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Strong Complementary Acyclic Domination of a Graph

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Abstract. Let G = (V,E) be a graph. A dominating set S of G is called a strong complementary acyclic dominating set if S is a strong dominating set and the induced subgraph $\langle V - S \rangle$ is acyclic. The minimum cardinality of a strong complementary acyclic dominating set of G is called the strong complementary acyclic domination number of G and is denoted by $\gamma_{c-a}^{st}(G)$. In this paper, we introduce and discuss the concept of strong complementary acyclic domination number of G.We determine this number for some standard graphs and obtain some bounds for general graphs. Its relationship with other graph theoretical parameters are also investigated.

Keywords: Domination number, strong domination number, strong complementary acyclic domination number

AMS Mathematics Subject Classification (2010): 05C35

1. Introduction

By a graph we mean a finite, simple, and undirected graph G(V,E) where V denotes its vertex set and E its edge set. Unless otherwise stated, the graph G has n vertices and e edges. Degree of a vertex v is denoted by d(v), the maximum degree of a graph G is denoted by $\Delta(G)$. We denote a cycle on n vertices by C_n, a path on n vertices by P_n, and a complete graph on n vertices by K_n. A graph G is connected if any two vertices of G are connected by a path. A maximal connected sub graph of a graph G is called a component of G. The number of components of G is denoted by $\omega(G)$. The complement $\overline{G}$ of G is the graph with vertex set V in which two vertices are adjacent if and only if they are not adjacent in G. A graph G is said to be acyclic if it has no cycles. A tree is a connected acyclic graph. A bipartite graph is a graph whose vertex set can be partitioned into two disjoint sets V_1 and V_2 such that every edge has one end in V_1 and another end in V_2. A Complete bipartite graph is a bipartite graph where every vertex of V_1 is adjacent to every vertex in V_2. The Complete bipartite graph with partitions of order $|V_1| = m$ and $|V_2| = n$, denoted by $K_{m,n}$. A star denoted by $K_{1,n-1}$ is a tree with one root vertex and n-1 pendant vertices. A bistar, denoted by D(r,s) is the graph obtained by joining the root vertices of the stars $K_{1,r}$ and $K_{1,s}$. A wheel graph denoted by W_n is a graph with n vertices formed by joining a single vertex to all vertices of C_{n-1}. A helm graph, denoted by H_n is a graph obtained from the wheel W_n by attaching a pendant vertex to each vertex in the outer cycle of W_n .Corona of two graphs G_1 and G_2, denoted by $G_1 \circ G_2$ is the graph obtained by taking one copy of G_1 and $|V(G_1)|$ copies of G_2 in which i th vertex of G_1 is joined to every vertex in the ith copy of G_2. . If S is a subset of V, then $\langle S \rangle$ denotes the vertex induced sub graph of G induced by S. The open neighborhood of a set S of vertices of graph G , denoted by N(S) is the set of all vertices adjacent to some vertex in S, and $N(S) \cup S$ is called the closed neighbourhood of S, denoted by N[S]. The diameter of a connected graph is the maximum distance between two vertices in G and is denoted by diam(G). A cut-vertex of a graph G is a vertex whose removal increases the number of components. A vertex cut of a connected graph G is a set of vertices whose removal results in a disconnected graph. The connectivity or vertex connectivity of a graph G, denoted by k(G)(where G is not complete) is the size of a smallest vertex cut. A connected sub graph H of a connected graph G is called a H-cut if $\omega(G-H) \geq 2$. The chromotic number of a graph G, denoted by $\chi(G)$ is the minimum number of colors needed to color all the vertices a graph G in which adjacent vertices receive distinct colors. For any real number $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. A Nordhaus-Gaddum type result is a lower or upper bound on the sum or product of a parameter of a graph and its complement. Terms not defined here are used in the sense of [2].

A subset of V is called a dominating set of G if every vertex in V-S is adjacent to at least one vertex in S. The domination number $\gamma(G)$ of G is the minimum cardinality taken over all dominating sets in G.A dominating set S of G is called a strong dominating set of G if for every $v \in V-S$ there exist a vertex $u \in S$ such that $uv \in E(G)$ and $d(u) \geq d(v)$. The minimum cardinality taken over all strong dominating sets is the strong domination number and is denoted by $\gamma_{st}(G)$.

A dominating set S of G is called a complementary acyclic dominating set of G if $\langle V-S \rangle$ is acyclic.The minimum cardinality taken over all complementary acyclic dominating sets is the complementary acyclic domination number and is denoted by $\gamma_{c-a}(G)$.

