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Neighborhood Sets and Neighborhood Polynomials of Cycles

J.Joseline Manora [†] **and *I.Paulraj Jayasimman*** [‡]

[†] PG and Research Department of Mathematics, T.B.M.L College, Porayar-609 307 Tamilnadu, India. e-mail:joseline _ manora@yahoo.co.in

[‡] Department of Mathematics, Indira Institute of Engineering and Technology, Pandur-631 203, Tamilnadu, India. e-mail: ipjayasimman@gmail.com

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Abstract. A set S of vertices in a graph G is a neighborhood set of G if $G=\bigcup_{v\in S}\langle N[v]\rangle$, where $\langle N[v]\rangle$ is the subgraph of G induced by v and all vertices adjcent to v. The neighborhood number $n_0(G)$ of G is the minimum number of vertices in a neighborhood of G Let C_n^i be the family of neighborhood sets of a cycle C_n with cardinality i. In this paper we construct family of neighborhood sets of cycles C_n^i and obtain a recursive formula for $n_0(C_n,i)=|C_n^i|$. In this paper, we obtain some properties of neighborhood sets and polynomials of Cycles.

Keywords: Neighborhood set, neighborhood number and neighborhood polynomials

AMS Mathematics Subject Classification (2010): 05C69

1. Introduction

Let G be the simple graph with vertex set $V=\{v_1,v_2,...,v_n\}$ and edge set $E=\{e_1,e_2,...,e_m\}$. A neighborhood set $S\subseteq V$ is a neighborhood set of G if $G=\bigcup_{v\in S}\langle N[v]\rangle$, where $\langle N[v]\rangle$ is induced subgraph of G. The neighborhood number $n_0(G)$.Let C_n^i be the family of neighborhood sets of a cycles C_n^i with cardinality i and let $n_0(C_n,i)=|C_n^i|$ and The polynomials are $N_0(C_n,i)=\sum_{i=\lceil\frac{n}{2}\rceil}^{n} n_0(C_n,i)x^i$ the polynomials of cycle.

In the next section we construct the families of neighborhood sets of C_n with cardinality i by the families of neighborhood sets of C_{n-1} and C_{n-2} with cardinality $i-1$ and also we investigate the neighborhood polynomial of cycle.

Definition 1.1. [3] Let $N(G,i)$ be the family of neighborhood sets of a graph G with cardinality i and let $n(G,i)=|N(G,i)|$. Then the neighborhood sets polynomial $N(G,x)$ of G is defined as $N(G,i)=\sum_{i=n_0}^{|V(G)|} n(G,i)x^i$ where $n_0(G)$ is the neighborhood number of G.

Theorem 1.2. [3] If a graph G consist of m components $G_1, G2, ..., Gm$ then $N(G,x)=\prod_{i=1}^{m} N(G_i,x)$.

Theorem 1.3. [3] Let G_1 and G_2 be connected graphs of order p_1 and p_2 respectively. Then $N(G_1 \vee G_2)=((1+x)^{p_1}-1)((1+x)^{p_2}-1)+N(G_1,x)+N(G_2,x)$.

2. Main results

2.1. Neighborhood sets of cycles

Let $C_n, n \geq 3$ be the cycle with n vertices $V(C_n)=n$ and *E(Cn)*={(1,2),(2,3),…,(*n-1*,*n*), (*n*,*1*)}. Let C_n^i be the neighborhood sets of C_n with cardinality i. We investigate neighborhood sets of cycles. Every cycle C_n consist a simple path. We need the following lemma to prove our main results in this section:

Lemma 2.1. *The following properties hold for cycles*

(i) $n_0(C_n)=\lceil \frac{n}{2} \rceil$

(ii) $C_j^i=\phi$ if and only if $i>j$ or $i<\lceil \frac{n}{2} \rceil$.

(iii) If a graph G contains a simple pathe of length $2k-1$, then every neighborhood set of G must contain at least k vertices of the path.

To find a dominating set of C_n with cardinality i, we do not need to consider the neighborhood sets of C_{n-3} and C_{n-4} with cardinality $i-1$. Therefore, we need to consider C_{n-1}^{i-1} and C_{n-2}^{i-1}. The families of these neighborhood sets can be empty or otherwise. Thus, we have four combinations of whether the two families are empty or not. The following two combinations are not possible $C_{n-1}^{i-1}=\phi$ then $C_{n-2}^{i-1}=\phi$ and $C_{n-1}^{i-1}=C_{n-2}^{i-1}=\phi$ then $C_n^i=\phi$, since $i=\lceil \frac{n}{2} \rceil$. Thus we consider two combinations or cases.

