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Strong Chromatic Number of Fuzzy Graphs

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Abstract. Many of the real life problems are solved using graph coloring techniques. This paper is an attempt to define coloring in a fuzzy graph based on strong arcs. The strong chromatic number of complete fuzzy graph and fuzzy tree are obtained. Solutions to traffic light problems using strong coloring is suggested.

Keywords: Fuzzy graph, strong arcs, strong coloring, strong chromatic number.

AMS Mathematics Subject Classification (2010): 05C72

1. Introduction

Fuzzy graphs were introduced by Rosenfeld [4], ten years after Zadeh's landmark paper fuzzy sets [13]. Fuzzy graph theory is now finding numerous applications in modern science and technology especially in the fields of information theory, neural networks, expert systems, cluster analysis, medical diagnosis, control theory, etc. Fuzzy modeling is an essential tool in all branches of science, engineering and medicine. Fuzzy models give more precision, flexibility and compatibility to the system when compared to the classic models [14,15].

Rosenfeld has obtained the fuzzy analogues of several basic graph-theoretic concepts like bridges, paths, cycles, trees and connectedness and established some of their properties [4]. Fuzzy trees were characterized by Sunitha and Vijayakumar [7]. The authors have characterized fuzzy trees using its unique maximum spanning tree. A sufficient condition for a node to be a fuzzy cutnode is also established in [7]. Center problems in fuzzy graphs [9], blocks in fuzzy graphs [8] and properties of self complementary fuzzy graphs [10] were also studied by the same authors. In [2], the authors have defined the concepts of strong arcs and strong paths. The authors have established the existence of a strong path between any two nodes of a fuzzy graph.

Depending on the strength of an arc, in [12], the authors classify strong arcs into two types namely α −strong and β −strong and introduce two other types of arcs in fuzzy graphs which are not strong and are termed as δ and δ^* arcs. In [11], Sunitha and Sunil has made a study on the different aspects of fuzzy graph theory developed so far.

In this paper, we define coloring of fuzzy graph based on strong arcs since the δ− arcs play negligible role in any network. Results on strong chromatic number of complete fuzzy graphs and fuzzy trees are obtained. Also the application of strong coloring in traffic lights problem is discussed.

2. Preliminaries

The following basic definitions are taken from [5]. A fuzzy graph is an ordered triple G : (V, σ, μ) where V is a set of vertices $\{u_1, u_2 \ldots u_n\}$, σ is a fuzzy subset of V i.e, $\sigma : V \rightarrow [0,1]$ and is denoted by $\sigma = \{(u_1, \sigma(u_1)), (u_2, \sigma(u_2)) \ldots (u_n, \sigma(u_n))\}$ and μ is a fuzzy relation on σ, i.e $\mu(u,v) \leq \sigma(u) \wedge \sigma(v)\ \forall\, u, v \in V$. We consider fuzzy graph G with no loops and assume that V is finite and nonempty, μ is reflexive $(i.e., \mu(u,u) = \sigma(u), \forall u \in V)$ and symmetric $(i.e., \mu(u,v) = \mu(v,u), \forall (u,v) \in V \times V)$. In all the examples σ is chosen suitably. Also, we denote the under-lying crisp graph of G by $G^* : (\sigma^*, \mu^*)$ where $\sigma^* = \{u \in V : \sigma(u) > 0\}$ *and* $\mu^* = \{(u,v) \in V \times V : \mu(u,v) > 0\}$. Throughout we assume that $\sigma^* = V$.

A path P of length n is a sequence of distinct nodes u_0 , u_1, $u_2 \ldots u_n$ such that $\mu(u_{i-1}, u_i) > 0$, $i = 1, 2, \ldots \ldots, n$ and the degree of membership of a weakest arc in P is defined as the strength of P. If $u_0 = u_n$ and $n \geq 3$ then P is called a cycle. P is called a fuzzy cycle if it contains more than one weakest arc.

The strength of a cycle is the strength of the weakest arc in it. A fuzzy cycle of length n is denoted by C_n. The maximum strength among all paths from u to v is denoted by $CONN_G(u,v)$. A fuzzy graph $G : (\sigma, \mu)$ is connected if for every u, v in V , $CONN_G(u,v) > 0$. A connected fuzzy graph G : (V, σ, μ) is a fuzzy tree if it has a fuzzy spanning subgraph F : (V, σ, ϑ), which is a tree where for all arcs (u, v) not in F there exists a path from u to v in F whose strength is more than μ(u, v). A maximum spanning tree of a connected fuzzy graph G : (V, σ, μ) is a fuzzy spanning subgraph T : (V, σ, ϑ), such that T^* is a tree, and for which $\sum_{u \neq v} \vartheta(u,v)$ is maximum. Also fuzzy graph is a fuzzy tree if and only if it has a unique maximum spanning tree [7]. A fuzzy graph G is said to be complete if $\mu(u,v) = \sigma(u) \wedge \sigma(v), \forall u, v \in V$.

