

On Solving Fuzzy Game Problem using Octagonal Fuzzy Numbers

*K.Selvakumari*¹ and *S.Lavanya*²

¹Department of Mathematics, Vel's University, Chennai, India.

E-mail: selvafeb6@gmail.com

²Department of Mathematics, D.G.Vaishnav College, Chennai. India.

E-mail: lavanypasad1@gmail.com

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Abstract. In this paper, we consider a two persons zero sum game with imprecise values in payoff matrix. All the imprecise values are assumed to be octagonal fuzzy numbers. An approach for solving problems by using ranking of the fuzzy numbers has been considered to solve the fuzzy game problem. By using ranking to the payoffs we convert the fuzzy valued game problem to crisp valued game problem, which can be solved using the traditional method.

Keywords: Fuzzy Number, Octagonal fuzzy number, ranking of fuzzy numbers

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1. Introduction

The mathematical treatment of the Game theory was made available in 1944, when John von Neumann and Oscar Morgenstern published the famous article 'Theory of games and economic Behavior' [12]. The problem of Game theory defined as a body of knowledge that deals with making decisions when two or more intelligent and rational opponents are involved under conditions of conflict and competition. Game theory has played an important role in the fields of decision making theory such as economics, management etc. When we apply the Game theory to model some practical problems which we encounter in real situations, we have to know the values of payoffs exactly. However, it is difficult to know the exact values of payoffs and we could only know the values of payoffs approximately. In such situations, it is useful to model the problems as games with fuzzy payoffs.

In a fuzzy game problem, all parameters are fuzzy numbers. Fuzzy numbers may be normal (or) abnormal, triangular or trapezoidal or octagonal. Basirzadeh [2] have proposed a method for ranking fuzzy numbers using α -cuts in which he has given a ranking for triangular and trapezoidal fuzzy numbers. A ranking using α -cuts was introduced on octagonal fuzzy numbers in [10]. Using this ranking, the fuzzy Game problem is converted to a crisp value problem, which can be solved using the traditional method.

1.1. Fuzzy set

Let X be a non empty set. A fuzzy set A in X is characterized by its membership function $A \rightarrow [0,1]$ and $A(x)$ is interpreted as the degree of membership of element x in fuzzy A for each $x \in X$.

The Value zero is used to represent complete non-membership; the value one is used to represent complete membership and values in between are used to represent intermediate degrees of membership. The mapping A is also called the membership function of fuzzy set A.

1.2. Crisp set

A crisp set is a special case of a fuzzy set, in which the membership function only takes two values, commonly defined as 0 and 1.

1.3. Fuzzy number

A fuzzy number \tilde{A} is a fuzzy set on the real line R, must satisfy the following conditions.

- (i) There exist atleast one $x_0 \in R$ with $\mu_{\tilde{A}}(x_0)=1$
- (ii) $\mu_{\tilde{A}}(x)$ is piecewise continuous.
- (iii) \tilde{A} must be normal and convex.

2. Octagonal fuzzy numbers [10]

Definition 2.1. An octagonal fuzzy number denoted by \tilde{A}_w is defined to be the ordered quadruple

- $\tilde{A}_w = (l_1(r), s_1(t), s_2(t), l_2(r))$ for $r \in [0,k]$ and $t \in [k,w]$ where
- 1. $l_1(r)$ is a bounded left continuous non-decreasing function over $[0,w_1], [0 \leq w_1 \leq k]$
 - 2. $s_1(t)$ is a bounded left continuous non-decreasing function over $[k,w_2], [k \leq w_2 \leq w]$
 - 3. $s_2(t)$ is a bounded left continuous non-increasing function over $[k,w_2], [k \leq w_2 \leq w]$
 - 4. $l_2(r)$ is a bounded left continuous non-increasing function over $[0,w_1], [0 \leq w_1 \leq k]$

Definition 2.2. A fuzzy number \tilde{A} is a normal octagonal fuzzy number denoted by $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$ where $a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5 \leq a_6 \leq a_7 \leq a_8$ are real numbers and its membership function $\mu_{\tilde{A}}(x)$ is given below