Lemma 2.2. *If* $Y \in C_{n-3}^{i-1}$ *and there exist* $x \in [n]$ *such that* $Y \cup \{x\} \in C_n^i$ *then* $Y \in C_{n-2}^{i-1}$.

Proof: Suppose that $Y \notin C_{n-2}^{i-1}$ since $Y \in C_{n-3}^{i-1}$ Y contains at least one vertex labled $n-3$ or $n-4$. If $n-3 \in Y$, then $Y \in C_{n-2}^{i-1}$ a contradiction. Hence $n-4 \in Y$ but then in this case $Y \cup x \notin C_n^i$ for every $x \in [n]$ also a contradiction.

Example 2.3. *consider* $C_n^i = C_7^4$, *Let* $Y = \{1,2,4\} \in C_4^3$ *then* $Y \cup \{x\} = \{1,2,4,6\} \in C_7^4$ $x \in [7]$ *then* $Y \in C_5^3$. *Suppose* $Y \notin C_5^3$ *,since* $Y \in C_4^3$ *If* $Y = \{1,2,3\}$ *at least one vertex labled* $n-4$ *(or)* $n-3$ *it imply that* $\{1,2,4\} \in C_5^3$ *it is a contradiction. In this case* $Y \cup \{x\} \notin C_7^4$.

Lemma 2.4. (i) *If* $C_{n-1}^{i-1} = \phi$ *then* $C_{n-2}^{i-1} = \phi$

(ii) If $C_{n-1}^{i-1} = C_{n-2}^{i-1} = \phi$ then $C_n^i = \phi$

Lemma 2.5. *Suppose that* $C_n^i \neq \phi$*, then we have*

(i) $C_{n-1}^{i-1} = \phi$ then $C_{n-2}^{i-1} \neq \phi$ if and only if $i = \lceil \frac{n}{2} \rceil$ or $n = 2k$ and $i = k$ for some $k \in N$

(ii) $C_{n-2}^{i-1} = \phi$ and $c^{i-1}n-1 \neq \phi$ if and only if $i = n$

(iii) $C_{n-1}^{i-1} \neq \phi$ then $C_{n-2}^{i-1} \neq \phi$ if and only if $n = 2k+1$ and $i = k+1$

Proof. (i) Since $C_{n-1}^{i-1} = \phi$ by lemma 2.1(ii) we have $i-1 > n-1$ or $i-1 < \lceil \frac{n-1}{2} \rceil$. If $i-1 > n-1$ then $i > n$ by lemma 2.1(ii), $C_n^i = \phi$ a contradiction. So we have $i < \lceil \frac{n-1}{2} \rceil + 1$ and since $C_n^i \neq \phi$, together we have $\lceil \frac{n}{2} \rceil \le i < \lceil \frac{n-1}{2} \rceil + 1$ which gives is $n = 2k$ and $i = k$ for some $k \in N$ so we have $C_{n-1}^{i-1} = \phi$ and $C_{n-2}^{i-1} \neq \phi$.

($\Leftarrow$) If $n = 2k$ and $i = k$ for some $k \in N$ then by Lemma 2.1 (ii), we have $C_{n-1}^{i-1} = \phi$ then $C_{n-2}^{i-1} \neq \phi$.

(ii) Since $C_{n-2}^{i-1} = \phi$ by lemma2.1(ii) $i-1 > n-2$ or $i-1 < \lceil \frac{n-2}{2} \rceil$. If $i-1 < \lceil \frac{n-2}{2} \rceil$ then $i-1 < \lceil \frac{n-1}{2} \rceil$ and hence $C_{n-1}^{i-1} = \phi$ a contradiction, so we must have $i > n-1$. Since $C_{n-1}^{i-1} \neq \phi$ we have $i-1 \le n-1$. Therefore we have $i = n$.

($\Leftarrow$) If $i = n$ then by lemma 2.1 (ii), $C_{n-2}^{i-1} = \phi$ and $C_{n-1}^{i-1} \neq \phi$.

(iii) Suppose $C_{n-1}^{i-1} = \phi$ by lemma we have $i-1 > n-1$ or $i-1 < \lceil \frac{n-1}{2} \rceil$. If $i-1 > n-1$ then $i-1 > n-2$ by lemma 2.1(ii) $C_{n-1}^{i-1} = C_{n-2}^{i-1} = \phi$ a contradiction. So we have $i < \lceil \frac{n-1}{2} \rceil + 1$ also we have $i < \lceil \frac{n-2}{2} \rceil + 1$ because $C_{n-2}^{i-1} \neq \phi$. Hence we have $\lceil \frac{n-2}{2} \rceil + 1 \leq i < \lceil \frac{n-1}{2} \rceil + 1$ therefore $n = 2k+1$ and $i = k+1 = \lceil \frac{2k+1}{2} \rceil$ for some $k \in N$. If $n = 2k+1$ and $i = k+1$ for some $k \in N$ then by lemma 2.1(ii) $C_{n-1}^{i-1} = C_{2k+1}^{k} \neq \phi$.

