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0-Ideals of a 0-Distributive Lattice

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Abstract. Somepropertiesof 0-ideals in 0-distributive lattices and quasi-complemented lattices are derived. Preservation of 0-ideals byan onto homomorphism defined on a 0-distributive lattice is discussed. It is proved that the set of all of0-ideals in a normal lattice forms a sub lattice of all of its ideals but not in general. The set of all of0-ideals in a 0-distributive lattice forms a distributivelattice under thespecially defined operations on it.

Keywords: 0-distributive lattice; prime ideal; minimal prime ideal; prime filter; maximal filter; 0-ideal; homomorphism; normal lattice; quasi complemented lattice.

AMS Mathematics Subject Classification (2010):06B99

1. Introduction

Cornish [1] introduced the concept of 0-ideal in a distributive lattice and studied their properties with the help of congruence relations. In [3], Sambsadaiva Rao studied prime 0-ideals in distributive lattices. As a generalization of the concept of distributive lattices, 0- distributive lattices are introduced by Varlet [6] and Almost distributive lattices are introduced by Swamy and Rao [5]. Recently, a study of 0-ideals and 0- homomorphisms of an Almost distributive lattice is carried out in [4], In this paper our aim is to study some properties of0-ideals in a 0- distributive lattice and the set of all0-ideals in a 0-distributive lattice.In section 2, we list some basic information on 0- distributive lattices which is needed for the development of this topic. In section 3, we study properties of 0-ideals in 0-distributive lattices.Here we give necessary and sufficient condition for a proper 0-ideal of a 0- distributive lattice to be prime and show that every0-ideal of a bounded 0-distributive lattice is the intersection of all minimal prime ideals containing it. In section 4, we discuss various situations in which image of a 0-ideal is a 0-ideal under lattice homomorphism of 0-ditributive lattices. In section 5, wetalk about the relation between the lattice of all ideals and the lattice of all 0-ideals of a 0- distributive lattice. Some properties of 0-ideals inaquasi complemented lattice are furnished insection 6.

2. Preliminaries

In this article we collect some basic concepts needed in the sequel for other non-explicitly stated elementary notions please refer to [6].Throughout L will denote a lattice with 0 unless otherwise specified. A lattice L with 0 is 0- distributive, if for $x, y, z \in L$, $x \wedge$

$y = 0$ and $x \wedge z = 0$ imply $x \wedge (y \vee z) = 0$. For any filter F of L define $0(F) = \{x \in L \mid x \wedge y = 0$, for some $y \in F\}$.An ideal I in L is called 0-ideal if $I = 0(F)$ for some filter F in L.For any prime ideal P of L, define $0(P) = \{x \in L \mid x \wedge y = 0, \text{forsome } y \notin P\}$. Note that for a minimal prime ideal P in L, $0(P) = P$. For any non empty subset A of L, the set $A^* = \{x \in L \mid x \wedge y = 0, \text{forall } y \in A\}$ is called an annihilator of A in L. An ideal I in L is called an annihilator ideal if $I = I^{**}$.An ideal I in L is called dense in L if $I^* = \{0\}$.An element $x \in L$ is said to be dense in L if, $(x]^* = \{x\}^* = \{0\}$.An ideal I of L is called an α- ideal if $(x]^{**} \subseteq I$ for each $x \in I$.Let $I(L)$ denote the set of all ideals of a bounded lattice L. Then $(I(L),\wedge,\vee)$ is a lattice where $I \wedge J = I \cap J$and $I \vee J = < I \bigcup J >$for any two ideals I and J of L. An ideal I of L is called a direct factor of L, if there exists an ideal J of L such that $I \vee J = L$ and $I \cap J = \{0\}$. A 0- distributive lattice L is said to be normal if $f \wedge g = 0 \Rightarrow (f \wedge g]^* = (f]^* \vee (g]^*$ for $f, g \in L$ .A 0-distributive lattice L is said to be quasi-complemented if for any $x \in L$,there exists y$\in L$ such that $(x]^* = (y]^{**}$. Note that a 0- distributive lattice L is quasi complemented, if for any $x \in L$, there exists y$\in$L such that $x \wedge y = 0$ and $x \vee y \in D$ where D denotes the set of all dense elements in L.

3. 0-Ideals

We begin with the following lemma.

Lemma 3.1. In any lattice L with 0, we have

(a) For any filter F of L, $0(F)$ is a semi ideal in L and $F \cap 0(F) \neq \emptyset \Rightarrow F = L = 0(F)$.

(b) If L contains a dense element, then $0(F) = L \Leftrightarrow F = L$, for any filter F of L.

(c) For a filter F of L, $0(F) = \{0\}$ if and only if L has a dense element.

(d) For any prime ideal P of L, $0(P)$is a semi ideal in L and $0(P) = 0(L \setminus P)$.

(e) For a proper filter $F$of L,$0(F)$ is contained in some minimal prime ideal of L.

(f) If M is a minimal prime ideal of L containing $0(F)$, then $M \cap F = \emptyset$ for any filter $F$of L.

Proof: (a) Obviously, for any filter$F$ of L, $0(F)$ is a semi ideal in L. Let F be a filter of L such that $F \cap 0(F) \neq \emptyset$. Select $x \in F \cap 0(F)$. $x \in 0(F) \Rightarrow x \wedge y = 0$, for some $y \in F$. As $x \in F$ and $y \in F$, $0 = x \wedge y \in F \Rightarrow F = L$ and hence$0(F) = L$.

(b) $F = L \Rightarrow 0(F) = L$,obviously. Let $0(F) = L$ and d be a dense element in L. $d \in 0(F) \Rightarrow d \wedge f = 0$, for some $f \in F$. As $f \in \{d\}^* = \{0\}$, we get $f = 0$. Thus $0 \in F$ and hence $F = L$.

(c) Assume that there exists a filter F in L such that $\{0\} = 0(F)$. But then $(f]^* = \{0\}$ for some $f \in F$. This shows that L has a dense element. Conversely, assume that L has a dense element. Then the set D of all dense elements in L is a filter with $0(D) = \{0\}$. Hence the result.

