Annals of Pure and Applied Mathematics
Vol. 7, No. 2, 2014, 1-5
ISSN: 2279-087X (P), 2279-0888(online)
Published on 26 August 2014
www.researchmathsci.org

Annals of
Pure and Applied Mathematics

Application of Green Function Method for Solution Advection Diffusion Equation of Nutrient Concentration in Groundwater

P.S.Avhale[1] and S.B.Kiwne[2]

[1]Shivaji Arts and Science College, Kannad, Aurangabad-431103, India
e-mail:- avhaleps@yahoo.com
[2]Department of Mathematics, Deogiri College Aurangabad-431001, India

Received 10 August 2014; 21 August 2014

Abstract. We apply Green function techniques for solving nonhomogeneous differential equations. The advection diffusion equation of nutrients in the porous ground water is nonhomogeneous partial differential equation can be solved by reducing the nonhomogeneous advection diffusion equation with substitution to diffusion equation with nonhomogeneous part and then general solution is derived by using Greens function method in infinite domain with initial condition. We solve advection diffusion equation in both form in cartesian and radial coordinate system.

Keywords: Solution of advection diffusion equation

AMS Mathematics subject classification (2010):76Rxx ,76Sxx ,76Txx , 97M60

1. Introduction

The model Barber Nye and Tinker given an equation for concentration of nutrient in the porous water which is of the form see [5,7]

$$\frac{\partial}{\partial t}(\phi c) + \nabla.(cu) = \nabla.[\phi D \nabla c] - d_s. \quad (1.1)$$

where ϕ porosity of the water, u is a Darcy flux of pore water, D is a diffusion coefficient of nutrients in pore water, d_s is the interfacial ion transport, which is non-zero if the liquid is below saturation. Now let Darcy flux u is given by $u = V\phi$ in three dimension. Then above equation becomes see [1]

$$\frac{\partial}{\partial t}(\phi c) + \nabla.(cV\phi) = \nabla.[\phi D \nabla c] - d_s. \quad (1.2)$$

For the constant velocity equation(1.2) become

$$\phi\frac{\partial c}{\partial t} + \phi V \nabla c = \phi D \nabla^2 c - d_s, \quad (1.3)$$

with reduces in following form

$$\frac{\partial c}{\partial t} + V \nabla c = D \nabla^2 c - \frac{d_s}{\phi}. \quad (1.4)$$

2. Reduction of advection diffusion equation by substitution

We substitute in the equation (1.4) $c = we^{[\frac{V}{2D}(x+y+z) - 3\frac{V^2}{4D}t]}$.

Then it become see [4]

$$\frac{\partial w}{\partial t} = D\nabla^2 w - \frac{d_s}{\phi} e^{[\frac{3V^2}{4D}t - \frac{V}{2D}(x+y+z)]}. \tag{2.1}$$

and

$$\frac{1}{D}\frac{\partial w}{\partial t} = \nabla^2 w - \frac{d_s}{\phi D} e^{[\frac{3V^2}{4D}t - \frac{V}{2D}(x+y+z)]}. \tag{2.2}$$

For the solution of equation (2.2) consider the homogeneous part

$$\frac{1}{D}\frac{\partial w}{\partial t} = \nabla^2 w. \tag{2.3}$$

Consider the equation (2.3) with the boundary conditions as

$$\frac{1}{D}\frac{\partial w}{\partial t} = \nabla^2 w,\ -\infty < x < \infty,\ -\infty < y < \infty \ and\ -\infty < z < \infty,\ t > 0, \tag{2.4}$$

$$w = f(x, y, x),\ t = 0. \tag{2.5}$$

Solution of equation (2.4) with boundary condition (2.5) is as see [2]

$$w = \frac{1}{(4\pi Dt)^{\frac{3}{2}}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x', y', z') e^{\frac{-[(x-x\prime)^2+(y-y\prime)^2+(z-z\prime)^2]}{4Dt}} dx'dy'dz'. \tag{2.6}$$

The solution (2.3) in terms of Green's function is

$$w(x, y, z) = \int_{x\prime=-\infty}^{\infty}\int_{y\prime=-\infty}^{\infty}\int_{z\prime=-\infty}^{\infty} G(x, y, z, t|x', y', z', \tau)|_{\tau=0} \\ .f(x', y', z')dx'dy'dz'. \tag{2.7}$$