An arc (u,v) is said to be strong if $\mu(u,v) \geq CONN_{G\setminus(u,v)}(u,v)$ [2]. Strong arcs are again classified as α − strong and β − strong arcs [12]. An arc (u,v) is said to be α − strong if $\mu(u,v) > CONN_{G\setminus(u,v)}(u,v)$ and if $\mu(u,v) = CONN_{G\setminus(u,v)}(u,v)$, it is said to be β − strong . An arc (u, v) is said to be δ− arc if $\mu(u,v) < CONN_{G\setminus(u,v)}(u,v)$.

The concept of chromatic number of fuzzy graph was introduced by Munoz et. al.[6]. The authors considered fuzzy graphs with crisp vertex set i.e fuzzy graphs for which $\sigma(x) = 1, \forall\, x \in V$ and edges with membership degree in [0, 1].

Definition 2.1. [6] If G : (V, σ, μ) is such a fuzzy graph where V = {1, 2, 3...n} and μ is a fuzzy number on the set of all subsets of V × V . Assume $I = A \cup \{0\}$ where $A = \{\alpha_1 < \alpha_2 < \ldots < \alpha_k\}$ is the fundamental set (level set) of G. For each α ∈ I, G_α denote the crisp graph $G_\alpha = (V, E_\alpha)$ where $E_\alpha = \{ij / 1 \leq i < j \leq n,\ \mu(i,j) \geq \alpha\}$ and $\chi_\alpha = \chi(G_\alpha)$ denote the chromatic number of crisp graph G_α. By this definition the chromatic number of fuzzy graph G is the fuzzy number $\chi(G) = \{(i, \nu(i)) / i \in X\}$ where $\nu(i) = max\,\{\alpha \in I / i \in A_\alpha\}$ and $A_\alpha = \{1, 2, 3..., \chi_\alpha\}$.

Definition 2.2. Later Eslahchi and Onagh [3] introduced fuzzy coloring of fuzzy graphs

as follows. A family $\Gamma = \{\gamma_1, \gamma_2, \ldots \gamma_k\}$ of fuzzy sets on a set V is called a k-fuzzy coloring of G = (V, σ, μ) if
(i) $\vee \Gamma = \sigma$,
(ii) $\gamma_i \wedge \gamma_j = 0$,
(iii) for every strong edge (x, y) (i.e $\mu(x, y) > 0$) of G, min $\{\gamma_i(x), \gamma_i(y)\} = 0$ $(1 \leq i \leq k)$.
(The strong arc here means $\mu(x, y) > 0$ according to the definition. But in fuzzy graphs it only means adjacency).

The minimum number k for which there exists a k-fuzzy coloring is called the fuzzy chromatic number of G, denoted as $\chi^f(G)$.

Example 2.3. Consider the following fuzzy graph [Fig. 1]

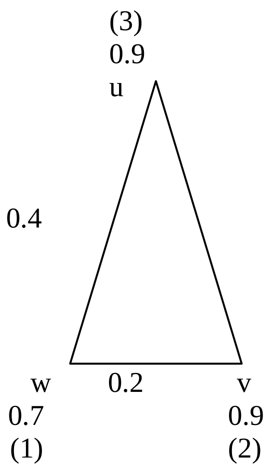

Figure 1:
The numbers inside the brackets represents colors assigned to the vertices.

Here $\chi^f(G) = 3$.

3. Strong chromatic number of fuzzy graphs
Coloring of graphs play a vital role in network problems. In any network, modeled as a fuzzy graph, the role of δ− arc is negligible, as the flow is minimum along δ− arc and there is an alternate strong path (maximum flow) between the corresponding nodes. Hence strong arcs are more significant in networks. This motivated us in defining a new coloring of fuzzy graphs based on strong arcs.

Definition 3.1. Consider a fuzzy graph G : (V, σ, μ).

The coloring $C : V(G) \to \mathbb{N}$ (where $\mathbb{N}$ is the set of all positive integers) such that $C(u) \neq C(v)$ if (u, v) is a strong arc (α − strong and β − strong) in G is called strong coloring.

A fuzzy graph G is k-strong colorable if there exists a strong coloring of G from a set of k colors.

The minimum number k for which G is k-strong colorable is called strong chromatic number of G denoted by $\chi_s(G)$.

Note that the end nodes of a δ− arc can be assigned the same color in strong coloring.

