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Some Extremal Problems in Weighted Graphs

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Abstract. Some new distance concepts in weighted graphs are introduced in this article. With respect to these distances, the concepts of center and self centered graphs are introduced and their properties are discussed. A characterization for these self centered graphs using the max – max composition of the corresponding distance matrices is obtained. Central properties of partial trees and partial blocks are also discussed.

Keywords: α- distance, β- distance, strong distance, self centered graphs

AMS Mathematics Subject Classifications (2010): 03E72, 03E75, 05C22, 05C38

1. Introduction and preliminaries

Graph theory has now become a major branch of applied mathematics and generally regarded as a branch of Combinatorics. Graph theory is a widely used tool for solving a combinatorial problem in different areas such as geometry, algebra, number theory, topology, optimization and computer science. Most important thing which is to be noted is that, any real life problem which can be solved by any graph technique can only be modelled by a weighted graph. Distance and center concepts play an important role in applications related with graphs and weighted graphs. Several authors including Bondy and Fan [1, 2, 3], Broersma, Zhang and Li [17], Sunil Mathew and Sunitha [9, 10, 11, 12, 13, 14] introduced many connectivity concepts in weighted graphs following the works of Dirac [4] and Grotschel [5]. More related works can be seen in [8, 15, 16].

In this article, we introduce three new distance concepts in weighted graphs. These concepts are derived by using the notion of connectivity in weighted graphs. In a weighted graph model, for example, in an information network or in an electric circuit, the reduction of flow between pairs of nods is more relevant and may frequently occur than the total disconnection of the entire network [7, 11, 12]. Finding the center of a graph is useful in facility location problems where the goal is to minimize the distance to the facility. For example, placing a hospital at a central point reduces the longest distance that the ambulance has to travel. This concept is our motivation. As weighted graphs are generalized structures of graphs, the concepts introduced in this article also generalize the classic ideas in graph theory.

A weighted graph *G: (V, E, W)* is a graph in which every arc *e* is assigned a non negative number *w(e)*, called the *weight* of *e* [1]. The *distance* between two nodes *u* and *v* in *G* is defined and denoted by $d(u,v)=\min\left\{\sum_{e\in P} w(e)/P \text{ is a } u-v \text{ path in } G\right\}$ [1,6]. The *eccentricity* of a node *u* in *G* is defined and denoted by $e(u)=\max\{d(u,v)/v \text{ is any other node of } G\}$[6]. The minimum and maximum eccentricities of nodes are respectively called *radius, r(G)* and *diameter, d(G)* of the graph [6]. A node *u* is called central if *e(u) = r(G)* and *diametral* or *pheripheral* if *e(u) = d(G)* [6]. *G* is called *self catered* if it is isomorphic with its center [6].

In a weighted graph *G:(V, E, W), the strength of a path* $P = v_0e_1v_1e_2v_2e_3v_3...e_nv_n$ is defined and denoted by $S(P) = min\{w(e_1), w(e_2), w(e_3), \dots, w(e_n)\}$ [13]. The *strength of connectedness* between a pair of nodes *u* and *v* in *G* is defined and denoted by $CONN_G(u, v) = max\{S(P)/P \text{ is a } u - v \text{ path in } G\}$[12]. A *u – v* path *P* is called a *strongest path* $S(P) = CONN_G(u, v)$[11]. A node *w* is called a *partial cut node (p- cut node)* of *G* if there exists a pair of nodes *u, v* in *G* such that $u \neq v \neq w$ and $CONN_{G-w}(u, v) < CONN_G(u, v)$ [12]. A graph without *p- cut nodes* is called a *partial block (p- block)* [12]. It is also proved in [12] that a node *w* in a weighted graph *G* is a *p- cut node* if and only if *w* is an internal node of every maximum spanning tree. A connected weighted graph *G:(V, E, W)* is called a partial tree *if G* has a spanning sub graph *F:(V, E', W')* which is a tree, where for all arcs *e = (u, v)* of *G* which are not in *F*, we have $CONN_G(u, v) > w(e)$ [12]. An arc e = (u, v) is called α- strong if $CONN_{G-e}(u, v) < w(e)$ and β- strong *if* $CONN_{G-e}(u, v) > w(e)$. An arc is called *strong* if it is either $\alpha-strong\ or\ \beta-strong$ [12]. The *max – max composition* of a square matrix with itself is again a square matrix of the same order whose $(i, j)^{th}$ entry is given by $d_{i,j} = max\{max(d_{i,1}, d_{1,j}), max(d_{i,2}, d_{2,j}), max(d_{i,3}, d_{3,j}), \dots, max(d_{i,n}, d_{n,j})\}$[7].