(d) Let P be a prime ideal of L. Then $L \setminus P$ is a filter of L. We have
$x \in 0(P) \Leftrightarrow x \wedge y = 0$ for some $y \notin P \Leftrightarrow x \wedge y = 0$ for some $y \in L \setminus P \Leftrightarrow x \in 0(L \setminus P)$.Therefore $0(P) = 0(L \setminus P)$.

(e) Let F be a proper filter of L. Then F must be contained in some maximal filter say M in L.Then $L \setminus M$ is a minimal prime ideal containing $0(F)$.

(f) Let M be a minimal prime ideal of L containing $0(F)$. Assume that $M \cap F \neq \emptyset$. Select $x \in M \cap F$. M being a minimal prime ideal, there exists y$\notin$ M such that $x \wedge y = 0$. As $x \wedge y = 0$ and $x \in F$ we get $y \in M$; a contradiction. Hence $M \cap F = \emptyset$.□

Remarks. (1) In L, for any proper filter $F, F \cap 0(F) = \emptyset$.

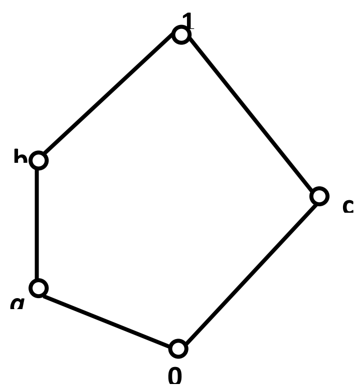


Figure 1:

(2) In L, a proper semi ideal $0(F)$ contains no dense element .

(3) If L is a 0- distributive lattice, then for any filter$F$ of L, $0(F)$ is an ideal in L andforany prime ideal P of $L, 0(P)$ is an ideal in L.

Consider the bounded 0-distributive lattice $L = \{0, a, b, c, 1\}$ as shown by the HasseDiagram of Fig.1.The ideal $[a)$ is not a 0- ideal of L. Hence the set Ω of all 0- ideals of L is a subset of the set of all ideals of$L$. The ideal $[0)$ is a 0- ideal of L which is notprime. The ideals $[b)$ and $[c)$ are prime 0- ideals of L.

In general about the 0-ideals of a bounded 0-distributive lattice we have

Theorem 3.2. For any bounded 0 –distributive lattice L, the following statements hold.

(a) A proper 0-ideal contains no dense elements.

(b) Every prime 0-ideal in L isminimal prime

(c) Every minimal prime ideal in L is an 0-ideal.

(d) Every non dense prime ideal in L is an 0-ideal.

(e) Every 0-ideal in L is an α- ideal.

(f) If L is a quasi- complemented lattice, then every prime ideal P not containing any dense element is a 0- ideal.

Proof. (a) Let a proper 0-ideal I contain a dense element say d in L. As I is a 0-ideal, $I = 0\,(F)$, for some proper filter F in L. But then, $d \in 0(F) \Longrightarrow d \wedge f = 0$ for some $f \in F$. As $f \in \{d\}^* = \{0\}$, we get $f = 0$. As $0 \in F, F = L$ and hence $0(F) = L$ (by Lemma 3.1 (b)). This contradicts the fact that$I$ is proper and the result follows.

(b) Let P be a prime 0-ideal in L. Then $P = 0(F)$ for some proper filter F in L. Select $x \in P = 0(F)$. Hence $x \wedge f = 0$, forsome $f \in F$..If $f \in P$, then $F \cap 0(F)$.Hence $F \cap 0(F) \neq \emptyset$.Then by Lemma3.1 (a),$P = 0(F) = F = L$;which is not true. Hence $f \notin P$. Therefore P is minimal prime.

(c) Let P be a minimal prime ideal in L. Then $L\setminus P$ is a filter of L Since P is a minimal prime ideal in L, we get $P = 0(P)$. Hence $P = 0\,(L\setminus P)$ (by Lemma 3.1(c)).

(d) Let P be a non dense prime ideal of L. As $P^* \neq \{0\}$, there exists $0 \neq x \in P^*$. Hence $P \subseteq P^{**} \subseteq (x]^*$. Now let $y \in (x]^*$. Then $x \wedge y = 0 \in P$ and $x \notin P$ imply $y \in P$. Thus $(x]^* \subseteq P$. From both the inclusions we get $P = (x]^*$. As $(x]^* = 0([x))$, we get $P = 0([x))$. Therefore P is a 0-ideal of L.

(e) Let I be a 0-ideal in L. Hence there exists a filter F in L such that $I = 0(F)$. Let $x \in 0\,(F)$. Then $x \in (f]^*$ for some$f \in F$.Hence$(x]^{**} \subseteq (f]^* \subseteq 0\,(F)$. This shows that the 0-ideal I in L is an α- ideal.

(f) Let L be a quasi-complemented lattice, P be a prime ideal of L with $P \cap D = \emptyset$. Let $x \in P$. Since L is a quasi- complemented lattice, there exists $y \in L$ such that $x \wedge y = 0$ and $x \vee y \in D$. But then $y \notin P$ as $P \cap D = \emptyset$. Thus $x \in 0(L \setminus P)$ shows that $P \subseteq 0(L \setminus P)$. As $0(L \setminus P) \subseteq P$ always, we get $P = 0(L \setminus P)$ and the result follows.□

Converse of Theorem 3.2(b) need not be true i.e. every 0-ideal need not be a minimal prime ideal in L. For this consider the 0-distributive lattice represented in Fig.1.{0} is a 0-ideal, but not a prime ideal in L and hence not a minimal prime ideal in L. Necessary and sufficient condition for a proper 0-ideal of a 0- distributive lattice to be prime is proved in the following theorem.

Theorem 3.3. Let I be a proper 0-ideal of a 0- distributive lattice L. Then I is a prime I ideal if and only if it contains a prime ideal.

