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Double Layered Fuzzy Graph

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Abstract. An interesting and new type of fuzzy graph can be obtained from a fuzzy graph whose crisp graph is a cycle. In this paper, we define a new fuzzy graph named Double Layered Fuzzy Graph (DLFG) and we have discussed some of its properties using order, size, μ - complement of fuzzy graphs, etc.

Keywords: Order, Size, vertex degree, μ - Complement, strong fuzzy graph, double layered fuzzy graph.

AMS Mathematics Subject Classification (2010): 94D05

1. Introduction

Fuzzy graph theory was introduced by Azriel Rosenfeld in 1975 [5]. Though introduced recently, it has been growing fast and has numerous applications in various fields. During the same time Yeh and Bang have also introduced various concepts in connectedness with fuzzy graphs [7]. Mordeson and Peng introduced the concept of operations on fuzzy graphs, Sunitha and Vijayakumar discussed about the operations of union, join, Cartesian product and composition on two fuzzy graphs [4]. The degree of a vertex in some fuzzy graphs was discussed by Nagoorgani and Radha [6]. Nagoorgani and Malarvizhi have defined different types of fuzzy graphs and discussed its relationships with isomerism in fuzzy graphs [3]. In this paper we define double layered fuzzy graph (DLFG) or 3 – D Fuzzy graph which gives a 3-D structure in fuzzy graph theory and some of its properties were discussed.

Section two contains the basic definitions in fuzzy graphs, in section three we introduce a new fuzzy graph called a double layered fuzzy graph, section four presents the theoretical concepts of DLFG and finally we give conclusion on DLFG

2. Preliminaries

Definition 2.1. [5] A fuzzy graph G is a pair of functions G:(σ,μ) where σ is a fuzzy subset of a non empty set S and μ is a symmetric fuzzy relation on σ . The underlying crisp graph of G:(σ,μ) is denoted by $G^*:(\sigma^*,\mu^*)$

Definition 2.2. [8] Let $G:(\sigma,\mu)$ be a fuzzy graph, the order of G is defined as $O(G)=\sum_{u\in V}\sigma(u)$

Definition 2.3. [8] Let $G:(\sigma,\mu)$ be a fuzzy graph, the size of G is defined as $S(G)=\sum_{u,v\in V}\mu(u,v)$

Definition 2.4. [10] Let $G:(\sigma,\mu)$ be a fuzzy graph, the degree of a vertex u in G is defined as $d_G(u)=\sum_{\substack{v\neq u\\ v\in V}}\mu(u,v)$ and is denoted as $d_G(u)$.

Definition 2.5. [9] A fuzzy graph $G:(\sigma,\mu)$ is said to be a strong fuzzy graph if $\mu(u,v)=\sigma(u)\wedge\sigma(v)$ for all (u,v) in μ^*.

Definition 2.6. [4] Let G be a fuzzy graph, the μ - complement of G is denoted as $G^\mu:(\sigma^\mu,\mu^\mu)$ where $\sigma^*\cup\mu^*$ and $\mu^\mu(u,v)=\begin{cases}\sigma(u)\wedge\sigma(v)-\mu(u,v) \text{ if } \mu(u,v)\succ 0\\ 0 \text{ if } \mu(u,v)=0\end{cases}$

3. Double layered fuzzy graph (DLFG)
3.1. Definition

Let $G:(\sigma,\mu)$ be a fuzzy graph with the underlying crisp graph $G^*:(\sigma^*,\mu^*)$. The pair $DL(G):(\sigma_{DL},\mu_{DL})$ is defined as follows. The node set of $DL(G)$ be $\sigma^*\cup\mu^*$. The fuzzy subset σ_{DL} is defined as $\sigma_{DL}=\begin{cases}\sigma(u) \text{ if } u\in\sigma^*\\ \mu(uv) \text{ if } uv\in\mu^*\end{cases}$

The fuzzy relation μ_{DL} on $\sigma^*\cup\mu^*$ is defined as

$$\mu_{DL}=\begin{cases}\mu(uv) \text{ if } u,v\in\sigma^*\\ \mu(e_i)\wedge\mu(e_j) \text{ if the edge } e_i \text{ and } e_j \textit{ have} \text{ a node in common between them}\\ \sigma(u_i)\wedge\mu(e_i) \text{ if } u_i\in\sigma^* \text{ and } e_i\in\mu^* \text{ and each } e_i \text{ is incident with single } u_i\\ \qquad\text{either clockwise or anticlockwise.}\\ 0 \text{ otherwise}\end{cases}$$

By definition $\mu_{DL}(u,v)\le\sigma_{DL}(u)\wedge\sigma_{DL}(v)$ for all u,v in $\sigma^*\cup\mu^*$. Here μ_{DL} is a fuzzy relation on the fuzzy subset σ_{DL}. Hence the pair $DL(G):(\sigma_{DL},\mu_{DL})$ is defined as **double layered fuzzy graph (DLFG)** or **3 – D Fuzzy Graph**. We prefer to label the graph as DLFG.