2. α, β and strong distances

In this section, we give the definitions of the distances along with examples.

Definition 2.1. *Let G: (V, E, W) be a connected weighted graph. Let u and v be any two nodes of G. Then the α- distance between the nodes u and v is defined and denoted by*

$$d_\alpha(u,v)=\begin{cases} min\sum_{e\in P} w(e); & if\ P\ is\ any\ \alpha-strong\ path\ between\ u\ and\ v\\ 0 & ;if\ u=v\\ \infty & ;if\ there\ exists\ no\ \alpha-strong\ u-v\ path\ in\ G\end{cases}$$

Clearly d_α satisfies all the axioms of a metric as follows.

1. $d_\alpha(u,v)\geq 0$ *for all u and v*
2. $d_\alpha(u,v)=0$ *if and only if* $u=v$
3. $d_\alpha(u,v)=d_\alpha(v,u)$ *for all u and v*
4. $d_\alpha(u,v)\leq d_\alpha(u,w)+d_\alpha(w,v)$ *for ll u, v and w.*

Hence $(V(G), d_\alpha)$ is a metric space.

Definition 2.2. *Let G: (V, E, W) be a connected weighted graph. Let u and v be any two nodes of G. Then the β- distance between the nodes u and v is defined and denoted by*

$$d_\beta(u,v) = \begin{cases} min \sum_{e\in P} w(e); & if\ P\ is\ any\ \beta - strong\ path\ between\ u\ and\ v \\ 0 & ;\ if\ u = v \\ \infty & ; if\ there\ exists\ no\ \beta - strong\ u - v\ path\ in\ G \end{cases}$$

Clearly d_β satisfies all the axioms of a metric as follows.

1. $d_\beta(u, v) \geq 0$ *for all u and v*
2. $d_\beta(u, v) = 0$ *if and only if* $u = v$
3. $d_\beta(u, v) = d_\beta(v, u)$ *for all u and v*
4. $d_\beta(u, v) \leq d_\beta(u, w) + d_\beta(w, v)$ *for ll u, v and w.*

Hence *(V(G),* d_β*)* is a metric space.

Definition 2.3. Let G: (V, E, W) be a connected weighted graph. Let u and v be any two nodes of G. Then the strong distance between the nodes u and v is defined and denoted by

$$d_s(u,v) = \begin{cases} min \sum_{e\in P} w(e); & if\ P\ is\ any\ strong\ path\ between\ u\ and\ v \\ 0 & ;\ if\ u = v \\ \infty & ; if\ there\ exists\ no\ strong\ u - v\ path\ in\ G \end{cases}$$

Clearly d_s satisfies all the axioms of a metric as follows.

1. $d_s(u, v) \geq 0$ *for all u and v*
2. $d_s(u, v) = 0$ *if and only if* $u = v$
3. $d_s(u, v) = d_\alpha(v, u)$ *for all u and v*
4. $d_s(u, v) \leq d_s(u, w) + d_s(w, v)$ *for ll u, v and w.*

Hence *(V(G),* d_s*)* is a metric space.

In the following example (figure 1), we find that these three distances are generally different.

Example 2.1.

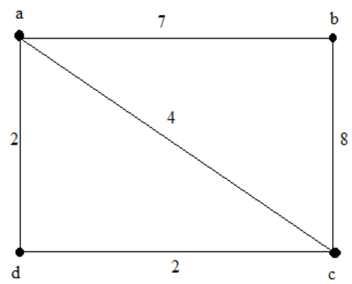

Figure 1: A weighted graph

The α, β and strong distances between different pairs of nodes are given below.