In Fig. 1, we find that arc (u, v) is α− strong since $\mu(u,v) = 0.5 > CONN_{G\setminus(u,v)}(u,v) = 0.2$. Similarly arc (u, w) is α−strong since $\mu(u,w) = 0.4 > CONN_{G\setminus(u,w)}(u,w) = 0.2$. But $CONN_{G\setminus(w,v)}(w,v) = 0.4 > \mu(w,v) = 0.2$ and hence (w,v) is a δ− arc. Hence strong coloring gives colors 1 for u and 2 for both w and v. Thus $\chi_s(G)$= 2.

The results on strong coloring of complete fuzzy graphs and fuzzy tree are given as follows.

Proposition 3.1. For a fuzzy graph G : (V,σ, μ), $\chi_s(G) = \chi^f(G)$, if G is complete or is a fuzzy cycle.

Proof: $\chi_s(G) = \chi^f(G)$ = p , if G is complete with p vertices, since there are no δ− arcs in a complete fuzzy graph [12] and if G is a fuzzy cycle on p vertices, then G contains only strong arcs [12]. Thus $\chi_s(G) = \chi^f(G)$ = 2 , when p is even and $\chi_s(G) = \chi^f(G)$ = 3, when p is odd.

The converse of the above proposition is not true.

Example 3.2. Consider the following fuzzy graph [Fig. 2]

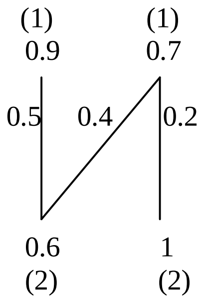

Figure 2:

Here $\chi_s(G) = \chi^f(G)$ = 2, but the fuzzy graph in Fig. 2 is neither complete nor is a fuzzy cycle.

Proposition 3.2. If G is a fuzzy tree, then $\chi_s(G) = \chi^f(T)$ where T is the maximum spanning tree of G.

Proof: Note that an arc (x,y) in a fuzzy tree G is strong ⇔ (x,y) is an arc in T [2]. Hence strong coloring of G is exactly same as coloring of T and hence $\chi_s(G) = \chi^f(T)$.

Example 3.3. Consider the following fuzzy tree G in Fig.3(a) and its maximum spanning tree T in Fig.3(b)

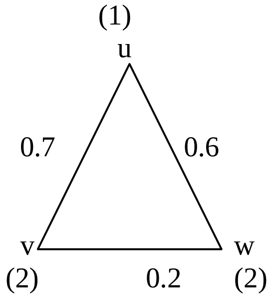

Figure 3(a):

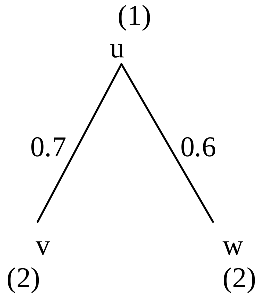

Figure 3(b):

Then $\chi_s(G)= \chi^f\ (T) = 2$.

4. Application

In this section we suggest solutions of two traffic lights problems using strong chromatic number.

Illustration 1. Consider a traffic problem as in the figure 4. In this problem we assume there is no right turn. Here D is having heavy traffic, B with medium and A and C, both have low traffic.

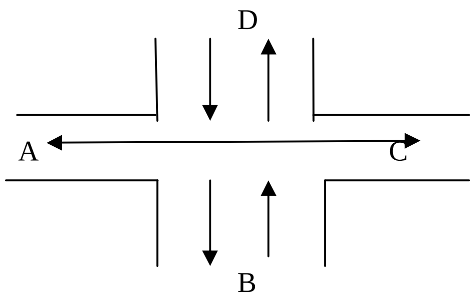

Figure 4:

Fuzzy Graph Modeling:

The corresponding fuzzy graph is constructed as follows. The nodes represent the four roads A, B, C and D. Two nodes are adjacent if they have intersecting traffic flows.

Membership Values:

Depending on the number of vehicles passing through each road, the corresponding node is given membership value say 0.9, 0.5 and 0.1 for high, medium and low traffic respectively which represents the intensity of traffic along the roads. The edge strength denotes the probability of occurrence of accidents. D has heavy traffic flow and B has medium flow. Hence the probability of collisions is medium and thus arc BD has membership value 0.5. A and C both have low traffic and hence probability of accidents is low. Hence AC is assigned 0.1 strength. Similarly strengths are assigned to the other arcs as in Fig.5.