The following theorem construct the families of dominating sets of C_n.

Theorem 2.6. *For every* $n \geq 4$ *and* $i \geq \lceil \frac{n}{2} \rceil$,

(i) If $C_{n-1}^{i-1} = \phi$ and $C_{n-2}^{i-1} \neq \phi$ then $C_n^i = \{(1,3,5,...n-1)(2,4,6,...n)\}$.

(ii) If $C_{n-1}^{i-1} \neq \phi$ and $C_{n-2}^{i-1} \neq \phi$ and $C_{n-2}^{i-1} = C_{n-2}^{n-2}$. Then $C_n^i = \{[n] - x/x \in [n]\}$.

(or) $C_n^i = \{\{X_1 \cup \{n\}/X_1 \in C_{n-1}^{i-1}\}, \{X_2 \cup \{n-1\}/X_2 \in C_{n-2}^{i-1} = C_{n-2}^{n-2}\}\}$

(iii) If $C_{n-1}^{i-1} \neq \phi$ and $C_{n-2}^{i-1} = \phi$, then $C_n^i = [n]$.

(iv) If $C_{n-1}^{i-1} \neq \phi$ and $C_{n-2}^{i-1} \neq \phi$ then

$C_n^i = \{\{X_1 \cup \{n\}/X_1 \in C_{n-1}^{i-1}\}, \{X_2 \cup \{n-1\}/1 \in X_2\}, \{X_2 \cup \{n\}/1 \notin X_2\}\}$.

Proof. (i) $C_{n-1}^{i-1} = \phi$ and $C_{n-2}^{i-1} \neq \phi$. By lemma 2.4 (i), $n = 2k$ for some $K \in N$ therefore, $C_n^i = C_n^{\lceil n/2 \rceil} = \{\{1,3,5,...,n-1\}, \{2,4,6,...n\}\}$.

(ii) $C_{n-1}^{i-1} \neq \phi$ and $C_{n-2}^{i-1} = \phi$. By lemma 2.4 (ii), $i = n$, therefore $C_n^i = C_n^n = [n]$.

(iii) By lemma 2.4 (iii), $i = n-1$. Therefore $C_n^i = C_n^{n-1} = \{[n] - x/x \in [n]\}$.

(iv) Since $C_{n-1}^{i-1} \neq \phi$, $C_{n-2}^{i-1} \neq \phi$. Suppose that $X_1 \in C_{n-1}^{i-1}$, then $X_1 \cup \{n\} \in C_n^i$.

Let $Y_1 = \{X_1 \cup \{n\}/X \in C_{n-1}^{i-1}\} \subseteq C_n^i$. Let $X_2 \in C_{n-1}^{i-1}$, we denote $\{\{X_2 \cup \{n\} if 1 \in X_2\}, \{X_2 \cup \{n-1\} \, if \, 1 \in X_2\}\}$ by Y_2. By Lemma 2.1 (iii), at least one of the vertices labled $n-2$ or 1 is in X_2, If $1 \in X_2$ then $X_2 \cup \{n-1\} \in C_n^i$, otherwise $X_2 \cup \{n\} \in C_n^i$. Therefore $Y_2 \subseteq Y$. Hence we have proved $Y_1 \cup Y_2 \subseteq C_n^i$.

Example 2.7. *Consider* C_6 *with* $V(C_6) = [6]$. *We use the theorem to construct* C_6^i *for* $3 \leq i \leq 6$.

Since $C_{n-1}^{i-1} = C_5^2 = \phi$ and $C_6^3 = \{\{1,3,5\}\{2,4,6\}\}$ By theorem 2.7 (i)

$C_5^5 = \{[5]\}, C_4^5 = \phi$, we get, $C_6^6 = \{[6]\}$ $C_6^5 = \{[6] - \{x\}/x \in [6]\}$ = $\{1,2,3,4,6\}, \{1,2,3,5,6\}, \{1,3,4,5,6\}, \{2,3,4,5,6\}, \{1,2,4,5,6\}, \{1,2,3,4,5\}$ and for the construction of C_6^4 by the theorem 2.6 (iv) , $C_{n-1}^{i-1} \neq \phi$ and $C_{n-2}^{i-1} \neq \phi$ and $n-2 \neq i-1$, then $C_n^i = X_1 \cup \{n\}/X_1 \in C_{n-1}^{i-1}, X_2 \cup \{n-1\}/1 \in X_2, X_2 \cup \{n\}/1 \notin X_2$

Therefore $C_5^3 = \{\{1,2,4\}, \{1,3,4\}\{1,3,5\}\{2,3,5\}\{2,4,5\}\}$.