$d_\alpha(a,b)=7$, $d_\alpha(a,c)=15$, $d_\alpha(a,d)=\infty$, $d_\alpha(b,c)=8$, $d_\alpha(b,d)=\infty$, $d_\alpha(c,d)=\infty$. $d_\beta(a,b)=\infty$, $d_\beta(a,c)=4$, $d_\beta(a,d)=2$, $d_\beta(b,c)=\infty$, $d_\beta(b,d)=\infty$, $d_\beta(c,d)=2$. $d_s(a,b)=7$, $d_s(a,c)=4$, $d_s(a,d)=2$, $d_s(b,c)=8$, $d_s(b,d)=9$, $d_s(c,d)=2$.

3. Strong Center of a weighted graph

In this section, we introduce the concepts of eccentricity, radius, diameter and center with respect to the distances which are defined in the above section.

Definition 3.1. *The α- eccentricity of a node u in G is defined and denoted by* $e_\alpha(u) = \max\{d_\alpha(u, v)/ v\in V, 0 \le d_\alpha(u, v) \le \infty\}$.

In the same manner the β- eccentricity and strong eccentricity are defined below.

$e_\beta(u) = \max\{d_\beta(u, v)/ v\in V, 0 \le d_\beta(u, v) \le \infty\}$.

$e_s(u) = \max\{d_s(u, v)/ v\in V, 0 \le d_s(u, v) \le \infty\}$.

Definition 3.2. *A node v is called the α- eccentric node of u if* $e_\alpha(u) = d_\alpha(u, v)$. *The set of all α- eccentric nodes of u is denoted by* u_α^*.

In the same manner, we can define *β- eccentric node* and the *strong eccentric node*.

Definition 3.3. *Among the α- eccentricities of all the nodes of a graph , the minimum is called the α- radius of G. It is denoted by* $r_\alpha(G)$. *That is* $r_\alpha(G) = \min\{e_\alpha(u) / u\in V\}$.
Also the *β- radius of G* is defined and denoted by $r_\beta(G) = \min\{e_\beta(u) / u\in V\}$ and the *strong radius of G is defined and denoted by* $r_s(G) = \min\{e_s(u) / u\in V\}$.

As the radius is the minimum eccentricity, the maximum eccentricity is called the diameter of the graph.

Definition 3.4. *Among the α- eccentricities of all the nodes of a graph, the maximum is called the α- diameter of G. It is denoted by* $d_\alpha(G)$. *That is* $d_\alpha(G) = \max\{e_\alpha(u) / u\in V\}$.
Also the *β-diameter* of *G* is defined and denoted by $d_\beta(G) = \max\{e_\beta(u) / u\in V\}$ and the *strong diameter* of *G* is $d_s(G) = \max\{e_s(u) / u\in V\}$.

Definition 3.5. *A node u of G is called α- central if* $e_\alpha(u) = r_\alpha(G)$, *called β- central if* $e_\beta(u) = r_\beta(G)$ *and called strong central if* $e_s(u) = r_s(G)$.

Definition 3.6. *A node u is called α- diameteral if* $e_\alpha(u) = d_\alpha(G)$, *called β- diameteral if* $e_\beta(u) = d_\beta(G)$ *and called strong diameteral if* $e_s(u) = d_s(G)$.

Definition 3.7. *The subgraph induced by the set of all α- central nodes is called the α- center of G. It is denoted by* $\langle C_\alpha(G)\rangle$. *Analogously the β- center of G is the subgraph induced by the set of all β- central nodes of G and is denoted by* $\langle C_\beta(G)\rangle$. *Also the* *strong center of G is denoted as* <$C_s(G)$> *and is defined as the subgraph of G induced by the set of all strong central nodes.*

Using example 2.1, we illustrate all the above definitions
$e_\alpha(a) = 15$, $e_\alpha(b) = 8$, $e_\alpha(c) = 15$, $e_\alpha(d) = \infty$, $e_\beta(a) = 4$, $e_\beta(b) = \infty$, $e_\beta(c) = 4$, $e_\beta(d) = 2$, $e_s(a) = 7$, $e_s(b) = 9$, $e_s(c) = 8$, $e_s(d) = 9$, $r_\alpha(G) = 8$, $r_\beta(G) = 2$, $r_s(G) = 7$, $d_\alpha(G) = 15$, $d_\beta(G) = 4$, $d_s(G) = 9$.

b is the *α*- central node, *d* is the *β*- central node and *a* is the strong central node. *a* and *c* are the *α* diameteral nodes. They are *β*- diameteral also. The strong diameteral nodes are *b* and *d.*

If we take the diameteral nodes instead of the central nodes, we get the periphery of the graph.