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & \text{for } x < a_1 \\ k \left(\frac{x-a_1}{a_2-a_1} \right) & \text{for } a_1 \leq x \leq a_2 \\ k & \text{for } a_2 \leq x \leq a_3 \\ k + (1-k) \left(\frac{x-a_3}{a_4-a_3} \right) & \text{for } a_3 \leq x \leq a_4 \\ 1 & \text{for } a_4 \leq x \leq a_5 \\ k + (1-k) \left(\frac{a_6-x}{a_6-a_5} \right) & \text{for } a_5 \leq x \leq a_6 \\ k & \text{for } a_6 \leq x \leq a_7 \\ k \left(\frac{a_8-x}{a_8-a_7} \right) & \text{for } a_7 \leq x \leq a_8 \\ 0 & \text{for } x \geq a_8 \end{cases}$$

where $0 < k < 1$.

2.3. α -cut of an octagonal fuzzy number

The α -cut of an octagonal fuzzy number $\tilde{A} = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$ is

$$[\tilde{A}]_{\alpha} = \begin{cases} \left[a_1 + \left(\frac{\alpha}{k}\right)(a_2 - a_1), a_8 - \left(\frac{\alpha}{k}\right)(a_8 - a_7) \right] & \text{for } \alpha \in [0, k] \\ \left[a_3 + \left(\frac{\alpha-k}{1-k}\right)(a_4 - a_3), a_6 - \left(\frac{\alpha-k}{1-k}\right)(a_6 - a_5) \right] & \text{for } \alpha \in (k, 1] \end{cases}$$

3. Ranking of octagonal fuzzy numbers [10]

A measure of fuzzy number \tilde{A}_w is a function $M_{\alpha} : R_{\omega}(I) \rightarrow R^+$ which assigns a non-negative real number $M_{\alpha}(\tilde{A}_w)$ that expresses the measure of \tilde{A}_w .

$$M_{\alpha}(\tilde{A}_w) = \frac{1}{2} \int_{\alpha}^k (l_1(r) + l_2(r)) dr + \frac{1}{2} \int_k^{\omega} (s_1(t) + s_2(t)) dt \quad \text{where } 0 \leq \alpha \leq 1.$$

Definition 3.1. The measure of an octagonal fuzzy number is obtained by the average of the two fuzzy side areas, left side area and right side area, from membership function to α -axis.

Definition 3.2. Let \tilde{A} be a normal octagonal fuzzy number. The value $M_0^{oct}(\tilde{A})$, called the measure of \tilde{A} is calculated as follows:

$$M_0^{oct}(\tilde{A}) = \frac{1}{2} \int_{\alpha}^k (l_1(r) + l_2(r)) dr + \frac{1}{2} \int_k^{\omega} (s_1(t) + s_2(t)) dt \quad \text{where } 0 \leq k \leq 1. \\ = \frac{1}{4} [(a_1 + a_2 + a_7 + a_8)k + (a_3 + a_4 + a_5 + a_6)(1 - k)] \quad \text{where } 0 \leq k \leq 1.$$

Definition 3.3. Pure strategy. Pure strategy is a decision making rule in which one particular course of action is selected.

For fuzzy games the min-max principle is described by Nishizaki[10]. The course of the fuzzy game is determined by the desire of A to maximize his gain and that of restrict his loss to a minimum.

Definition 3.4. Saddle point. If the maxmin value equals the minimax value, then the game is said to have a saddle point and the corresponding strategies which give the saddle point are called optimal strategies. The amount of payoff at an equilibrium point is called the crisp game value of the game matrix.

3.1. Solution of all 2x2 matrix game

Consider the general 2x2 game matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

To solve this game we proceed as follows:

- (i) Test for a saddle point.
- (ii) If there is no saddle point, solve by finding equalizing strategies.