$C_4^3 = \{\{1,2,3\},\{1,2,4\}\{1,3,4\}\{2,3,4\}\}$

$Y_1 = X_1 \cup \{n\}/X_1 \in C_5^3 = \{1,2,4,6\},\{1,3,4,6\},\{1,3,5,6\},\{2,3,5,6\},\{2,4,5,6\}$

$Y_2 = X_2 \cup \{n-1\}/1 \in X_2, X_2 \in C_4^3 = \{1,2,3,5\},\{1,2,4,5\},\{1,3,4,5\}$

$Y_3 = X_2 \cup \{n\}/1 \notin X_2, X_2 \in C_4^3 = \{2,3,4,6\}$

Finally, $C_6^4 = \{1,2,4,6\},\{1,3,4,6\},\{1,3,5,6\},\{2,3,5,6\},\{2,4,5,6\}$
$\{1,2,3,5\},\{1,2,4,5\},\{1,3,4,5\},\{2,3,4,6\}$

3. Neighborhood polynomial of cycle

In this section we introduce and investigate the neighborhood polynomial of cycles.

Definition 3.1. *Let* C_n^i *be the family of neighborhood sets of a cycle* C_n *with cardinality* i *and let* $n_0(C_n, i) = |C_n^i|$. *Then the neighborhood polynomial* $N(C_n, x)$ *of* C_n *is defined as* $N(C_n, x) = \sum_{i=\lceil \frac{n}{2} \rceil}^{n} n(C_n, i) x^i$.

Theorem 3.2. (i) *If* C_n^i *is the family of neighborhood set with cardinality* i *of* C_n *then*

$$|C_n^i| = |C_{n-1}^{i-1}| + |C_{n-2}^{i-1}|$$

(ii) For every $n \geq 4$, $N(C_n, x) = x[N(C_{n-1}, x) + N(C_{n-2}, x)]$ with the intial values $N(C_1, x) = x, N(C_2, x) = x^2 + 2x, N(C_3, x) = x^3 + 3x^2 + 3x$.

Proof. (i) It follows from the theorem 2.6.
(ii) It follows from the (i) and definition.

We obtain the coefficients of $N(C_n, x)$ for $1 \leq n \leq 12$ in the table 1.

Let $n(C_n, j) = |C_n^j|$. There are relationships between the numbers $n(C_n, j)(\frac{n}{2} \leq j \leq n)$ in the table 1.

n\ j	1	2	3	4	5	6	7	8	9	10	11	12
1	1											
2	2	1										
3	3	3	1									
4	0	2	4	1								
5	0	0	5	5	1							
6	0	0	2	9	6	1						
7	0	0	0	7	14	7	1					
8	0	0	0	2	16	20	8	1				
9	0	0	0	0	9	30	27	9	1			
10	0	0	0	0	2	25	50	35	10	1		
11	0	0	0	0	0	11	55	77	44	11	1	
12	0	0	0	0	0	2	36	105	112	54	12	1

In the following theorem, we obtain some properties of $n(C_n, j)$:

Theorem 3.3. *The following properties hold for coeficients of* $N(C_n, x)$*:*

(i) For every $n \in N, n(C_{2n}, n) = 2$.

(ii) For every $n \geq 4j \geq \lceil \frac{n}{2} \rceil, n_0(C_n, j) = n_0(C_{n-1}, j-1) + n_0(C_{n-2}, j-1)$.

(iii) For every $n \in N, n_0(C_{2n+1}, n+1) = 2n+1$

(iv) For every $n \in N, n_0(C_{2n+2}, n+1) = 2$

(v) For every $n \in N, n_0(C_n, n) = 1$

(vi) For every $n \in N, n_0(C_n, n-1) = n$

(vii) For every $n \in N, n_0(C_n, n-2) = \dfrac{n(n-3)}{2}$

(viii) For every $n \in N, n_0(C_n, n-3) = \dfrac{n(n-4)(n-5)}{6}$

(ix) For every $j > 2, \sum_{2j}^{i=j} n_0(C_i, j) = 2\sum_{i=j-1}^{2(j-1)} n_0(C_i, j-1)$

(x) If $S_n = \sum_{j=\lceil \frac{n}{2} \rceil}^{n} n_0(C_n, j)$ then for every $n \geq 6$,

$S_n = S_{n-1} + S_{n-2}$ with the intial values $S_1 = 1, S_2 = 3, S_3 = 7, S_4 = 7$ and $S_5 = 11$.