Comparing equation (2.6) and (2.7) we have a Green's function

$$G(x, y, z, t|x', y', z', \tau)|_{\tau=0} = \frac{1}{(4\pi Dt)^{\frac{3}{2}}} . e^{\frac{-[(x-x\prime)^2+(y-y\prime)^2+(z-z\prime)^2]}{4Dt}}. \tag{2.8}$$

The desired Green's function is obtained by substituting $(t - \tau)$ in the place of t in equation (2.8) see [3]

$$G(x, y, z, t|x', y', z', \tau) = \frac{1}{(4\pi D(t-\tau))^{\frac{3}{2}}} . e^{\frac{-[(x-x\prime)^2+(y-y\prime)^2+(z-z\prime)^2]}{4D(t-\tau)}}. \tag{2.9}$$

Then solution of (2.1) by application of Green's function method is given by

$$w = \int_{x\prime=-\infty}^{\infty}\int_{y\prime=-\infty}^{\infty}\int_{z\prime=-\infty}^{\infty} G(x, y, z, t|x', y', z', \tau)|_{\tau=0} f(x', y', z')dx'dy'dz' \\ + \frac{1}{\phi}\int_{\tau=0}^{\tau} d\tau \int_{x\prime=-\infty}^{\infty}\int_{y\prime=-\infty}^{\infty}\int_{z\prime=-\infty}^{\infty} . G(x, y, z, t|x', y', z', \tau) d_s \\ e^{[\frac{3V^2}{4D}t - \frac{V}{2D}(x+y+z)]} dx'dy'dz'.$$

Using the value of Green's function in (2.10)

$$w = \frac{1}{(4\pi Dt)^{\frac{3}{2}}} \int_{x\prime=-\infty}^{\infty}\int_{y\prime=-\infty}^{\infty}\int_{z\prime=-\infty}^{\infty} e^{\frac{-[(x-x\prime)^2+(y-y\prime)^2+(z-z\prime)^2]}{4D(t-\tau)}} f(x', y', z')dx'dy'dz'$$

$$+ \frac{1}{\phi}\int_{\tau=0}^{\tau} d\tau \int_{x\prime=-\infty}^{\infty}\int_{y\prime=-\infty}^{\infty}\int_{z\prime=-\infty}^{\infty} \frac{1}{(4D\pi(t-\tau))^{\frac{3}{2}}} e^{[\frac{-[(x-x\prime)^2+(y-y\prime)^2+(z-z\prime)^2]}{4D(t-\tau)} + \frac{3V^2}{4D}t - \frac{V}{2D}(x+y+z)]} \\ d_s dx'dy'dz'.$$

Now $c = w. e^{[\frac{V}{2D}(x+y+z) - \frac{3V^2}{4D}t]}$.

Hence the general solution of (2.1) is

$$c = \left[\frac{1}{(4\pi Dt)^{\frac{3}{2}}}\int_{x'=-\infty}^{\infty}\int_{y'=-\infty}^{\infty}\int_{z'=-\infty}^{\infty} e^{\frac{-[(x-x')^2+(y-y')^2+(z-z')^2]}{4D(t-\tau)}} f(x',y',z')dx'dy'dz'\right.$$

$$+\frac{1}{\phi}\int_{\tau=0}^{\tau} d\tau \int_{x'=-\infty}^{\infty}\int_{y'=-\infty}^{\infty}\int_{z'=-\infty}^{\infty} \frac{1}{(4D\pi(t-\tau))^{\frac{3}{2}}} e^{\left[\frac{-[(x-x')^2+(y-y')^2+(z-z')^2]}{4D(t-\tau)}+\frac{3V^2}{4D}t-\frac{V}{2D}(x+y+z)\right]}$$

$$\left. d_s dx'dy'dz'\right]. e^{\left[\frac{V}{2D}(x+y+z)-\frac{3V^2}{4D}t\right]}.$$