Many authors have introduced different types of domination parameters by imposing conditions on the dominating set [2].The concept of strong domination has been introduced by Sampathkumar and Pushpalatha [5] .

In this paper, we use this idea to develop the concept of strong complementary acyclic domination number of a graph..

2. Strong complementary acyclic domination

Definition 2.1. A dominating set S of G is called a strong complementary acyclic dominating set if S is a strong dominating set and the induced subgraph $\langle V-S \rangle$ is acyclic. The minimum cardinality of a strong complementary acyclic dominating set of G is called the strong complementary acyclic domination number of G and is denoted by $\gamma_{c-a}^{st}(G)$.

Example 2.2. A Strong complementary acyclic dominating set of a graph G is given below:

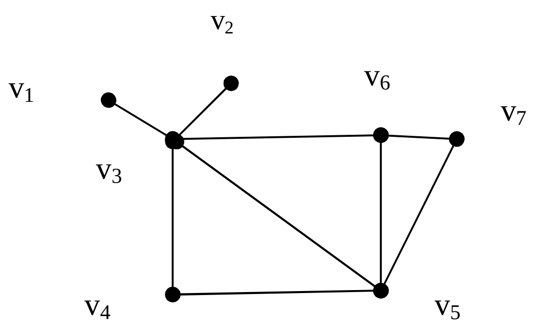

Figure 2.1: G_1

$\{v_3,v_5\}$ is a strong complementary acyclic dominating set of G.
For any graph G, V(G) is a strong complementary acyclic dominating set.

Remark 2.3. Throughout this paper we consider only graphs for which strong complementary acyclic dominating set exists. The complement of the strong complementary acyclic dominating set need not be a strong complementary acyclic dominating set.

Example 2.4.

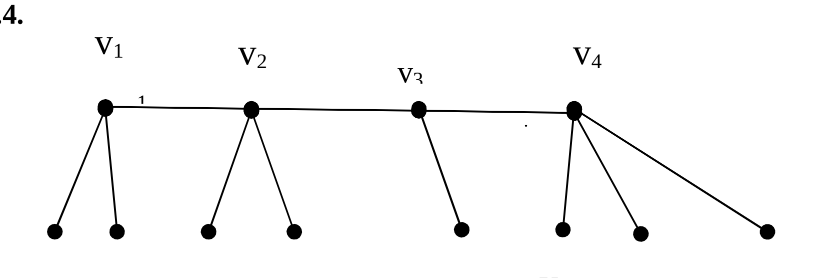

Figure 2.2: G_2

$\{v_1,v_2,v_3,v_4\}$ is a strong complementary acyclic dominating set. But its complement is not a strong complementary acyclic dominating set.

Definition 2.5. A Strong Complementary acyclic dominating set S of G is minimal if no proper subset of S is a Strong Complementary acyclic dominating set of G.
Remark 2.6. Any superset of strong complementary acyclic dominating set of G is also a strong complementary acyclic dominating set of G. Since if S is a strong complementary acyclic dominating set of G and $u \in V - S$, then $S \cup \{u\}$ is a strong complementary

acyclic dominating set of G. Therefore strong complementary acyclic domination is super hereditary.

Remark 2.7. A strong complementary acyclic dominating set of G is minimal iff it is 1-minimal.

Theorem 2.8. A strong complementary acyclic dominating set of G is minimal if and only if for each vertex $u \in S$ one of the following conditions holds:

1. u has a strong private neighbor in $V - S$.
2. $\langle (V - S) \cup \{u\} \rangle$ contains a cycle.

Proof: Let S be a strong complementary acyclic dominating set of G. Suppose S is minimal. Let $u \in S$. Then $S - \{u\}$ is not a strong complementary acyclic dominating set of G. Therefore $\langle (V - S) \cup \{u\} \rangle$ contains a cycle or u has a strong private neighbour in V-S with respect to S. Conversely, suppose for every u in S, one of the conditions holds.
If (1) hods, then $S - \{u\}$ is not a strong dominating set.
If (2) holds, then $S - \{u\}$ is not a complementary acyclic. Therefore, S is a minimal strong complementary acyclic dominating set of G.

Remark 2.9. Every strong complementary acyclic dominating set is a dominating set. But every dominating set need not be a strong complementary acyclic dominating set.

Example 2.10.