Definition 3.8. *The subgraph induced by the set of all α- diameteral nodes is called the α- periphery of G. It is denoted by* <$P_\alpha(G)$>. *Analogously the β- periphery of G is the subgraph induced by the set of all β- diameteral nodes of G, denoted by* <$P_\beta(G)$>. *Also the strong periphery of G is denoted by* <$P_s(G)$> *and is defined as the subgraph induced by the set of all strong diameteral nodes.*

Definition 3.9. *A node u is called α- isolated if there is no α- strong arc is incident on u, β- isolated if there is no β- strong arc is incident on u and strong isolated if no strong arc is incident on u.*

Definition 3.10. *An arc e = (u,* ***v****) is called α- isolated if there is no α- strong arc adjacent with e. Similarly β- isolated if there is no β- strong arc adjacent with e and called strong isolated if there is no strong arc adjacent with e.*
From the last two definitions, we have the following preposition.

Preposition 3.1. *Let G: (V, E, W) be a weighted graph. If e = (u, v) is an α- isolated arc in G, then* $e_\alpha(u) - e_\alpha(v) = 0$.
Proof: Since *e = (u, v)* is *α*- isolated, *e* is adjacent with no *α*- strong arcs. That means *e* is the only *α*- strong arc, which is incident on *u* and *v*. While calculating the *α*- eccentricity of *u*, the farthest node from *u* is *v* and vice – versa. Thus $e_\alpha(u) = e_\alpha(v) = w(e)$ and hence $e_\alpha(u) - e_\alpha(v) = 0$. This completes the proof.

It can easily be seen that, the above preposition is valid for both *β*- isolated and strong isolated arcs.

Preposition 3.2. *Let G: (V, E, W) be a connected weighted graph such that every arc e in G have weight* $w(e) \geq k \geq 1$. *Then* $e_s(u) \geq e(u)$ *for every node u in G, where e(u) is the eccentricity of u in the underlying graph of G.*
Proof: Let *G: (V, E, W)* be a connected weighted graph such that each of the arc has weight $k \geq 1$. Let *u* be any node of *G*. Let *v* be the farthest node of *u* in the underlying graph of *G*. We know that, between any pair of nodes *u* and *v* of *G*, there exists a strong path [12]. Now *e(u)* is the number of arcs in the shortest path connecting *u* and *v* in the underlying graph of *G*. But $e_s(u)$ is the sum of the weights of all arcs in the shortest strong

path connecting *u* and *v* in G. Since each edge *e* is with weight $w(e) \geq 1$, it is Clear that $e_s(u) \geq e(u)$.

The above preposition is trivial and valid for both α and β distances. As in the classical concept of distance in graphs, we have the following inequalities. We omit their proof as they are obvious.

Theorem 3.1. *Let G: (V, E, W) be a connected weighted graph. Then the following inequalities hold.*

1. $r_\alpha(G) \leq d_\alpha(G) \leq 2r_\alpha(G)$
2. $r_\beta(G) \leq d_\beta(G) \leq 2r_\beta(G)$
3. $r_s(G) \leq d_s(G) \leq 2r_s(G)$.

4. Self centered graphs

In this section, we present the idea of self centered graphs with respect to the distances which are introduced in section 2. Here we present some necessary conditions and a characterization of self centered graphs. Throughout this section, *G* is a connected weighted graph.

Definition 4.1. *G is called α- self centered if G is isomorphic with* $<C_\alpha(G)>$, *G is β- self centered if G is isomorphic with* $<C_\beta(G)>$, *and is called strong self centered if G is isomorphic with* $<C_s(G)>$.