If the interfacial ion transport d_s is zero when the liquid is saturated. Then equation (2.1) becomes

$$\frac{\partial}{\partial t}(\phi c) + \nabla.(cV\phi) = \nabla.[\phi D\nabla c], \tag{2.10}$$

then equation (2.13) by substitution $c = we^{\left[\frac{V}{2D}(x+y+z)-3\frac{V^2}{4D}t\right]}$ reduces to

$$\frac{\partial w}{\partial t} = D\nabla^2 w. \tag{2.11}$$

Equation (2.14) is a the diffusion equation with the boundary conditions as

$$\frac{1}{D}\frac{\partial w}{\partial t} = \nabla^2 w,\ -\infty < x < \infty,\ -\infty < y < \infty\ and\ -\infty < z < \infty,\ t > 0, \tag{2.12}$$

$$w = f(x,y,x),\ t = 0. \tag{2.13}$$

Solution of equation (2.15) with boundary condition (2.16) is workout by similar procedure as

$$c = \left[\frac{1}{(4\pi Dt)^{\frac{3}{2}}}\int_{x'=-\infty}^{\infty}\int_{y'=-\infty}^{\infty}\int_{z'=-\infty}^{\infty} e^{\frac{-[(x-x')^2+(y-y')^2+(z-z')^2]}{4D(t-\tau)}} f(x',y',z')\right.$$

$$\left. dx'dy'dz'\right] e^{\left[\frac{V}{2D}(x+y+z)-\frac{3V^2}{4D}t\right]}.$$

3. Nonhomogeneous Advection diffusion equation in cylindrical form

Consider the nonhomogeneous advection diffusion equation in radial form[3][6]

$$\frac{\partial^2 c}{\partial r^2} + \frac{1}{r}\frac{\partial c}{\partial r} + \frac{1}{k}g(r,t) = \frac{1}{D}\frac{\partial c}{\partial t},\ in\ 0 \le r < \infty, t > 0, \tag{3.1}$$

$$c = F(r), for\ t = 0. \tag{3.2}$$

Considering the homogenous part of the above equation separating the variables, solution for time variable is $e^{(-D\beta^2 t)}$, where β is separation variable.

The space-variable function $R_o(\beta, r)$ is the solution of the equation see [3]

$$\frac{d^2 R_o}{dr^2} + \frac{1}{r}\frac{dR_o}{dr} + \beta^2 R_o = 0,\ in\ 0 \le r < \infty. \tag{3.3}$$

The solution of equation (3.3), which is finite at $r = 0$, is

$$R_o(\beta, r) = J_0(\beta r). \tag{3.4}$$

Then the complete solution of homogenous part is given by $\psi(r,t)$ is given by

$$\psi(r,t) = \int_{\beta=0}^{\infty} k(\beta)e^{-D\beta^2 t} J_o(\beta r)d\beta. \tag{3.5}$$

The application of the initial condition (3.2) to equation (3.5)

$$F(r) = \int_{\beta=0}^{\infty} k(\beta) J_o(\beta r) d\beta, \ in \ \ 0 \le r < \infty. \tag{3.6}$$

This is an expansion of an arbitrary function $F(r)$ defined in the interval $0 \le r < \infty$ in terms of $J_o(\beta_r)$ functions. Therefore we obtain

$$F(r) = \int_{\beta}^{\infty} \beta J_o(\beta r) d\beta \int_{r'=0}^{\infty} J_o(\beta r') F(r' dr') dr', in \ \ 0 \le r < \infty. \tag{3.7}$$

Comparing the equation (3.6) and (3.7) we get

$$k(\beta) = \beta \int_{r'=0}^{\infty} r' J_o(\beta r') F(r') dr'. \tag{3.8}$$

Then the complete integral is given by

$$\psi(r,t) = \int_{\beta=0}^{\infty} e^{-D\beta^2 t} \beta J_o(\beta r) d\beta \int_{r'=0}^{\infty} r' J_o(\beta r') F(r') dr', \tag{3.9}$$

by changing the order of integration see [3] and use of integrals we have

$$\int_{\beta=0}^{\infty} e^{-D\beta^2 t} \beta J_o(\beta r) J_0(\beta r') d\beta = \frac{1}{2Dt} e^{(-\frac{r^2+r'^2}{4Dt})} I_o(\frac{rr'}{2Dt}). \tag{3.10}$$