Proof. (i) Since $C_{2n}^n = \{\{2,4,6,8...2n\}\{1,3,5,7...2n-1\}\}$ we get the result $n_0(C_{2n}, n) = 2$.

(ii) $|C_n^i| = |C_{n-1}^{i-1}| + |C_{n-1}^{i-2}|$ from this we get the result $n \geq 4j \geq \lceil \frac{n}{2} \rceil$,

$$n_0(C_n, j) = n_0(C_{n-1}, j-1) + n_0(C_{n-2}, j-1)$$

(iii) By induction on n, the result is true of $n = 1$, we get $C_3^2 = \{\{1,2\},\{1,3\},\{2,3\}\}$ This is true for all natural numbers less than n and we prove it for n by $(i)(ii)$ and the induction we have,

$$n_0(C_{2n+1}, n+1) = n_0(C_{2n}, n) + n_0(C_{2n-1}, n) = 2n+1.$$

(iv) By induction on n. Since $C_2^4 = \{(1,3),(2,4)\}$. So $n_0(C_4, 2) = 2$. This result is true for all natural numbers n and we prove it for n by $(i),(ii)$ and (iii).

$$n_0(C_{2n+2}, n+1) = n_0(C_{2n+1}, n) + n_0(C_{2n}, n) = 2.$$

(v) For any graph with n vertices, $n_0(G, n) = 1$, then we have the result.

(vi) For any graph with n vertices, $n_0(G, n-1) = n$, then we have the result.

(vii) By induction on n. This result is true for $n = 5$, Since $n_0(C_5, 3) = 5$. Suppose that the result is true for all natural numbers less than n, and we prove it for n. By parts (ii),(vi),(v) and induction hypothesis we have,

$$n_0(C_n, n-2) = n_0(C_{n-1}, n-3) + n_0(C_{n-2}, n-3)$$
$$= \frac{(n-1)(n-4)}{2} + (n-2) = \frac{n(n-3)}{2}$$

(viii) By induction on n. This result is true for $n = 6$, Since $n_0(C_6, 3) = 2$. Suppose the result true for all natural numbers less than n and prove it for n. By parts (ii),(iv),(vii) and induction hypothesis we have,

$$n_0(C_n, n-3) = n_0(C_{n-1}, n-4) + n_0(C_{n-2}, n-4)$$
$$= \frac{(n-1)(n-5)(n-6)}{6} + \frac{(n-2)(n-5)}{2} \quad = \frac{n(n-4)(n-5)}{6}$$

(ix) By induction on j. Suppose that $j = 2$. Then

$\sum_{i=3}^{6} n_0(C_i, 3) = 12 = 2\sum_{i=2}^{4} n_0(C_i, 2)$. Now suppose that the result is true for every $j < k$, and we prove for $j = k$:

$$\sum_{i=k}^{2k} n_0(C_i, k) = \sum_{i=k}^{2k} n_0(C_{i-1}, k-1) + \sum_{i=k}^{2k} n_0(C_{i-2}, k-1)$$

$$= 2\sum_{i=k-1}^{2(k-1)} n_0(C_{i-1}, k-2) + 2\sum_{i=k-1}^{2(k-1)} n_0(C_{i-2}, k-2) = 2\sum_{i=k-1}^{2(k-1)} n_0(C_i, k-1)$$

(x) By the theorem 3.2 $|C_n^i| = |C_{n-1}^{i-1}| + |C_{n-1}^{i-2}|$, we have

$$S_n = \sum_{j=\lceil \frac{n}{2} \rceil}^{n} n_0(C_n, j) = \sum_{j=\lceil \frac{n}{2} \rceil}^{n} n_0(C_{n-1}, j-1) + \sum_{j=\lceil \frac{n}{2} \rceil}^{n} n_0(C_{n-2}, j-1)$$
$$= \sum_{j=\lceil \frac{n}{2} \rceil - 1}^{n-1} n_0(C_{n-1}, j) + \sum_{j=\lceil \frac{n}{2} \rceil - 1}^{n-2} n_0(C_{n-2}, j-1)$$
$$S_n = S_{n-1} + S_{n-2}.$$

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