Proof: If I is a prime ideal, then obviously it contains a minimal prime ideal.Now assume that I contains a prime ideal P but I is not prime. Select $a \notin I$, $b \notin I$ such that $a \wedge b \in I$. As $P \subseteq I$ and P is prime, we have $a \notin P, b \notin P$ with $a \wedge b \notin P$. Thus $(a \wedge b]^* \subseteq P \subseteq I$.As I is a 0-ideal of L, there exists a filter F in L such that $I = 0(F)$. Now $a \wedge b \in I = 0(\mathrm{F}) \Longrightarrow a \wedge b \wedge y = 0$ for some $y \in F$. Hence $y \in (a \wedge b]^* \subseteq I = O(F) \Longrightarrow y \in F \cap O(F) \Longrightarrow F \cap O(F) \neq \emptyset$. ByLemma3.1(a), $F = 0(F) = L$. Hence $I = L$; which is absurd. Hence I is prime. I being a prime 0-ideal of L,it is minimal prime, by Theorem 3.2(b).Hence the result. □

It is well known that everyideal of a bounded 0-distributive lattice can not be expressed as the intersection of all prime ideals containing it but for 0-ideals of a bounded 0-distributive lattice we have

Theorem 3.4. Every0-ideal of a bounded 0-distributive lattice is the intersection of all minimal prime ideals containing it.

Proof: Let I be a 0-ideal of L. Hence there exists a filter F in L such that $I = 0(F)$. Define $J = \bigcap \{ M \,/\, M$ is a minimal prime ideal containing I }.Clearly, $I \subseteq J$. Suppose $I \subset J$. Select $x \in J$ such that $x \notin I = 0(F)$. Hence $x \wedge y \neq 0$ for each $y \in F$. Fix up any $y \in F. x \wedge y \neq 0 \Longrightarrow x \wedge y \in G$, for some maximalfilter G of L. As $L \setminus G$ is a minimal prime ideal, $y \notin L \setminus G \Longrightarrow (y]^* \subseteq L \setminus G$. Again we know that $0(F) = \bigcup\{(f]^* \mid \in F\}$.Hence $I = 0(F) \subseteq L \setminus G$.This in turn shows that $x \in L \setminus G$; which is absurd. Hence $I = J$ and the result follows.□

Theorem 3.5. Every proper 0-ideal of a bounded 0-distributive lattice is contained in aminimal prime ideal.

Proof: Let I be a prime 0-ideal in L. Then $I = 0(F)$ for some proper filter F in L.Clearly, $I \cap F = 0(F) \cap F = \emptyset$. Let $\Psi = \{ G \mid G$ is a filter of L such that $F \subseteq G$ and $I \cap G = \emptyset\}$.Clearly, $F \in \Psi$ and Ψ satisfies the Zorn's lemma. Let M be a maximal element of Ψ. We claim that M is a maximal filter in L. Suppose K is a proper filter of L such that $M \subset K$. By maximality of M and $F \subseteq M \subseteq K$ we get $I \cap K \neq \emptyset$. Select $\mathrm{x} \in I \cap K$ .As $x \in I = 0(F)$, $x \wedge y = 0$ for some $y \in F$.But then $x \wedge y = 0 \in K$; a contradiction. Hence M is a maximal filter of L. L being a 0-distributive lattice, M is a prime filter. Therefore M is a minimal prime ideal of L such that $I \subseteq L \setminus M$. □

Immediately, by Theorem 3.5, we have

Corollary 3.6. A proper filter $F$of a bounded 0-distributive lattice is maximal if and only if $L\backslash F$is a 0-ideal.
Proof: Let F be amaximal filterof L. Then $L\backslash F$ is aminimal prime idealof L and hence a 0-ideal.Conversely, let $L\backslash F$ be a 0-ideal. Then $L\backslash F$ being a proper 0-ideal, it must be contained in some minimal prime ideal say M of L(byTheorem 3.5). Thus $L\backslash M \subseteq F$.□
We have the following simple property of 0- ideals**.**

Theorem 3.7. Intersection of any two 0-ideals in a 0- distributive lattice L is a 0-ideal of L.
Proof: It is enough to prove that for any two filters$F$ and G of L, $0(F) \cap 0(G) = 0(F \cap G)$. Obviously, $0(F \cap G) \subseteq 0(F) \cap 0(G)$. Let $x \in 0(F) \cap 0$ (G). But then $x \wedge f = 0$ for some $f \in F$ and $x \wedge g = 0$ for some $g \in G$. As L is 0- distributive,$x \wedge (f \vee g) = 0$. As $f \vee g \in F \cap G$, we get $x \in 0(F \cap G)$. Thus $0(F) \cap 0(G) \subseteq O(F \cap$ G) . Combining both the inclusions we get $0(F) \cap 0(G) = 0(F \cap G)$.□

Corollary 3.8. Intersection of any family of 0-ideals in a 0- distributive lattice L is a 0-ideal of L.

4. Epimorphism and 0-ideals
We begin this section by the following remark.

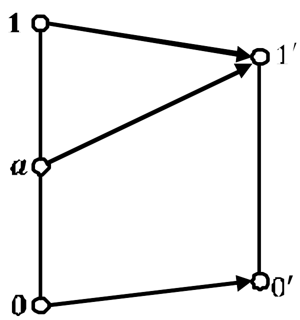


Figure 2:

If f is a ring epimorphism, then it is an isomorphism if and only if kernel of $f$is{ 0 }.But it is not true in the case of distributive lattices and hence for 0-distributive lattices also. It may be seen from the following example.
Consider the bounded distributive lattices L_1= {0,a,1} and L_2= $\{0', 1'\}$ as shown by the Hasse Diagram of Fig.2. Define $f : L_1 \to L_2$ as shown in the figure. Clearly, f is onto and $K(f) = \{ x \in L_1 / f(x) = \{ 0 \}$, but f is not one –one.Now we state a result that we need frequently in this article.

Result 4.1. Let L_1 and L_2 be boundedlattices and let $\lambda : L_1 \to L_2$ be a lattice epimorphism such that $\lambda(0) = 0$ and $\lambda(1) = 1$.Then we have the following:
(1) Forany filter F of L_2 , $\lambda^{-1}(F)$ is a filter of L_1.
(2) Forany filter G of L_1, $\lambda(G)$is a filter of L_2.