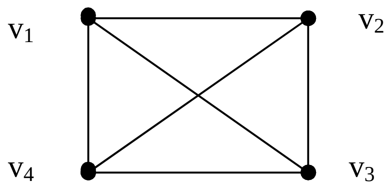

Figure 2.3: G_3

Here $\{v_1,v_2\}$ is a strong complementary acyclic dominating set and also a dominating set. Also $\{v_1\}$ is a dominating set but it is not a strong complementary acyclic dominating set.

Theorem 2.11. For any graph G, $\gamma(G) \leq \gamma_{st}(G) \leq \gamma_{c-a}^{st}(G)$ and the bounds are sharp.
Let S be a minimum strong complementary acyclic dominating set of G.
Let $v \in V - S$. Then there exists $u \in S$ such that u and v are adjacent and $\deg(u) \geq \deg(v)$.

Therefore S is a strong dominating set of G and hence S is a dominating set of G.
Therefore $\gamma(G) \leq \gamma_{st}(G) \leq \gamma_{c-a}^{st}(G)$.

Example 2.12.

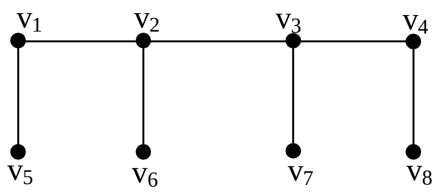

Figure 2.4: G_4

Here $\gamma(G) = \gamma_{st}(G) = \gamma_{c-a}^{st}(G) = 4$.

Theorem 2.13.

1. $\gamma_{c-a}^{st}(K_2) = n-1$.

1. $\gamma_{c-a}^{st}(K_n) = n-2, n \geq 3$.

2. $\gamma_{c-a}^{st}(K_{1,n}) = 1$.

3. $\gamma_{c-a}^{st}(D_{r,s}) = 2$.

4. $\gamma_{c-a}^{st}(W_n) = 2$.

5. $\gamma_{c-a}^{st}(K_{m,n}) = \min\{m,n\}$.

Theorem 2.14. For any path P_m

$$\gamma_{c-a}^{st}(P_m) = n \text{ if } \mathrm{m = 3n, n \in N}$$
$$= n+1 \;\; if \;\; m = 3n+1 \, or \, 3n+2, n \in N$$

Proof:

Case (i) Let $G = P_{3n}$, $n \in N$. Let $v_1, v_2, v_3, \ldots v_{3n}$ be the vertices of $V(P_{3n})$.
$\{v_2, v_5, v_8 \ldots v_{3n-1}\}$ is the unique strong dominating set of P_{3n}. It is also the strong complementary acyclic dominating set of P_{3n}. Therefore $\gamma_{c-a}^{st}(P_{3n}) = n$, for all n□N.

Case (ii) Let $G = P_{3n} + 1$, $n \in N$. Let$\{v_1, v_2, v_3, \ldots v_{3n}, v_{3n+1}\}$.
$S_1 = \{v_2, v_5, v_8 \ldots v_{3n-1}, v_{3n+1}\}$ and $S_2 = \{v_1, v_3, v_6, v_9 \ldots v_{3n}\}$are two strong complementary acyclic dominating sets of G.

Now $|S_1| = |\{v_2, v_5, v_8, \ldots v_{3n-1}\}| + |v_{3n+1}| = n+1$. Also

$|S_2| = |v_1| + |\{v_3, v_6, \ldots v_{3n}\}| = n+1$.

Therefore, $\gamma_{c-a}^{st}(P_{3n}) \leq n+1$.

Also $\gamma_{st}(P_{3n+1}) = n+1$ and by Theorem 2.10, we have $\gamma_{st}(G) \leq \gamma_{c-a}^{st}(G)$.

Hence $\gamma_{c-a}^{st}(P_{3n+1}) = n+1$.

Case (iii) Let $G = P_{3n+2}, n \in N$. Let{$v_1, v_2, v_3, \ldots v_{3n}, v_{3n+1}, v_{3n+2}$}.
S={$v_2, v_5, v_8 \ldots v_{3n-1}, v_{3n+1}$} is a strong complementary acyclic dominating ses of G.
Now $|S| = |\{v_2, v_5, v_8, \ldots v_{3n-1}\}| + |v_{3n+1}| = n+1$.
Therefore $\gamma_{c-a}^{st}(P_{3n+2}) \le n+1$
Also $\gamma_{st}(P_{3n+2}) \le n+1$ and by Theorem 2.10 $\gamma_{st}(G) \le \gamma_{c-a}^{st}(G)$.
Hence $\gamma_{c-a}^{st}(P_{3n+2}) = n+1$.

