applications

In real life Ramsey numbers are used .As noted by Slany, combinatorial games "serve as models that simplify the analysis of competitive situations as models that simplify the analysis of competitive situations with opposing parties that pursue different interests" and finding a winning strategy to a combinatorial game can be translated into finding a strategy to cope with many kinds of real world problems such as found in telecommunications, circuit design, scheduling, as well as a large number of other problems of industrial relevance."

If we consider this problem in terms of people at a party, Ramsey's Theorem guarantees that there is some smallest number of people at the party required to ensure that there is either a set of p mutual acquaintances or q mutual strangers. Thus, the old puzzle that asks us to prove that with any six people at a party, among them there must be a set of three mutual acquaintances or a set of three mutual strangers actually requires us to show that R(3,3) = 6.

We should also note that Ramsey's Theorem can be generalized to account for colorings in any finite number of colors, not just 2-colorings.

Ramsey's Theorem guarantees that this smallest integer R(p, q) exists but does little to help us determine what its value is, given some positive integers, p and q. In general, this is actually an exceedingly difficult problem.

Ramsey graph games start with a complete graph K_n and the players color an edge on their turn, each player uses a unique color. In an avoidance game each player is given a graph which if she colors a sub graph isomorphic to this graph monochromatically in her color, she losses.

In an achievement game she would win by coloring such a sub graph monochromatically. The avoidance version is called Sim after Simmons who introduced it in 1969. Sim has been studied extensively, notably by Frank Harary who is known for his work in graph theory.

Starting with the achievement game, which resembles the popular Tic-Tac-Toe, suppose as in the case of R(3; 3) = 6 that there are 2 players and they are trying to form monochromatic K_3s. If the graph they are coloring is K_6, then there will be a winner because It has already proved that R(3; 3) = 6 which means that any way the two players color their graph, someone will eventually color a K_3.

However, if they are playing on a K_5 then the unique R(3; 3) critical graph shows that they may tie, meaning neither achieves their goal. If they play on K_3 they must tie as there is only a single k_3 to color and each will color at least one of the edges.

Theorem 4.1. In the achievement Ramsey games, if all players are trying to achieve the same graph, and there is a winning strategy, then it belongs to the first player.

Proof: This is a standard strategy stealing argument. If someone other than the first player, say the k^{th}-player, had a winning strategy then the first player imagines she is this player. On each turn she imagines k-1 additional edges have been colored one each in the colors of the k-1 players preceding her, and then selects an edge to color dictated by the k^{th}-player's wining strategy. But in following this strategy, the first player does not lose because each of her moves were part of a winning strategy which means that she could not have lost before her next turn.

If there are not k edges left on player one's turn, this means player one did not follow player K's strategy as player K wins at latest on the turn before. If the first player does not lose, then the k^{th} -player does not win. So only player one may have a winning strategy, continuing the example above.

If the players are attempting to form monochromatic K_{3s} on K_4, then as K_4 has only 6 edges, if the first player wins, she must do so on her third turn. However, after two turns she threatens to complete at most one K_3 on her next turn, and if player 2 after seeing player one's second move, is sure to color the missing edge in the K_3 that player one threatens to complete, player 2 achieves at least a draw. So again, with best play on a K4 the result is a tie because the above theorem indicates player one should not lose.

If they play on a K_5 player one wins. Player one need only stop player 2 and avoid forming a 5-cycle as this is the unique R(3; 3) critical graph. It turns out that this is always possible; the observation that achieving 3 edges of the same color incident to a vertex disallows 5-cycles will help in executing this strategy. Finally, for n $\geq$6 if they play on K_n,

Ramsey theory promises that one of the player wins, when the theorem indicates that the first player has a winning strategy. So Ramsey theory is intimately linked to the outcome of this combinatorial game.

5. Open problems

Clearly this field still offers a huge number of open problems. The most obvious of which are finding more Ramsey numbers and improving the bounds we currently know. However, there are many related open problems.

Proposition 5.1. The limit $n \to \infty$, $R(n, n)^{\frac{1}{n}}$ exists.

Problem 5.2. Determine the value of c = limit $n \to \infty$, $R(n, n)^{\frac{1}{n}}$

Problem 5.3. Prove or disprove that R(4, n) > $\frac{n^3}{\log_c n}$ for some c, provided n is sufficiently large.

Problem 5.4. We have the bounds $2^k \leq R_k(3,3,\ldots,3) \leq (k+1)!$,do these Ramsey numbers grow faster than exponential in k?

6. Conclusion

In this paper, we were discussed about the examples of Ramsey numbers and their bounds. Here we gave the comparison between the Ramsey numbers for paths and wheels with harmonious chromatic number barbell graph .Also we discussed about the open problems in Ramsey theory.

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