The complete integral of homogeneous part equation (3.1) is

$$\psi(r,t) = \frac{1}{2Dt} \int_{r'=0}^{\infty} r' e^{(-\frac{r^2+r'^2}{4Dt})} F(r') I_o(\frac{rr'}{2Dt}) dr'. \tag{3.11}$$

The solution of the homogenous part of equation(3.1) in terms of Green's function is given by see [3]

$$\psi(r,t) = \int_{r'=0}^{\infty} r' G(r,t|r',\tau)|_{\tau=0} F(r') dr' \tag{3.12}$$

where r' is the Sturm-Liouville weight function.Comparison of equation (3.11) and (3.12) yield

$$G(r,t|r',\tau)|_{\tau=0} = \frac{1}{2Dt} e^{\left(-\frac{r^2+r'^2}{4Dt}\right)} I_o\left(\frac{rr'}{2Dt}\right). \tag{3.13}$$

The desired Green's function is obtained by replacing t by $(t-\tau)$ in equation (3.13)

$$G(r,t|r',\tau) = \frac{1}{2Dt} e^{(-\frac{r^2+r'^2}{2D(t-\tau)})} I_o(\frac{rr'}{2D(t-\tau)}), \tag{3.14}$$

Then the solution of the nonhomogeneous problem (3.1) in terms of the above Green's function is given as

$$c(r,t) = \int_{r'=0}^{\infty} r' G(r,t|r',\tau)|_{\tau=0} F(r') dr' + \frac{D}{k} \int_{\tau=0}^{\tau} d\tau \int_{r'=0}^{\infty} r' G(r,t|r',\tau) g(r',\tau) dr' - D \int_{\tau=0}^{\tau} [r' \frac{\partial G}{\partial r'}]_{r'=0} . f(\tau) d\tau.$$

Introducing the the Green's function in equation (3.15) we have the complete integral as follows

$$c(r,t) = \frac{1}{2Dt} \int_{r'=0}^{\infty} r' e^{(-\frac{r^2+r'^2}{4Dt})} I_o(\frac{rr'}{2Dt}) F(r') dr' + \frac{D}{k} \int_{\tau=0}^{\infty} d\tau \int_{r'=0}^{\infty} r' \frac{1}{2Dt} e^{[-\frac{r^2+r'^2}{2D(t-\tau)}]} I_o(\frac{rr'}{2D(t-\tau)}) g(r',\tau) dr' - D \int_{\tau=0}^{\tau} [r' \frac{\partial [\frac{1}{2Dt} e^{(-\frac{r^2+r'^2}{4Dt})} I_o(\frac{rr'}{2Dt})]}{\partial r'}]_{r'=0} . f(\tau) d\tau.$$

4. Conclusion

The Green's functions method discussed above for solution of advection diffusion equation in an infinite domain. The Green's function is used to find the solution of an inhomogeneous differential equation and/or boundary conditions from the solution of the homogenous part of advection diffusion equation.

REFERENCES

1. J.Bear, *Hydraulics of Groundwater*, McGraw-Hiil, 1959
2. H.S.Carslaw and J.C.Jaeger, *Conduction of heat in solid*, Oxford University press, 1959.
3. M.Necati Ozisik, *Heat Conduction,* A Wiley-Interscience Publication, 1993.
4. R.N.Singh, Advection diffusion equation models in near-surface geophysical and environmental sciences, *J. Ind. Geophys Union,* 17(2) (2013) 117-127.
5. T.Roose, A.C.Fowler.P.R.Darrah, A mathematical model of plant nutrient uptake, *J. Math. Biol*, 42 (2001) 347-360.
6. R.S. Yadav and P.R. Sharma, effects of porous medium on MHD fluid flow along a stretching cylinder, *Annals of Pure and Applied Mathematics*, 6(1) (2014) 104-113.
7. Md. Hasanuzzaman, B.Mandal and M.M.Touhid Hossain, A study of similarity solution of unsteady combined free and force convective laminar boundary layer flow about a vertical porous surface with suction and blowing, *Annals of Pure and Applied Mathematics,* 6(1) (2014) 85-97.