The following theorem is true for both β and strong self centered graphs.

Theorem 4.1. *Let G: (V, E, W) be a connected weighted graph such that there exists exactly one α- strong arc incident on every node and that all the α- strong arcs are of equal weight, then G is α- self centered.*

Proof: Given that all the nodes of *G* are incident with exactly one α- strong arc, and all the α- strong arcs are of equal weight. That means if $e = (u, v)$ is α- strong, then there will be no other α- strong arcs incident on *u* and *v*. Hence $e_\alpha(u) = w(e) = e_\alpha(v)$. By this same argument we get this same equality for any other α- strong arc. Thus $e_\alpha(u) = w(e)$ for every node *u* in *G*.

This proves that *G* is α- self centered.

The next theorem is a characterization for these self centered graphs.

Theorem 4.2. *A connected weighted graph G : (V, E, W) is α- self centered if and only if for any two nodes u and v of G such that u is an α- eccentric node of v, then v should be one of the α- eccentric nodes of u.*

Proof: First assume that *G* is α- self centered. Also assume that *u* is an α- eccentric node of *v*. That means $e_\alpha(v) = d_\alpha(v, u)$. Since *G* is α- self centered, all the nodes of *G* will be having the same α- eccentricity. Therefore $e_\alpha(u) = e_\alpha(v)$. From the above two equations, we get $e_\alpha(u) = d_\alpha(v, u) = d_\alpha(u, v)$. Thus $e_\alpha(u) = d_\alpha(u, v)$. This proves that *v* is an α- eccentric node of *u*.

Conversely assume that "if *u* is an α- eccentric node of *v*, then *v* should be one of the α- eccentric nodes of *u*". That means $e_\alpha(u) = d_\alpha(u, v)$ and $e_\alpha(v) = d_\alpha(v, u)$. But due to symmetry of d_α, we have $d_\alpha(u, v) = d_\alpha(v, u)$. Therefore $e_\alpha(u) = e_\alpha(v)$ for any two arbitrary nodes u and v of G. Hence all the nodes of G have the same α- eccentricity and hence G is α- self centered.

This completes the proof of the theorem.

In the same manner, we can prove this result for β and strong self centered graphs.

5. The distance matrix and the max – max composition

In this section, we present an easy check for a weighted graph G to see whether it is α- self centered or not. This result can also be applied for both β and strong self centered graphs.

Definition 5.1. *Let $G: (V, E, W)$ be a connected weighted graph with n nodes. The α- distance matrix of G is defined and denoted as $D_\alpha(G) = (d_{i,j})$ is the square matrix of order n and $d_{i,j} = d_\alpha(v_i, v_j)$. Note that the α- distance matrix is a symmetric matrix.*

Instead of $D_\alpha(G)$, we simply write D_α when there is no confusion regarding the name of the weighted graph.

Analogously we can define β and strong distance matrices. In the following example (figure 2), we give the three distance matrices.

Example 5.1.

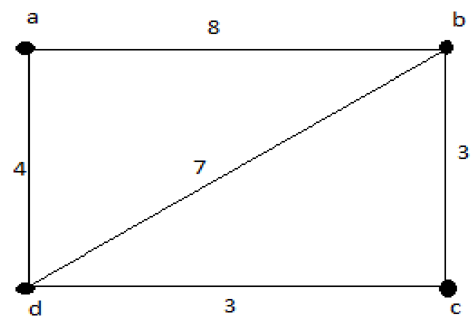

Figure 2:

The α, β and strong distance matrices are given below.

$$D_\alpha = \begin{bmatrix} 0 & 8 & \infty & 15 \\ 8 & 0 & \infty & 7 \\ \infty & \infty & 0 & \infty \\ 15 & 7 & \infty & 0 \end{bmatrix}, \quad D_\beta = \begin{bmatrix} 0 & \infty & \infty & \infty \\ \infty & 0 & 3 & 6 \\ \infty & 3 & 0 & 3 \\ \infty & 6 & 3 & 0 \end{bmatrix} \quad D_s = \begin{bmatrix} 0 & 8 & 11 & 14 \\ 8 & 0 & 3 & 6 \\ 11 & 3 & 0 & 3 \\ 14 & 6 & 3 & 0 \end{bmatrix}$$

Next we have a theorem regarding the eccentricities of nodes using the max – max composition of the distance matrices.