The Optimal mixed strategies for player A = (p_1, p_2) and

For player B = (q_1, q_2)

$$\text{where } p_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} ; p_2 = 1 - p_1$$

$$q_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} ; q_2 = 1 - q_1 \text{ and}$$

Value of the game $V = \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11}+a_{22}) - (a_{12}+a_{21})}$

Example 3.1. Consider the following fuzzy game problem.

$$\begin{matrix} & & \text{Player B} \\ \text{Player A} & \begin{pmatrix} (2,4,5,6,7,8,9,11) & (2,3,4,5,6,7,8,9) \\ (-1,0,1,2,3,4,5,6) & (-3,-2,-1,0,1,2,3,4) \end{pmatrix} \end{matrix}$$

Solution:

By definition of octagonal fuzzy number \tilde{A} is calculated as

$$M_0^{oct}(\tilde{A}) = \frac{1}{2} \int_{\alpha}^k (l_1(r) + l_2(r)) dr + \frac{1}{2} \int_k^{\omega} (s_1(t) + s_2(t)) dt \quad \text{where } 0 \leq k \leq 1.$$

$$= \frac{1}{4} [(a_1 + a_2 + a_7 + a_8)k + (a_3 + a_4 + a_5 + a_6)(1 - k)] \quad \text{where } 0 \leq k \leq 1.$$

Step 1:

Convert the given fuzzy problem into a crisp value problem.

This problem is done by taking the value of k as 0.4, we obtain the value of $M_0^{oct}(a_{ij})$.

$a_{11}=(2,4,5,6,7,8,9,11)$	$M_0^{oct}(a_{11}) = \frac{1}{4} [(2+4+9+11)(0.4)+(5+6+7+8)(1-0.4)]$ $= \frac{1}{4} [26(0.4)+26(0.6)] = 6.5$
$a_{12}=(2,3,4,5,6,7,8,9)$	$M_0^{oct}(a_{12}) = \frac{1}{4} [(2+3+8+9)(0.4)+(4+5+6+7)(1-0.4)]$ $= \frac{1}{4} [22(0.4)+22(0.6)] = 5.5$
$a_{21}=(-1,0,1,2,3,4,5,6)$	$M_0^{oct}(a_{21}) = \frac{1}{4} [(-1+0+5+6)(0.4)+(1+2+3+4)(1-0.4)]$ $= \frac{1}{4} [10(0.4)+10(0.6)] = 2.5$
$a_{22}=(-3,-2,-1,0,1,2,3,4)$	$M_0^{oct}(a_{22}) = \frac{1}{4} [(-3-2+3+4)(0.4)+(-1+0+1+2)(1-0.4)]$ $= \frac{1}{4} [2(0.4)+2(0.6)] = 0.5$

Since the condition $a_1 + a_2 + a_7 + a_8 = a_3 + a_4 + a_5 + a_6$ is satisfied by all the octagonal numbers for any value of k. We will get the same matrix as below.

Step 2:

The pay-off matrix is

$$\begin{matrix} & & \text{Player B} \\ & & \begin{pmatrix} 6.5 & 5.5 \\ 2.5 & 0.5 \end{pmatrix} \\ \text{Player A} & & \end{matrix}$$

Minimum of 1st row = 5.5
 Minimum of 2nd row = 0.5
 Maximum of 1st column = 6.5
 Maximum of 2nd column = 5.5
 Max(min)=5.5; Mini(max)=5.5
 It has saddle Point.

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The Crisp solution to the problem is saddle point= (A₁,B₂), Value of the game=5.5.

Example 3.2. Consider the following fuzzy game problem.

	PlayerB	
Player A	(-1,0,1,2,3,4,5,6)	(8,9,10,11,12,13,14,15)
	(4,5,6,7,8,9,10,11)	(0, 1, 2, 3, 4, 5, 6, 7)

Solution:

By definition of octagonal fuzzy number \tilde{A} is calculated as

$$M_0^{oct}(\tilde{A}) = \frac{1}{2} \int_{\alpha}^k (l_1(r) + l_2(r)) dr + \frac{1}{2} \int_k^{\omega} (s_1(t) + s_2(t)) dt \quad \text{where } 0 \leq k \leq 1.$$

$$= \frac{1}{4} [(a_1 + a_2 + a_7 + a_8)k + (a_3 + a_4 + a_5 + a_6)(1 - k)] \quad \text{where } 0 \leq k \leq 1.$$

Step 1:

Convert the given fuzzy problem into a crisp value problem.