Remark 3.1.1. Here the crisp graph G^* is a cycle and the above definition is applicable for n number of cycles.

Example 3.1.1. Consider the fuzzy graph G, whose crisp graph G^* is a cycle with n = 3 vertices.

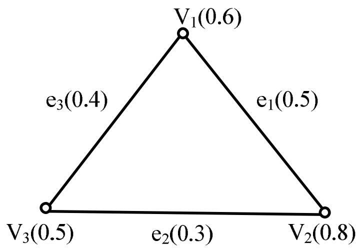

Figure 1: A Fuzzy graph G:(σ,μ)

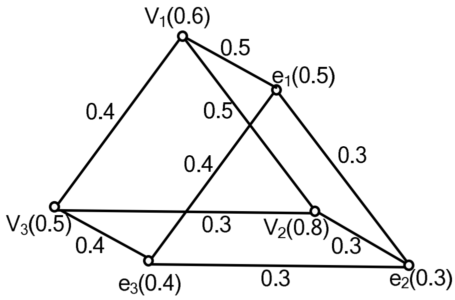

Figure 2: Double layered fuzzy graph $DL(G) = (\sigma_{DL}, \mu_{DL})$

Consider the fuzzy graph with n = 4 vertices.

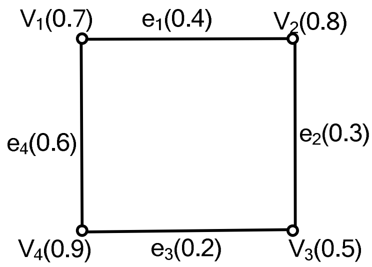

Figure 3: A Fuzzy graph G:(σ,μ)

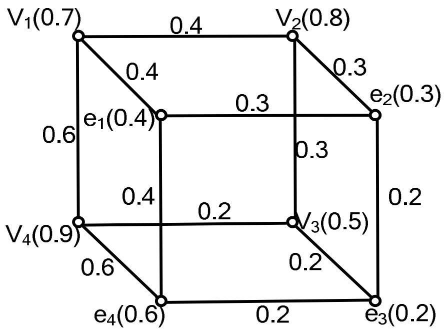

Figure 4: Double layered fuzzy graph $DL(G) = (\sigma_{DL}, \mu_{DL})$

Also consider the fuzzy graph with n = 5 vertices.

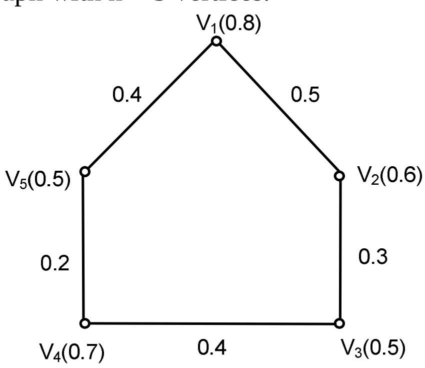

Figure 5: A Fuzzy graph G:(σ,μ)

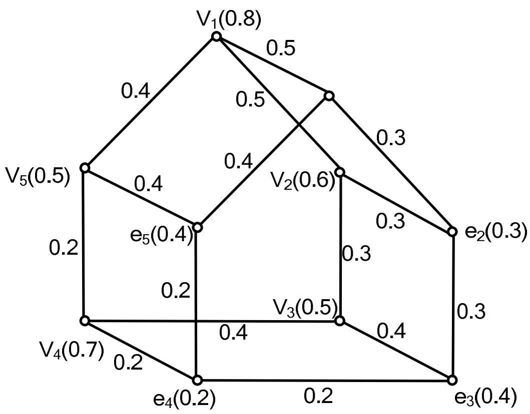

Figure 6: Double layered fuzzy graph $DL(G) = (\sigma_{DL}, \mu_{DL})$

Similarly we can get different double layered fuzzy graphs for a given fuzzy graph G, whose crisp graph is a cycle.

4. Theoretical concepts

Theorem 4.1. Order DL(G) = Order(G) + Size(G), where G is a fuzzy graph.