Theorem 5.1. *Let G: (V, E, W) be a connected weighted graph. The diagonal elements of the max – max composition of the α- distance matrix of G with itself are the α-eccentricities of the nodes of G.*

Proof: Let $D_\alpha = (d_{i,j})$ be the α- distance matrix of G. Then $d_{i,j} = d_\alpha(v_i, v_j)$. In the max – max composition $D_\alpha 0 D_\alpha$, the i^{th} entry in the principal diagonal $d_{i,i} = max\{max(d_{i,1}, d_{1,i}), max(d_{i,2}, d_{2i}), max(d_{i,3}, d_{3,i}), ..., max(d_{i,n}, d_{n,i})\}$. But due to symmetry of D_α, we have $d_{i,i} = max\{d_{i,1}, d_{i,2}, d_{i,3}, ..., d_{i,n}\} = max\{d_\alpha(v_i, v_1), d_\alpha(v_i, v_2), d_\alpha(v_i, v_3), ..., d_\alpha(v_i, v_n)\} = e_\alpha(v_i)$.

This completes the proof of the theorem.

Theorem 5.2. *A connected weighted graph G: (V, E, W) is α- self centered if and only if all the entries in the principal diagonal of the max – max composition of the- distance matrix with itself are the same.*

Proof: As proved in *theorem 5.1*, the principal diagonal entries in the max – max composition of the α- distance matrix with itself are the α- eccentricities of the nodes. If they are same, that means $e_\alpha(u)$ is the same for all u in G, then G is α- self centered. Hence the proof is completed.

We illustrate the above theorem in the following examples.

Example 5.2.

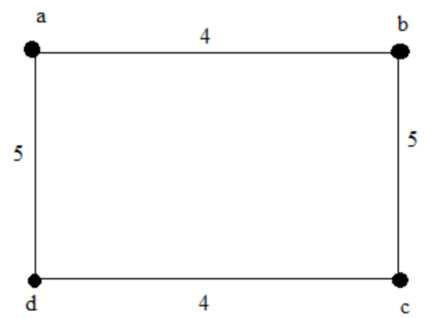

Figure 3:

The α- distance matrix and the max – max composition are given below.

$$D_\alpha = \begin{bmatrix} 0 & \infty & \infty & 5 \\ \infty & 0 & 5 & \infty \\ \infty & 5 & 0 & \infty \\ 5 & \infty & \infty & 0 \end{bmatrix} \quad D_\alpha 0 D_\alpha = \begin{bmatrix} 5 & \infty & \infty & 5 \\ \infty & 5 & 5 & \infty \\ \infty & 5 & 5 & \infty \\ 5 & \infty & \infty & 5 \end{bmatrix}$$

Clearly all the diagonal elements of the composition are same and hence G is α- self centered.

Example 5.3.

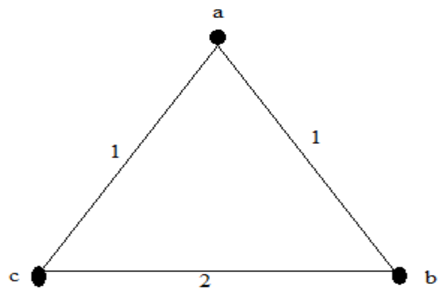

Figure 4:

The α- distance matrix and the max – max composition are given below.

$$D_\alpha = \begin{bmatrix} 0 & \infty & \infty \\ \infty & 0 & 2 \\ \infty & 2 & 0 \end{bmatrix} \qquad D_\alpha 0 D_\alpha = \begin{bmatrix} \infty & \infty & \infty \\ \infty & 2 & 2 \\ \infty & 2 & 2 \end{bmatrix}.$$

Clearly all diagonal elements in the composition are not same, and hence the graph is not α- self centered.

Remark 5.1. From the above two examples it is clear that, a partial block may or may not be α- self center.