Notation. Let , $< L_1 ,\wedge, \vee >$and $< L_2 , \bar{\wedge}, \bar{\vee} >$ denote bounded 0-distributive lattices. Let $\lambda : L_1 \to L_2$ be a lattice epimorphism such that $\lambda(0) = 0$ and $\lambda(1) = 1$.Throughout this article we assume that $K(\lambda) = \{ 0 \}$, where $K(\lambda) = \{x \in L_1 / \lambda (x) = 0\}$.

Theorem 4.2. For any filter F of $L_1, \lambda [0(F)] = O[\lambda(F)]$.
Proof: Let $x \in \lambda[0(F)]$. Then $x = \lambda(a)$ for some $a \in 0(F)$. Now, $a \in 0(F)$ implies $a \wedge s = 0$ for some $s \in F$. Therefore $x \bar{\wedge} \lambda(s) = \lambda(a) \bar{\wedge} \lambda(s) = \lambda(a \wedge s) =$

$\lambda(0) = 0$. As $\lambda(s) \in \lambda(F$, we get $x \in 0[\lambda(F)]$. This shows that, $\lambda[O(F)] \subseteq O[\lambda(F)]$. Now let$x \in O[\lambda(F)]$. As $x \in L_2$ and λ is surjective there exists $y \in L_1$ such that $x = \lambda(y)$.Now $\lambda(y) \in 0[\lambda(F)] \Rightarrow \lambda(y) \barwedge \lambda(s) = 0$ for some $s \in F \Rightarrow \lambda(y \wedge s) = 0 \Rightarrow y \wedge s \in K(\lambda) = \{0\} \Rightarrow (y \wedge s) = 0 \Rightarrow y \in 0(F) \Rightarrow x = \lambda(y) \in \lambda[0(F)]$. This shows that,$0[\lambda(F)] \subseteq \lambda[0(F)$. Combining both the inclusionswe get $\lambda[0(F)] = 0[\lambda(F)]$. □

From the proof of Theorem 4.2, it follows that

Corollary 4.3. Forany filter $F$of a bounded lattice L, $\lambda[O(F)] \subseteq O[\lambda(F)]$.

Corollary 4.4. For any 0- ideal T of L_1, $\lambda(T)$ is an 0- ideal of L_2.

Proof: Let T be a 0- ideal of L_1. Hence there exists a filter G in L_1 such that $T = 0(G)$. Then by Result 4.1, $\lambda(G)$ is a filter in L_2.Hence $\lambda(T) = \lambda[0(G)] = 0[\lambda(G)]$ by Theorem 4.2.This shows that $\lambda(T)$ is an 0- ideal inL_2.□

Remark. The condition that $K(\lambda) = \{0\}$ is indispensible in Theorem 4.2.

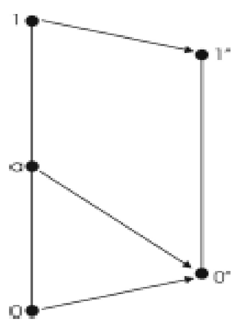


Figure 3

The condition that $K(\lambda) = \{0\}$ is indispensible in Theorem 4.2.For this consider the bounded 0- distributive lattices $L_1 = \{0, a, 1\}$ and $L_2 = \{0', 1'\}$ as shown by the Hasse Diagram of Fig.3. Define $f: L_1 \to L_2$ as shown in the figure. Clearly, f is onto and $K(f) \neq \{0\}$.For the filter $F = \{a, 1\}$ in L_1, we have $\lambda[0(F)] = \lambda\{0\} = \{0'\}$ and $0[\lambda(F)] = 0\{0', 1'\} = \{0', 1'\}$. Hence $\lambda[0(F)] \neq 0[\lambda(F)]$.

Theorem 4.5. For any filters F and G of L_1, $0(F) = 0(G)$ if and only if $0[\lambda(F)] = 0[\lambda(G)]$.

Proof: Let F and G beany filters of L_1 such that $0(F) = 0(G)$. Then obviously, $\lambda[0(F)] = \lambda[0(G)]$. By Theorem 4.2,we get $0[\lambda(F)] = 0[\lambda(G)]$. Conversely,let $0[\lambda(F)] = 0[\lambda(G)]$. Now, $t \in 0(F) \Rightarrow \lambda(t) \in \lambda[0(F)] \Rightarrow \lambda(t) \in 0[\lambda(F)] \Rightarrow \lambda(t) \in 0[\lambda(G)] \Rightarrow \lambda(t) \barwedge \lambda(s) = 0$ for some $s \in G \Rightarrow \lambda(t \wedge s) = 0 \Rightarrow t \wedge s \in K(\lambda) = \{0\} \Rightarrow t \wedge s = 0 \Rightarrow t \in 0(G)$. Therefore $0(F) \subseteq 0(G)$. Similarly we can show that $0(G) \subseteq 0(F)$. Therefore $0(F) = 0(G)$. □

Theorem 4.6. Forany 0- ideal J of L_2, $\lambda^{-1}(J)$ is an 0-ideal ofL_1.

Proof: Let J be a 0- ideal of L_2. Then there exists a filter G in L_2 such that $J = 0(G)$. Then by Result 4.1, $\lambda^{-1}(G)$ is a filter in L_1. Claim that $\lambda^{-1}[0(G)] = 0[\lambda^{-1}(G)]$. Let $x \in 0[\lambda^{-1}(G)]$.Then $x \wedge s = 0$ for some $s \in \lambda^{-1}(G)$. Now $x \wedge s = 0 \Rightarrow \lambda(x \wedge s) = 0 \Rightarrow \lambda(x) \barwedge \lambda(s) = 0$ and $\lambda(s) \in G \Rightarrow \lambda(x) \in 0(G) \Rightarrow x \in \lambda^{-1}[0(G)]$. This shows that $0[\lambda^{-1}(G)] \subseteq \lambda^{-1}[0(G)]$.