Proof: As the node set of DL(G) is $\sigma^* \cup \mu^*$ and the fuzzy subset σ_{DL} on $\sigma^* \cup \mu^*$ is defined as $\sigma_{DL} = \begin{cases} \sigma(u) \text{ if } u \in \sigma^* \\ \mu(uv) \text{ if } uv \in \mu^* \end{cases}$

$$\begin{aligned} \text{Order DL(G)} &= \sum_{u \in VUE} \sigma_{DL}(u) \text{ (by definition 2.2)} \\ &= \sum_{u \in V} \sigma_{DL}(u) + \sum_{u \in E} \sigma_{DL}(u) \\ &= \sum_{u \in V} \sigma(u) + \sum_{u \in E} \mu(u) \text{ (by definition of } \sigma_{DL}(u)\text{)} \\ &= \text{Order (G) + Size (G).} \end{aligned}$$

Theorem 4.2. $Size\ DL(G) = 2Size\ (G) + \sum_{e_i, e_j \in E} \mu(e_i) \wedge \mu(e_j)$, where G is a fuzzy graph and i, j ∈ N.

Proof: $Size\ DL(G) = \sum_{u,v \in VUE} \mu_{DL}(u,v)$ (by defenition 2.3)

$$= \sum_{u,v \in V} \mu_{DL}(u,v) + \sum_{e_i, e_j \in E} \mu_{DL}(e_i, e_j) + \sum_{u_i \in V, e_i \in E} \mu_{DL}(u_i, e_i)$$

(u_i is in one of the end node of e_i in the third summation)

$$= \text{size (G)} + \sum_{e_i, e_j \in E} \mu(e_i) \wedge \mu(e_j) + \sum_{u_i \in V, e_i \in E} \sigma(u_i) \wedge \mu(e_i)$$

$$= \text{size (G)} + \sum_{e_i, e_j \in E} \mu(e_i) \wedge \mu(e_j) + \sum_{e \in E} \mu(e)$$

Since in the third summation, we are considering only one vertex in each edge either clockwise or anticlockwise direction, its membership value is less than the value of the vertices.

$$\begin{aligned} Size\ DL(G) &= \text{size (G)} + \sum_{e_i, e_j \in E} \mu(e_i) \wedge \mu(e_j) + size \text{ (G)} \\ &= \text{2size (G)} + \sum_{e_i, e_j \in E} \mu(e_i) \wedge \mu(e_j) \end{aligned}$$

Theorem 4.3. $|E_{DL}(G)| = 2|E(G)| + |E(L(G))|$

Proof: Each edge in G is replaced by a new vertex in DL(G). The pair of adjacent edges in G contributes a new edge in DL(G) and each edge in G is neighbourhood of only one vertex either clockwise or anticlockwise. Also the vertex which are adjacent in G is also adjacent in DL(G).

Thus, we have $\left|E_{DL}(G)\right| = 2\left|E(G)\right| +$ no of pairwise adjacent edges in G^*

$$= \left|E(G)\right| + \left|E(L(G))\right|.$$

Theorem 4.4. If G is a strong fuzzy graph then DL(G) is also a strong fuzzy graph.

Proof: By the definition of strong fuzzy graph we have $\mu(u,v) = \sigma(u) \wedge \sigma(v)$ for all (u,v) in μ^*.

Assume G is a strong fuzzy graph. we need to prove DL(G) is a strong graph. Consider an edge (u,v) in DL(G). Then

$$\mu_{DL} = \begin{cases} \mu(uv) \text{ if } u,v \in \sigma^* \\ \mu(e_i) \wedge \mu(e_j) \text{ if the edge } e_i \text{ and } e_j \textit{ have} \text{ a node in common between them} \\ \sigma(u_i) \wedge \mu(e_i) \text{ if } u_i \in \sigma^* \text{ and } e_i \in \mu^* \text{ and} \\ \qquad \text{each } e_i \text{ is incident with single } u_i \text{ either clockwise or anticlockwise.} \end{cases}$$

Case i: It is trivial from our assumption that G is a strong graph. Thus $\mu(u,v) = \sigma(u) \wedge \sigma(v)$ for all (u,v) in μ_{DL}^*.

Case ii: If $\mu_{DL}(u,v) = \mu(e_i) \wedge \mu(e_j)$ if $u = e_i,\ v = e_j \in \mu^*$ are adjacent in in G^* then $\mu_{DL}(u,v) = \sigma_{DL}(e_i) \wedge \sigma_{DL}(e_j)$ (by the definition of σ_{DL})

Case iii: If $\mu_{DL}(u,v) = \sigma(u_i) \wedge \mu(e_i)$ and each e_i is incident to single u_i in G^*. Then $\mu_{DL}(u,v) = \sigma_{DL}(u_i) \wedge \sigma_{DL}(e_j)$ (by the definition of σ_{DL})

Hence if G is a strong fuzzy graph, by case i, ii and iii we have $\mu_{DL}(u,v) = \sigma_{DL}(u) \wedge \sigma_{DL}(v)$ for all (u,v) in μ_{DL}^*.