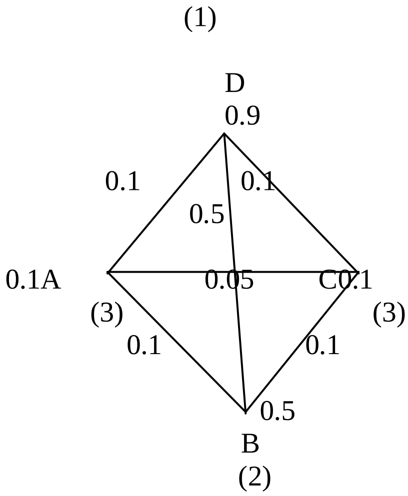

Figure 5:

<u>Types of arcs:</u>

AC is a $\delta -$ arc while all others are strong arcs.

<u>Strong Coloring:</u>
The numbers inside parenthesis represents colors of vertices.
The strong chromatic number $\chi_s(G)$= 3.

<u>Solution:</u> The number of traffic lights required = 3.

The two roads B and D cannot have same clearance signal. Also if B or D is having green signal, traffic along all other roads should be stopped. But A and C can have yellow signal at the same time. Thus the number of traffic lights required is exactly equal to strong chromatic number of corresponding fuzzy graph.

Next consider another traffic problem as follows.

Illustration 2. Consider the following traffic problem (Fig.6) in which each arrow represents the direction of flow of vehicles. We suggest a solution of this problem based on strong coloring.

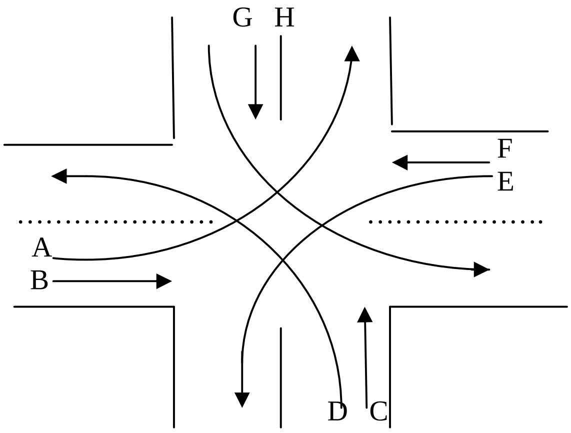

Figure 6:

<u>Fuzzy Graph Modeling:</u>

The problem is modeled as a fuzzy graph (Fig.7) as follows. The nodes are given membership values say 0.9, 0.5 and 0.1 for high, medium and low traffic respectively as in illustration 1. In [1], the authors have assigned "medium" flow along an edge with end nodes having "high" and "low" flow. But for fuzzy graphs $\mu(x,y) \leq \sigma(x) \wedge \sigma(y) \forall x,y \in V$. Hence we have considered the edge membership values as the probability of occurrence of accidents as in the above illustration. Consider arc BC. B has strength 0.9 and C has strength 0.5. Hence the probability of collisions is 0.5, minimum of the two strengths. Thus the arcs AC, BC, CE, BE, AF, AG and GF are assigned a value of 0.5 as strength. All other arcs are assigned strength 0.1 since one of their end nodes has strength 0.1 or low traffic flow.

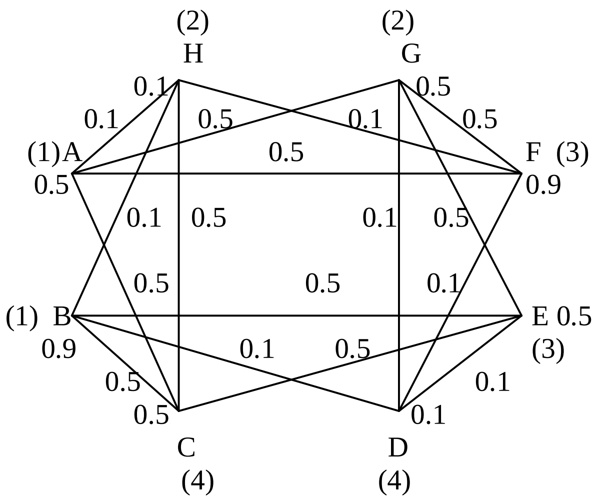

Figure 7:

Types of arcs:
All the arcs are strong arcs. There are no δ - arcs.

Strong Coloring:

The numbers inside parenthesis represents coloring of vertices. The strong chromatic number $\chi_s(G)$= 4.

Solution: The number of traffic lights for the problem = 4. In this traffic flow, there are no δ - arcs and no two routes having traffic intersections can be allowed to have traffic flow simultaneously. Hence the number of traffic lights required is 4.

5. Conclusion

The modeling of real life situations using fuzzy graphs is a widely investigated area of current research. This paper is an attempt to solve network flow problems using the concept of strong coloring. The significance of strong arcs in networks is studied. We try to extend our concept to other application domains.

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