Conversely, let $x \in \lambda^{-1}[0(G)]$. Then $\lambda(x) \in [0(G)] \Rightarrow \lambda(x) \barwedge \lambda(s) = 0$ and $\lambda(s) \in G \Rightarrow \lambda(x \wedge s) = 0 \Rightarrow x \wedge s \in K(\lambda) = \{0\} \Rightarrow x \wedge s = 0$ and $s \in \lambda^{-1}(G) \Rightarrow x \in 0[\lambda^{-1}(G)]$.This shows that$\lambda^{-1}[0(G)] \subseteq 0\ [\lambda^{-1}(G)]$. Combining both the inclusions we get $\lambda^{-1}(J) = \lambda^{-1}[0(G)] = 0\ [\lambda^{-1}(G)]$. Hence λ^{-1}(J)is a 0-ideal of L_1.□

5. The set Ω of all 0-ideals

Let L be a 0-distributive lattice and let Ω denote the poset $(\Omega, \subseteq)$ of all 0-ideals of$L$.In this article we prove that the poset $(\Omega, \subseteq)$ need not be a sub lattice of the lattice(L),∧,∨) of all ideals of Lin general.But under the condition of normality of L the poset$(\Omega, \subseteq)$ will be a sub lattice of the lattice$(I(L)$,∧,∨).

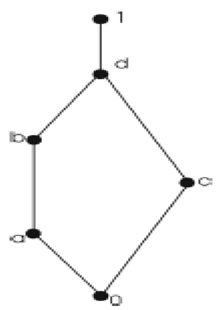


Figure 4:

Consider the bounded 0-distributive lattice L={0,a,b,c,d,1} as shown by the Hasse Diagram of Fig.4. For the filters $F = [b)$ and $G = [c), 0(F) \vee 0(G) = \{0, a, b, c, d\}$ isnot a 0-ideal of L. As join of two 0- ideals of a 0-distributive lattice L is not a 0-ideal of L, the set Ω of all 0-ideals of L is not a sub lattice ofthe lattice $(I\ (L)$,∧,∨).

In the following theorems we prove some properties of 0- ideals in a normal lattice.

Theorem 5.1. The poset $(\Omega, \subseteq)$ is a sub lattice of the lattice $I(L)$ provided L is a normal lattice.

Proof: For any two filters$F$ and $G$of L, $0(F) \wedge 0(G) = 0(F) \cap 0(G) = 0(F \cap G)$ (see Theorem 3.7). Hence $0(F) \cap 0(G) \in \Omega$ .Now we prove that $0(F) \vee 0(G) = 0(F \vee G)$. Obviously, $0(F) \vee 0(G) \subseteq 0(F \vee G)$. Let $x \in 0(F \vee G)$. Then $x \wedge t = 0$ for some $t \in F \vee G$. Hence $t \geq f \wedge g$ for some $f \in F$ and $g \in G$. Hence $x \wedge f \wedge g = 0 \Longrightarrow x \in (f \wedge g]^* \Longrightarrow x \in (f]^* \vee (g]^*$ (since L is normal) $\Longrightarrow x \in 0(F) \vee 0(G)$ (since $(f]^* \subseteq 0(F)$ and$(g]^* \subseteq 0(G)$.Thus $0(F \vee G \subseteq 0(F) \vee 0(G)$. Combining both the inclusions we get $0(F) \vee 0(G) = 0(F \vee G)$.Hence (Ω, ∧,∨) is a sub lattice of the lattice$(I(L)$,∧,∨).□

Theorem 5.2. Let L be a normal lattice. Then for any ideal I which contains a 0- ideal K, there exists thelargest 0-ideal containing K and contained in I.

Proof: Define β = { J | J is an 0-idal such that $K \subseteq J \subseteq I$ }.Clearly $K \in$ β. Let $\{J_i$ / i $\in \Delta\}$ be a chain in β. Then $\bigcup\{J_i$ / i $\in \Delta\}$ is a 0-idal and $K \subseteq \bigcup J_I \subseteq I$.So by Zorn's lemma β contains a maximal element, say M. We now prove that M is unique. Suppose there exists a maximal element $M_1 \neq M$ in β. Then we have $K \subseteq M_1 \vee M \subseteq I$. As L is a normal lattice, $M_1 \vee M \in$ β (see Theorem 5.1). But then $M_1 = M_1 \vee M = M$; and hence the uniqueness. Thus in a normal lattice L for any ideal I which contains a 0- ideal K, there exists a largest 0-ideal containing K and contained in I. □

We know that {0} is a 0-ideal in a lattice L with 0, if it contains a dense element. Hence by Theorem 5.2, it follows that

Corollary 5.3. In a normal lattice L, containing dense elements, there exists the largest 0-ideal in L.

Corollary 5.4. In a bounded normal lattice L, there exists the largest 0-ideal in L.

Theorem 5.5. Let L be a normal lattice .If $\{I\ \alpha\ /\ \alpha \in \Delta\}$ is a family of 0-ideals in L, then $\vee\ I\ \alpha$ is a 0-ideal of L.

Proof. As $I\ \alpha\ is\ a\ 0-$ ideal of L ,let $I\alpha = 0\ (F\alpha)$ for some filter $F\alpha$ of L, for each $\alpha \in \Delta$. $0\ (F\alpha) \subseteq 0\ (\vee F\alpha)$ for each $\alpha \in \Delta \Longrightarrow \vee\ 0\ (F\alpha) \subseteq 0\ (\vee\ F\alpha)$. Conversely let $x \in 0\ (\vee\ F\alpha)$. Then $x \wedge t = 0$ for some t$\in (\vee\ F\alpha)$. But then $t \geq f_1 \wedge f_2 \wedge f_3 \wedge \ldots \wedge fn$ for some $f\ i \in Fi\ (1 \leq i \leq n\ , n\ finite)$. Hence, $x \wedge (f_1 \wedge f_2 \wedge f_3 \wedge \ldots \wedge fn\ ) = 0 \Longrightarrow (x \wedge f_1) \wedge (x \wedge f_2) \wedge (x \wedge f_3) \wedge \ldots \wedge (x \wedge fn\ ) = 0$. As L is a normal lattice $((x \wedge f_1\ )]^* \vee \big((x \wedge f_2)\big]^* \ldots \vee \big((x \wedge fn)\big]^* = L.\ x \in L \Longrightarrow x \in ((x \wedge f1\ )]^* \vee \big((x \wedge f_2)\big]^* \ldots \vee ((x \wedge fn)\ ]^* \Longrightarrow x \leq a_1 \vee a_2 \ldots a_n$ where $a_i \in \big((x \wedge fi\ )\big]^* (1 \leq i \leq n\ )$.

