

A Triangular Fuzzy Model for Assessing Problem Solving Skills

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Abstract. We apply a Triangular Fuzzy Model (TFM) for assessing students' problem solving skills. The TFM is a variation of a special form of the Centre of Gravity (COG) defuzzification technique that we have used in earlier papers for assessing students' performance in several mathematical tasks. The main idea of the TFM is the replacement of the rectangles appearing in the graph of the COG technique by isosceles triangles sharing common parts. In this way we cover the ambiguous cases of students' scores being in the limits between two successive grades (e.g. between A and B). A classroom experiment is also presented illustrating our results in practice.

Keywords: *Fuzzy Sets, Centre of Gravity defuzzification technique, Triangular Fuzzy Model, Problem Solving.*

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1. Introduction

In 1999 Voskoglou developed a fuzzy model for the description of the learning process [8] and later [9] he used the *total uncertainty* of the corresponding fuzzy system for assessing the students' skills in learning mathematics. Meanwhile Subbotin et al. [2], based on Voskoglou's fuzzy model, adopted properly the widely used in Fuzzy Mathematics *Center of Gravity (COG) defuzzification technique* to provide an alternative measure for the assessment of students' learning skills. Since then, both Voskoglou and Subbotin, either collaborating or independently to each other, utilized the COG technique in assessing other students' competencies (e.g. see [3], [6], [10-13], etc) as well as the Bridge player's performance [14] and in testing the effectiveness of a CBR system [4].

A first attempt to apply the Triangular Fuzzy Model (TFM) was made in [5], while more recently Subbotin and Voskoglou [6] presented an improved version of it for assessing students' critical thinking skills. The basic idea of the TFM is to replace the rectangles appearing in the membership function's graph of the COG model by triangles having common parts, which must be considered twice in calculating the COG of the level's section lying between the resulting graph and the OX axis. In this way one

Michael Gr. Voskoglou

succeeds to cover the ambiguous cases of students' scores being at the limits of two successive grades (e.g. A and B, B and C, etc). It is a very common approach in such cases to divide the interval of the specific grades in three parts and to assign the corresponding grade using + and - . For example, $80 - 82 = B-$, $83 - 86 = B$, $87 - 89 = B+$. However, this consideration does not reflect the common situation, where the teacher is not sure about the grading of the students whose performance could be assessed as marginal between and close to two adjacent grades; for example, something like between 81 and 79 percent. The TFM fits this situation.

This paper aims at using the TFM in obtaining a fuzzy measure of students' Problem Solving (PS) skills. The text is organized as follows: In section 2 we develop the TFM using a scale of five grades (A, B, C, D and F) instead of three grades (A, B, C-F) used in [5] and [7]. In this way the fuzzy measure obtained becomes more accurate. In section 3 we present a classroom experiment performed recently with students of the Graduate Technological Educational Institute (T. E. I.) of Western Greece illustrating our results in practice. Finally, section 4 is devoted to conclusions and discussion about our plans for further research on the subject.

For general facts on fuzzy sets we refer to the book [1].

2. The triangular fuzzy model

Let $U = \{A, B, C, D, F\}$ be the set of students' grades $A =$ excellent, $B =$ very good, $C =$ good, $D =$ satisfactory and $F =$ less than satisfactory. In applying the COG as an assessment method, we represent the student group under assessment as a fuzzy set in U and we associate to each x in U an interval of the OX axis. Then, we construct the graph of the corresponding membership function, which in this case is a bar graph consisting of five rectangles with one side lying on the OX axis (e.g. see Figure 3 of [12]) and we calculate, using the well known from Mechanics formulas, the coordinates (X_c, Y_c) of the COG of the level's section lying between this graph and the OX axis (defuzzification of our data). Further, using elementary algebraic inequalities we determine the area (a triangle) where the COG lies and by elementary geometric observations we obtain a criterion about the student group's performance (for details see, for example, section 3 of [12]).

As said above, in applying the TFM instead of the COG method, we replace the rectangles appearing in the graph of the COG method by triangles. Therefore, we shall have five such triangles in the resulting scheme, each one corresponding to a students' grade (F, D, C, B and A respectively). Without loss of generality and for making our calculations easier we consider isosceles triangles with bases of length 10 units lying on the OX axis. The height to the base of each triangle is equal to the percentage of students' of the group under assessment who achieved the corresponding grade. We allow for any two adjacent triangles to have 30% of their bases belonging to both of them. In this way, we cover the situation of uncertainty in assessing marginal students' scores, as we have described above.

The resulting scheme is presented in Figure 1. The student group under assessment can be represented again, as in the COG method, as a fuzzy set in U , whose membership function $y=m(x)$ has as graph the line $OA_1B_1A_2B_2A_3B_3A_4B_4A_5C_9$. It is easy to calculate the coordinates (b_{i1}, b_{i2}) of the points B_i , $i = 1, 2, 3, 4, 5$. In fact, B_1 is the intersection of

A Triangular Fuzzy Model for Assessing Problem Solving Skills

the straight line segments A_1C_2 , C_1A_2 , B_2 is the intersection of C_3A_3 , A_2C_4 and so on.. Therefore, it is straightforward to determine the analytic form of $y=m(x)$ consisting of 10 branches, corresponding to the equations of the straight lines OA_1 , A_1B_1 , B_1A_2 , A_2B_2 , B_2A_3 , A_3B_3 , B_3A_4 , A_4B_4 , B_4A_5 and A_5C_9 in the intervals $[0, 5)$, $[5, b_1)$, $[b_1, 12)$, $[12, b_2)$, $[b_2, 19)$, $[19, b_3)$, $[b_3, 26)$, $[26, b_4)$, $[b_4, 33)$ and $[33, 38]$ respectively.