Theorem 4.5. Let G be a fuzzy graph then

$$d_{DL(G)}(u) = \begin{cases} d_G(u) \ + \ (\sigma(u_i) \wedge \mu(e_i)) \text{ if } u \in \sigma^* \\ \sum_{e_i \in \mu^*} \mu(e_i) \wedge \mu(e_j) \ + \ (\sigma(u_i) \wedge \mu(e_i)) \text{ if } u \in \mu^* \end{cases}$$

Proof: By definition 1.10, we have $d_G(u) = \sum_{\substack{v \neq u \\ v \in V}} \mu(u,v)$

Case i: Let $u \in \sigma^*$, then

$$d_{DL(G)}(u) = \sum_{v \in \sigma^*} \mu_{DL}(u,v) + \mu_{DL}(u_i, e_i)$$

$$= \sum_{v \in \sigma^*} \mu(u,v) + \sigma(u_i) \wedge \mu(e_i) \ (\because \text{in the first summation the vertices}$$

which are adjacent in G is also adjacent in DLFG)

$$= d_G(u) + \sigma(\mathrm{u}_i) \wedge \mu(e_i)$$

Case ii: Let $u \in \mu^*$, then

$$d_{DL(G)}(u) = \sum_{e_i, e_j \in \mu^*} \mu_{DL}(e_i, e_j) + \mu_{DL}(u_i, e_i) = \sum_{e_i, e_j \in \mu^*} \mu(e_i) \wedge \mu(e_j) + \sigma(\mathrm{u}_i) \wedge \mu(e_i)$$

Remark 4.1. The μ-complement of DL(G) is the fuzzy graph G with its edge membership value is always less than G.

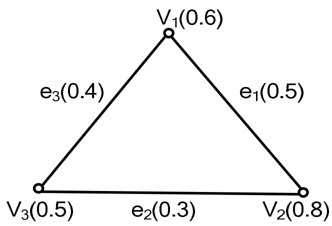

Figure 7: A Fuzzy graph G:(σ,μ)

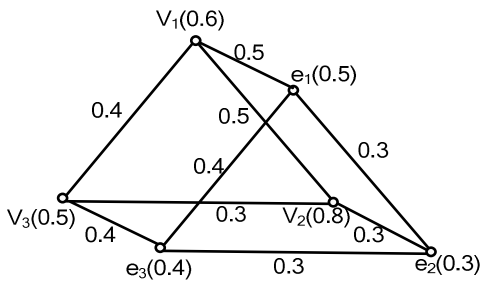

Figure 8: Double layered fuzzy graph $DL(G) = (\sigma_{DL}, \mu_{DL})$

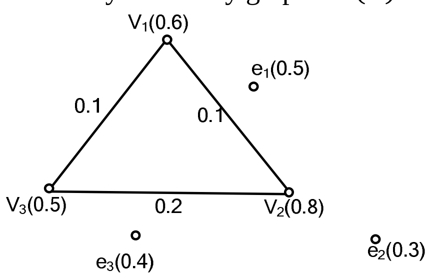

Figure 9: Complement of double layered fuzzy graph $DL(G^{\mu}) = (\sigma_{DL}^{\mu}, \mu_{DL}^{\mu})$

Remark 4.2. If G is a strong fuzzy graph then the μ - complement of DL(G) is isolated vertices. Thus $d_{DL}(u) = 0$ for all u in $\sigma^* \cup \mu^*$.

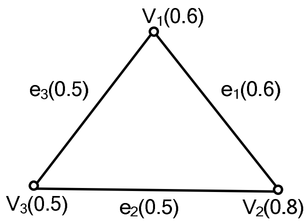

Figure 10: A Fuzzy graph G:(σ,μ)

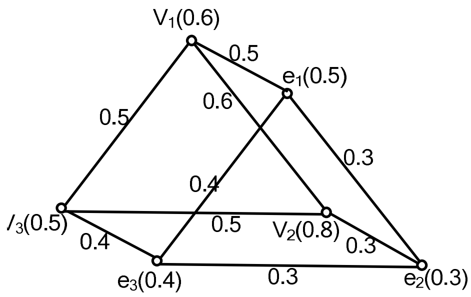

Figure 11: Double layered fuzzy graph $DL(G) = (\sigma_{DL}, \mu_{DL})$

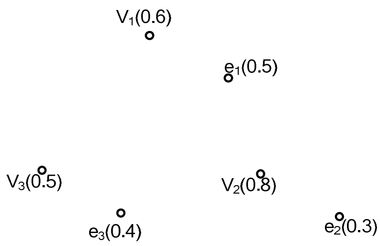

Figure 12: Complement of double layered fuzzy graph $DL(G^{\mu}) = (\sigma_{DL}^{\mu}, \mu_{DL}^{\mu})$

5. Conclusion

In this paper, we have defined a new fuzzy graph namely double layered fuzzy graph and illustrated with some examples. Further structures can be developed by increasing number of vertices and embedded cycles. These structural patterns with the inherent cycles give us an indicator to prove further into different patterns in networking models or classification tools.

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