Thus $x \wedge f_i \wedge a_i i = 0$, $(1 \leq i \leq$ n). Hence $x \wedge (\wedge_{i=1}^{n} f_i\ ) \wedge a_{\,i} = 0$. As L is 0-distributive , $x \wedge (\wedge_{i=1}^{n} f_i) \wedge (\vee_{i=1}^{n} a_{\,i}) = 0$. But then $x \wedge (\wedge_{i=1}^{n} fi\ ) = 0$ (as $x \leq \vee_{i=1}^{n} a\ i) \Longrightarrow x \in ((\wedge_{i=1}^{n} fi\ )\ ]^* \Longrightarrow x \in (\ f1\ )\ ]^* \vee (\ f2\ )\ ]^* \ldots \vee (\ fn\ )\ ]^*$ (since L is a normal lattice) $\Longrightarrow x \in 0\ (F1\ ) \vee 0\ (F2\ ) \vee \ldots 0\ (Fn\ )$ as $(\ fi\ \ ] \subseteq 0\ (F\ i\ )$ $(1 \leq i \leq$ n). This shows that $(\vee\ F\alpha\ ) \subseteq \vee\ 0\ (F\alpha\ )$. Combining both the inclusions we get $\vee\ I\ \alpha = \vee\ 0\ (F\alpha\ ) = 0\ (\vee\ F\alpha\ )$; and the result follows.□

Though the poset $(\Omega, \subseteq)$ need not be a sub lattice of $I\ (L)$, interestingly we prove that the poset $(\Omega, \subseteq)$ forms a distributive lattice under the special operations $\sqcap$ and $\sqcup$ defined on it.

Theorem 5.6. Let L be a 0-distributive lattice.The poset $(\Omega, \subseteq)$ forms a distributive lattice on its own.

Proof. Let F and G be any filters in L.

Claim I: The poset $(\Omega, \subseteq)$ is a lattice.

(1) $O(F \cap G)$ is an infimum of $0(F)$and $0(G)$ in Ω . Let $0(K) \subseteq O(F)$ and $0(K) \subseteq O(G)$ for some filter K in L. Hence $0\ (K)$ is in Ω. Let $x \in 0(K)$. Then $x \in 0(F)$ and $x \in 0\ (G)$ imply $x \wedge y = 0$ for some $y \in F$ and $x \wedge z = 0$ for some $z \in G$. As L is a 0 - distributive lattice, $x \wedge (y \vee z\ ) = 0$. But then $x \in 0(F \cap G)$ as $y \vee z \in F \cap G$. Hence $0(K) \subseteq 0(F \cap G)$. So $0(\ F \cap G\ )$ is an infimum of $0(F)$ and 0(G) in ΩIf we denote the infimum of $0(F)$ and 0(G) in Ω by $0\ (\ F) \sqcap 0(G)$, then we have $0(F) \sqcap 0(G\ ) = 0(\ F \cap\ G)$.

(2) $O(F \vee \mathrm{G})$ is the supermum of $0(F)$and $0(G)$in Ω . Let $O(F) \subseteq 0(K)$ and $0(G) \subseteq 0(K)$ for some filter K in L.Let $x \in 0(F \vee \mathrm{G}\ )$. Then$x \wedge f \wedge g = 0$ for some $f \in$ F and $g \in G$. As $x \wedge f \in 0(G) \subseteq 0(K)$, we get $x \wedge f \wedge k = 0$, for some $k \in K$. But then $x \wedge k \in 0(F) \subseteq 0(K)$. Hence $x \wedge k \wedge s = 0$ for some $s \in K$. As$k \wedge s \in K$, we get $x \in 0(K)$. Therefore $0(F \vee \mathrm{G}\ )$ is the supermom of (F) and $0(\mathrm{G}\ )$in Ω . If we denote the supremumof $0(F)$and 0(G) in Ω by $0\ (F) \sqcup 0(\mathrm{G})$, then we have $0(F) \sqcup 0(\mathrm{G}) = 0(F \vee \mathrm{G})$.

From **(1)** and **(2)** we get that the poset $(\ \Omega\ , \subseteq)$ is a lattice under the binary operations $\sqcap$ and $\sqcup$ defined on it.

Claim II: The lattice $(\ \Omega, \sqcap, \sqcup\ )$ is a distributive lattice.

Now x$\in 0(F)\ (0(K) \sqcup 0(G)) \Longrightarrow$ x $\in 0(F) \cap (0(K \vee G)) \Longrightarrow x \wedge f = 0, x \wedge k = 0$ and $x \wedge g = 0$ for some $f \in$ F, $k \in$ K and $g \in$ G $. \Longrightarrow x \wedge (f \vee k) = 0$ and $x \wedge (f \vee g) = 0$ (as L is 0- distributive) $\Longrightarrow x \in (0(F) \sqcap 0(K))$ and $x \in (0(F) \sqcap (0(G))$ (as $f \vee k \in (0(\ F\ ) \sqcap 0(K))$ and $f \vee g \in (0(F) \sqcap 0(\ G\ )) \Longrightarrow$ x$\in (O(F) \sqcap (O(K)) \sqcup (O(F) \sqcap (O(G)$

(as $x \wedge (f \vee \mathrm{k} \vee \mathrm{g}) = 0$) $\Longrightarrow O(F) \sqcap O(K) \sqcup \mathrm{O}(\mathrm{G})) \subseteq (O(F) \sqcap O(K)) \sqcup (O(F) \sqcap O(G))$.
$As\ (O(F) \sqcap O(K)) \sqcup (O(F) \sqcap O(G)) \subseteq O(F) \sqcap (O(K) \sqcup \mathrm{O(G)})$ always, we get $O(F) \sqcap (O(K) \sqcup \mathrm{O(G)}) = (O(F) \sqcap (O(K)) \sqcup (O(F) \sqcap (O(G))$. Hence (Ω,⊓,⊔) is a distributive lattice.□