This problem is done by taking the value of k as 0.4, we obtain the value of $M_0^{oct}(a_{ij})$.

$a_{11}=(-1,0,1,2,3,4,5,6)$	$M_0^{oct}(a_{11}) = \frac{1}{4} [(-1+0+5+6)(0.4)+(1+2+3+4)(1-0.4)]$ $= \frac{1}{4} [10(0.4)+10(0.6) = 2.5$
$a_{12}=(8,9,10,11,12,13,14,15)$	$M_0^{oct}(a_{12}) = \frac{1}{4} [(8+9+14+15)(0.4)+(10+11+12+13)(1-0.4)]$ $= \frac{1}{4} [46(0.4)+46(0.6) = 11.5$
$a_{21}=(4,5,6,7,8,9,10,11)$	$M_0^{oct}(a_{21}) = \frac{1}{4} [(4+5+10+11)(0.4)+(6+7+8+9)(1-0.4)]$ $= \frac{1}{4} [30(0.4)+30(0.6) = 7.5$
$a_{22}=(0, 1, 2, 3, 4, 5, 6, 7)$	$M_0^{oct}(a_{22}) = \frac{1}{4} [(0+1+6+7)(0.4)+(2+3+4+5)(1-0.4)]$ $= \frac{1}{4} [14(0.4)+14(0.6) = 3.5$

Since the condition $a_1 + a_2 + a_7 + a_8 = a_3 + a_4 + a_5 + a_6$ is satisfied by all the octagonal numbers for any value of k. We will get the same matrix as below.

Step 2:

The pay-off matrix is

		Player B	
	Player A	(2.5	11.5)
		(7.5	3.5)
Minimum of 1 st row	=	2.5	
Minimum of 2 nd row	=	3.5	

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Maximum of 1st column = 7.5
 Maximum of 2nd column = 11.5
 Max(min)=3.5; Mini(max)=7.5
 It has no saddle Point.

Step 3: To find Optimum mixed strategy and value of the game:

Here $a_{11}=2.5$, $a_{12}=11.5$, $a_{21}=7.5$, $a_{22}=3.5$

$$p_1 = \frac{a_{22}-a_{12}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{3.5-11.5}{(2.5+3.5)-(7.5+11.5)} = \frac{8}{13} ; p_2=1-p_1=1-\frac{8}{13}=\frac{5}{13}$$

$$q_1 = \frac{a_{22}-a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{3.5-7.5}{(2.5+3.5)-(7.5+11.5)} = \frac{4}{13} ; q_2=1-q_1=1-\frac{4}{13}=\frac{9}{13}$$

$$\text{Strategy for player A} = (p_1, p_2) = \left(\frac{8}{13}, \frac{5}{13}\right)$$

$$\text{Strategy for player B} = (q_1, q_2) = \left(\frac{4}{13}, \frac{9}{13}\right) \text{ and}$$

$$\begin{aligned} \text{Value of the game } V &= \frac{a_{11}a_{22}-a_{12}a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})} \\ &= \frac{(2.5)(3.5)-(7.5)(11.5)}{(2.5+3.5)-(7.5+11.5)} = \frac{8.75-86.25}{6-19} = \frac{-77.5}{-13} = \frac{77.5}{13} \end{aligned}$$

Remark 1. If the Octagonal numbers are slightly modified so that the condition $a_1 + a_2 + a_7 + a_8 \neq a_3 + a_4 + a_5 + a_6$ is not satisfied, then for such a problem the solution for different values of k ($0 \leq k \leq 1$) can be easily checked to lie in a finite interval.

Remark 2. In the above two examples we have considered only 2×2 fuzzy games but the method applied here can be used to solve any $m \times n$ fuzzy game.

4. Conclusion

In this paper, a method of solving fuzzy game problem using ranking of fuzzy numbers has been considered. The parameter k can be modified suitably by the decision maker to get the desired result. We may get different fuzzy game value for different values of k for the same fuzzy game.

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