Theorem 2.15.
$\gamma_{c-a}^{st}(C_m) = n$ if $m = 3n, n \in N$
$= n+1$ *if* $m = 3n+1$ *or* $3n+2, n \in N$
Proof: The proof follows from Theorem 2.11.

Observation 2.16. If a spanning sub graph H of a graph G has a strong complementary acyclic dominating set then G has a strong complementary acyclic dominating set.

Observation 2.17. Let G be a connected graph and H be a spanning sub graph of G. If H has a strong complementary acyclic dominating set, then $\gamma_{c-a}^{st}(G) \le \gamma_{c-a}^{st}(H)$ and the bounds are sharp.

Theorem 2.18. For any connected graph G, with $n \ge 3$ vertices, $1 \le \gamma_{c-a}^{st}(G) \le n-2$ and the bounds are sharp.
Proof: The lower and upper bounds follows from definition. For $K_{1,n-1}$ the lower bound is attained and for K_4, the upper bound is attained.
Observation 2.19. For any connected graph G with 3 vertices, $\gamma_{c-a}^{st}(G) = n-2$ if and only if $G \cong P_4, C_3$.

The Nordhaus-Gaddum type result is given below.
Theorem 2.20. Let G be a graph such that G and $\overline{G}$ no isolates of order $n \ge 3$. Then $\gamma_{c-a}^{st}(G) + \gamma_{c-a}^{st}(\overline{G}) \le 2n-4$ and $\gamma_{c-a}^{st}(G) \cdot \gamma_{c-a}^{st}(\overline{G}) \le (n-2)^2$.
Proof: The bound directly follows from Theorem 2.15.

Relationship with other graph theoretical parameters.
Theorem 2.21. For any connected graph with $n \ge 3$ vertices $\gamma_{c-a}^{st}(G) + k(G) \le 2n-3$ and the bound is sharp if and only if $G \cong K_n$.
Proof: Let G be a connected graph with n vertices.We know that $k(G) \le n-1$ and by theorem 2.18, $\gamma_{c-a}^{st}(G) \le n-2$. Hence $\gamma_{c-a}^{st}(G) + k(G) \le 2n-3$. Suppose G is isomorphic K_n. Then clearly $\gamma_{c-a}^{st}(G) + k(G) = 2n-3$. Conversely, let

$\gamma_{c-a}^{st}(G)+k(G)=2n-3$. This is possible only if $\gamma_{c-a}^{st}(G)=n-2$ and $k(G)=n-1$. But $k(G)=n-1$ and so $G\cong K_n$ for which $\gamma_{c-a}^{st}(G)=n-2$ .Hence $G\cong K_n$.

Theorem 2.22. For any connected graph G with $n\geq 3$ vertices, $\gamma_{c-a}^{st}(G)+\chi(G)\leq 2n-2$ and the bound is sharp if and only if $G\cong K_n$.
Proof: Let G be a connected graph with n vertices. We know that $\chi(G)\leq n$ and by Theorem 2.18, $\gamma_{c-a}^{st}(G)\leq n-2$. Hence $\gamma_{c-a}^{st}(G)+\chi(G)\leq 2n-2$. Suppose G is isomorphic to K_n. Then clearly $\gamma_{c-a}^{st}(G)+\chi(G)=2n-2$. Conversely, let $\gamma_{c-a}^{st}(G)+\chi(G)=2n-2$. This is possible only if $\gamma_{c-a}^{st}(G)=n-2$ and $\chi(G)=n$. Since $\chi(G)=n$, G is isomorphic to K_n for which $\gamma_{c-a}^{st}(G)=n-2$.
Hence $G\cong K_n$.

Theorem 2.23. For any connected graph G with $n\geq 3$ vertices, $\gamma_{c-a}^{st}(G)+\Delta(G)\leq 2n-3$ and the bound is sharp.
Proof: Let G be a connected graph with n vertices, $\Delta(G)\leq n-1$ and by Theorem 2.18, $\gamma_{c-a}^{st}(G)\leq n-2$. Hence $\gamma_{c-a}^{st}(G)+\Delta(G)\leq 2n-3$.For K_5 the bound is sharp.

3. Conclusion

We found strong complementary acyclic domination number for some standard graphs and obtained some bounds for general graphs. Its relationship with other graph theoretical parameters are also investigated.

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