Corollary 5.7. If a 0-distributive lattice L contains dense elements, then the lattice (Ω,⊓,⊔) is a bounded, complete distributive lattice .
Proof: The lattice (Ω ,⊓,⊔) is distributive lattice (by Theorem 5.6). Clearly, {0} and L are the bounds of the poset (Ω, ⊆) Let $\{ Fi \mid i \in \Delta\}$ be any family of filters of L. Then $0(\cap Fi) = \cap\, 0(Fi)$ (see Corollary 3.8). Hence the poset (Ω , ⊆) is a complete lattice. Thus it follows that thelattice (Ω ,⊓,⊔) is a bounded ,complete, distributive lattice. □

Corollary 5.8. For abounded 0-distributive lattice L, the lattice (Ω,⊓,⊔) is a bounded, complete distributive lattice.
Any two distinct 0-ideals I and J of a lattice L are said to be ⊔ co-maximal if $I \sqcup J = L$.

Lemma 5.9. Let L be a 0- distributive lattice. $x \wedge y = 0 \Longrightarrow (x]^* \sqcup (y]^* = L$ for $x, y \in L$ i.e. $(x]^*$ and $(y]^*$ are ⊔ co- maximal.
Proof: As $(x]^* = 0\,([x))$ and $(y]^* = 0\,([y))$, we get $(x]^*, (y]^* \in \Omega$. Now $(x]^* \sqcup (y]^* = 0\,([x)) \sqcup 0\,([y)) = 0\,([x) \vee [y)) = 0\,(x \wedge y) = (x \wedge y]^* = (0]^* = L$.□
In the following theorem we show that any two distinct prime 0-ideals of a 0-distributive lattice L are ⊔ co- maximal.

Theorem 5.10. Any two distinct prime 0-ideals P and Q ofa 0- distributive lattice L are ⊔ co-maximal.
Proof. P and Q are distinct prime 0-ideals of $L \Longrightarrow P$ and Q are distinct minimal prime ideals of L. Select $a \in P \setminus Q$ and $b \in Q \setminus P$. As $a \in P$ and P is minimal there exists $x \notin P$ such that$x \wedge a = 0$. Similarly for $b \in Q$, there exists $y \notin Q$such that $\mathrm{y} \wedge b = 0$. Now P being a prime ideal,$x \notin P$ and $b \notin P \Longrightarrow x \wedge b \notin P$. Similarly $a \notin Q$ and $y \notin Q \Longrightarrow$ **y**$\wedge a \notin Q$.But then $(x \wedge b]^* \subseteq P$ and (**y**∧ a]* $\subseteq Q$.Again $(x \wedge b) \wedge$ (**y**∧a) = $(x \wedge a) \wedge (\mathrm{y} \wedge b) = 0 \Longrightarrow (x \wedge b]^* \sqcup$ (**y**∧a]* $= L$, by Lemma 3.12. As $L = (x \wedge b]^* \sqcup$ (**y**∧a]* ⊆ P ⊔ Q, we get P ⊔ Q = L.□

Corollary 5.11. Any two distinct minimal prime ideals of a 0-distributive lattice L are ⊔ co- maximal.

6. Quasi-complemented lattices
In this article we derive necessary and sufficient conditions for every 0- ideal to be an annihilator ideal in a quasi-complemented lattice.

Theorem 6.1. Let L be a quasi-complemented, bounded lattice. Then for the following statements, **(a)** ⟹ **(b)** ⟹ **(c)** ⟹ **(d)** ⟹ **(e)** and all these statements are equivalent in L provided $I \vee I^* = L$ for any 0 - ideal I of L.
(a) Every 0- ideal is an annihilator ideal.
(b) Every minimal prime ideal is an annihilator ideal.

(c) Every prime 0- ideal is of the form $(x]^{**}$ for some $x \in L$
(d) Every 0- ideal is of the form $(x]^{**}$ for some $x \in L$.
(e) Every minimal prime ideal is non dense.
Proof.
(a) ⟹ (b). Since everyminimal prime ideal isan0- ideal, the implication follows.
(b) ⟹(c). Let P be a prime 0- ideal of L. Then by Theorem 3.3(b),P is a minimal prime ideal of L. Hence by assumption, P is an annihilator ideal. Thus P being non dense, $P = (a]^*$ for some $a \notin D$. As L is quasi-complemented, there exists $b \in L$ such that $(a]^{**} = (b]^{**}$ and the implication follows.
(c) ⟹ (d). Suppose the condition (d) does not hold**.** Hence there exists an 0- ideal I which cannot be expressed in the form of$(x]^{**}$ for some $x \in L$. Define K = $\{J \mid J$ is a 0- ideal which cannot be expressed in the form of $(x]^{**}$ for some $x \in L\}$.Then clearly K $\neq \emptyset$. Let $\{J_i \,/\, i \in \Delta\}$ be a chain of 0- ideals in K .Then $\cup J_i$ is a 0- ideal of L. Suppose if possible $\cup J_i = (x]^{**}$ for some $x \in L$. Now $x \in (x]^{**} = \cup J_i \Rightarrow x \in J_i$ for some $i \in \Delta$. Hence$(x]^{**} \subseteq J_i$ since every 0-ideal in L is an α- ideal. Further as $J_i \subseteq \cup J_i = (x]^{**}$, we get as $J_i = (x]^{**}$, a contradiction . Hence $\cup J_i \neq (x]^{**}$ for any $x \in L$. Thus $\cup J_i \in$ K. Hence by Zorn's lemma K contains a maximal element say M. Claim that M is prime. Let $a, b \in L$ such that $a \notin M$ and $b \notin M$. As L is quasi-complemented, there exist $s, t \in L$ such that $(a]^{**} = (s]^*$ and $(b]^{**} = (t]^*$. But then we have $M \vee (a] \subseteq M \vee (a]^{**} = M \vee (s]^* \subseteq M \sqcup (s]^*$. Similarly we get $M \subset M \sqcup (t]^*$. By maximality of $M, M \sqcup (s]^* = (x]^{**}$ and $M\ (t]^* = (y]^{**}$ for some $x, y \in L$. Now, $M \sqcup (a \wedge b]^{**} = M \sqcup [(a]^{**} \wedge (b]^{**}] = M \sqcup [(a]^{**} \cap (b]^{**}] = M \sqcup [(s]^* \cap (t]^*] = [M \sqcup (s]^*] \cap [M \sqcup (t]^*] = (x]^{**} \cap (y]^{**} = (x]^{**} \wedge (y]^{**} = (x \wedge y]^{**}$.