6. The center of p-trees and p-blocks

In this section, we give a discussion about the central nodes of partial trees and partial blocks.

In the following theorem, α- central nodes of partial trees are characterized.

Theorem 6.1. *If a node of a partial tree is α- central, then it is a common node of at least two α- strong arcs.*

Proof: Let *G: (V, E, W)* be a partial tree. Then *G* has no β- strong arcs. We know that between any two nodes of a connected weighted graph *G*, there exists a strong path [12]. As *G* is independent of β- strong arcs, there exists an α- strong path between any two nodes of *G*. Let *u* be an α- central node of *G*. We want to prove that two or more α- strong arcs are incident on *u*. If possible suppose the contrary. Let there be exactly one α- strong arc, namely *e* incident on *u*. Therefore any α- strong path between *u* and any other node of *G* will contain the arc *e*. This proves that $e_\alpha(u) > r_\alpha(G)$, which is a contradiction to the fact that *u* is α- central. Therefore our assumption is wrong. Thus the proof of the theorem is completed.

Remark 6.1. If *u* is a common node of at least two α- strong arcs, then *u* is a partial cut node of *G* [12]. So from the above theorem it is clear that, if a node *u* of a partial tree *G* is α- central, then it is a partial cut node of *G*.

Remark 6.2. As partial trees are free from β- strong arcs, $e_\beta(u) = 0$ for every node u. Hence the equality $e_s(u) = e_\beta(u) + e_\alpha(u)$ is trivially true in all partial trees.

The next theorem is about the α- center of partial blocks.

Theorem 6.2. The α- center of a partial block G contains all α- strong arcs with minimum weight.

Proof: Suppose that $G: (V, E, W)$ is a partial block. Therefore G has no partial cut nodes. We know that, if a node u in a connected weighted graph is common to more than one α- strong arcs, then it is a partial cut node [12]. As G is free from partial cut nodes, at most one α- strong arc can be incident on every node of G. Thus the α- eccentricity, e_α of a node u is the weight of the α- strong arc incident on u.

So the α- radius of G, that is $r_\alpha(G)$ is the weight of the smallest α- strong arc. Hence the α- center of G, $< C_\alpha(G) >$ contains all α- strong arcs of G with minimum weight. This completes the proof of the theorem.

The next theorem helps us to find the number of connected components in the α- center of a partial block.

Theorem 6.3. Let $G: (V, E, W)$ be a partial block. If there exists a path containing all α- strong arcs of G with minimum weight alternatively, then $< C_\alpha(G) >$ will be connected.

Proof: By the previous theorem, $< C_\alpha(G) >$ consists of all α- strong arcs of G with minimum weight. Also in a partial block, not more than one α- strong arc can be incident on any node. So if there are k number of α- strong arcs present in G with minimum weight, all these arcs will be in $< C_\alpha(G) >$, moreover they are not adjacent also. Hence if we can find a path containing all α- strong arcs with minimum weight alternatively, $< C_\alpha(G) >$ will be connected. Thus the proof is completed.

Theorem 6.4. If a connected weighted graph $G: (V, E, W)$ is a partial block with k number of α- strong arcs. Then $k \leq \frac{|V|}{2}$.

Proof: Suppose that $G :(V, E, W)$ is a partial block. Then G has no partial cut nodes. Let k be the number of α- strong arcs in G. We have to prove that $k \leq \frac{|V|}{2}$. If possible suppose the contrary. Let $k > \frac{|V|}{2}$. Then there will be at least $\left\lfloor k - \frac{|V|}{2} \right\rfloor$ number of nodes with more than one α- strong arc incident on them. Clearly these nodes are partial cut nodes of G, a contradiction to the fact that G is free from partial cut nodes. So our assumption is wrong. This proves the theorem.

7. Conclusion

In this article, three new distances in weighted graphs are introduced. As reduction in strength between two nodes is more important than total disconnection of the graph, the authors made use of the connectivity concepts in defining the distances. A special focus on self centered graphs can be seen as they are applied widely. The max – max composition, which is presented in section 5 is very useful in characterizing the three types of self centered graphs. Studies and characterizations for both partial trees and partial blocks are also made.

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