If $a \wedge b \in M$, then $M = (x \wedge y]^{**}$ will contradicts the fact that $M \in$ K. Hence $a \wedge b \notin M$. This shows that M is a prime ideal. But then M is a prime 0- ideal which cannot be expressed in the form of $(x]^{**}$ for any $x \in L$; which is absurd to our assumption .Hence the implication follows.

(d) ⟹ (e). Let P be aminimal prime ideal in L.We know that every minimal prime ideal in L is a 0- ideal. Hence by condition **(d)** P $= (x]^{**}$ for some $x \in L$.Thus $P^* = (x]^{***} = (x]^*$. If P^* is dense , then x must be a dense element contained in proper 0- ideal P, which is absurd. Hence P is non dense. Thus we have proved that **(a) ⟹ (b) ⟹ (c) ⟹ (d) ⟹ (e).**
Now we prove that **(e) ⟹ (a)** provided $I \vee I^* = L$ for any 0 - ideal I of L.Let I be a 0-ideal of L. Then by assumption $I \vee I^* = L$. As $1 \in L$,,$1 = a \vee b$ for some $a \in I$ and $b \in I^*$. Hence $(a]^* \cap (b]^* = (a \vee b]^* = (1]^* = (0]$. Thus $(b]^* \subseteq (a]^{**}$. Select $c \in I^{**}$. Then as $b \in I^*$, we get $c \wedge b = 0$. Also $a \in I = 0(F) \Rightarrow a \wedge f = 0$ for some $f \in F$.

Now, $c \in (b]^* \subseteq (a]^{**} \subseteq (f]^* \subseteq 0(F) = I$ shows that $I^{**} \subseteq I$. As $I \subseteq I^{**}$ always, we get $I = I^{**}$ and hence the 0- ideal I is an annihilator ideal.

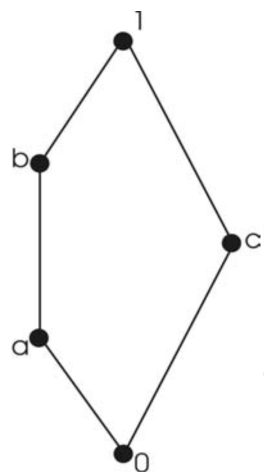
Corollary 6.2. Let$L$ be a 0 - distributive lattice containing dense elements.Then **(e) ⟹(a)** also holds provided $I \vee I^*$ contains a dense element for any 0 - ideal I of L.
Proof: By assumption $(I \vee I^*) \cap D \neq \emptyset$ **.** Select $d \in (I \vee I^*) \cap D . d \in D \Rightarrow \{d\}* = \{0\}$. $d \in (I \vee I^*) \Rightarrow d \leq a \vee b$ for some $a \in I$ and $b \in I^*$. Hence $a \wedge b = 0$. Also $a \in I = 0(F) \Rightarrow a \wedge f = 0$ for some $f \in F$.Select $c \in I^{**}$. Then as $b \in I^*$, we get

$c \wedge b = 0$. Now $\in (b]^* \subseteq (a]^{**}. \subseteq (f]^* \subseteq 0(F) = I$ shows that $I^{**} \subseteq I$. As $I \subseteq I^{**}$ always, we get $I = I^{**}$ and hence the 0- ideal I is an annihilator ideal. □

Remarks 6.3. **(1)** The condition $I \vee I^* = L$ need not hold for all 0-ideals I of a bounded0 - distributive lattice L. Consider the bounded 0 - distributive lattices as shown by the Hasse Diagrams of Fig.5. and Fig.6.

(i) In a bounded 0-distributive lattice L represented in Fig 5, for each 0-ideal I of $L, I \vee I^* = L$ holds.

(ii) In a bounded 0-distributive lattice L represented in Fig 6, for an 0-ideal $I = \{0,a\}$, $I \vee I^* \neq L$.



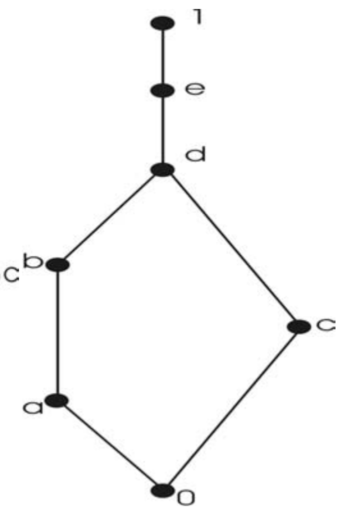


Figure 5: **Figure 6:**

(2) Recall that an ideal I of L is adirect factor of L if there exists an ideal J in L such that $I \vee J = L$ and $I \cap J = \{0\}$. If $I \vee I^* = L$ holds for 0- ideal I of L, then I is a direct factor of L.

(3) If every 0-ideal I of a bounded0 - distributive lattice L is a direct factor of L, then $I \vee I^* = L$ holds for all 0- ideals of$L$.

(4) If $I \vee I^* = L$ holds for all 0- ideals I of a bounded 0-distributive lattice L, then $(x]^* \vee (x]^{**} = L$ holds for all $x \in L$.

(5) If $I \vee I^* = L$ holds for all 0- ideals I of a bounded 0-distributive lattice L, then L is a normal lattice.

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