However, in applying the TFM the use of the analytic form of $y = m(x)$ is not needed (in contrast to the COG method) for the calculation of the COG of the resulting level's area. In fact, since the marginal cases of students' grades should be considered as common parts for any pair of the adjacent triangles, it is logical to not subtract the areas of the intersections from the area of the corresponding level's section, although in this way we count them twice; e.g. placing the ambiguous cases B+ and A- in both regions B and A. In other words, the COG method, which calculates the coordinates of the COG of the area between the graph of the membership function (line $OA_1B_1A_2B_2A_3B_3A_4B_4A_5C_9$) and the OX axis (see Figure 1), thus considering the areas of the "common" triangles $C_1B_1C_2$, $C_3B_2C_4$, $C_5B_3C_6$ and $C_7B_4C_8$ only once, is not the proper one to be applied in the above situation.

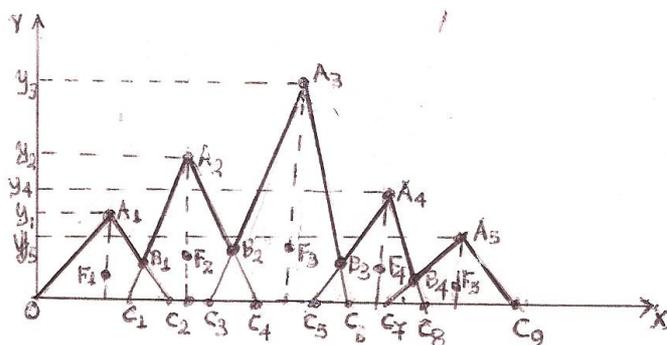


Figure 1: The membership function's graph of TFM

Indeed, in this case it is reasonable to represent each one of the five triangles OA_1C_2 , $C_1A_2C_4$, $C_3A_3C_6$, $C_5A_4C_8$ and $C_7A_5C_9$ of Figure 1 by their centers of gravities F_i , $i=1, 2, 3, 4, 5$ and to consider the entire level's section as the system of these points-centers. More explicitly, the steps of the whole construction of the TFM are the following:

1. Let y_1, y_2, y_3, y_4, y_5 be the percentages of the students in the group getting F, D, C, B, and A grades respectively, then $y_1 + y_2 + y_3 + y_4 + y_5 = 1$ (100%).
2. We consider the isosceles triangles with bases having lengths of 10 units each and their heights y_1, y_2, y_3, y_4, y_5 in the way that has been illustrated in Figure 1. Each pair of adjacent triangles has common parts in the base with length 3 units.
3. We calculate the coordinates (x_{c_i}, y_{c_i}) of the COG F_i , $i=1,2,3$, of each triangle as follows: The COG of a triangle is the point of intersection of its medians, and since this point divides the median in proportion 2:1 from the vertex, we find,

Michael Gr. Voskoglou

taking also into account that the triangles are isosceles, that $y_{c_i} = \frac{1}{3} y_i$. Also,

since the triangles' bases have a length of 10 units, we observe that $x_{c_i} = 7i - 2$.

4. We consider the system of the centers F_i , $i=1, 2, 3, 4, 5$ and we calculate the coordinates (X_c, Y_c) of the COG F of the whole level's area considered in Figure 1 from the following formulas, derived from the commonly used in such cases definition:

$$X_c = \frac{1}{S} \sum_{i=1}^5 S_i x_{c_i}, Y_c = \frac{1}{S} \sum_{i=1}^5 S_i y_{c_i} \quad (1).$$

In the above formulas (1) S denotes the whole area of the considered level's area and S_i , $i=1, 2, 3, 4, 5$ denote the areas of the corresponding triangles. Therefore $S_i = 5y_i$ and

$S = \sum_{i=1}^5 S_i = 5 \sum_{i=1}^5 y_i = 5$. Thus, from formulas (1) we finally get

$$X_c = \frac{1}{5} \sum_{i=1}^5 5y_i(7i-2) = 7 \sum_{i=1}^5 iy_i - 2 \quad \text{and} \quad Y_c = \frac{1}{5} \sum_{i=1}^5 5y_i \left(\frac{1}{3} y_i\right) = \frac{1}{5} \sum_{i=1}^5 y_i^2 \quad (2).$$

But, for $i, j=1, 2, 3, 4, 5$, we have that $0 \leq (y_i - y_j)^2 = y_i^2 + y_j^2 - 2y_i y_j$, therefore $y_i^2 + y_j^2 \geq 2y_i y_j$,

with the equality holding if, and only if, $y_i = y_j$. Therefore $1 = \left(\sum_{i=1}^5 y_i\right)^2 = \sum_{i=1}^5 y_i^2 +$

$$+ 2 \sum_{\substack{i,j=1, \\ i \neq j}}^5 y_i y_j \leq \sum_{i=1}^5 y_i^2 + \sum_{\substack{i,j=1, \\ i \neq j}}^5 (y_i^2 + y_j^2) = 5 \sum_{i=1}^5 y_i^2, \quad \text{or} \quad \sum_{i=1}^5 y_i^2 \geq \frac{1}{5} \quad (3), \quad \text{with the equality}$$

holding if, and only if, $y_1 = y_2 = y_3 = y_4 = y_5 = \frac{1}{5}$.

In the case of equality the first of formulas (2) gives that $X_c = 7\left(\frac{1}{5} + \frac{2}{5} + \frac{3}{5} + \frac{4}{5} + \frac{5}{5}\right) - 2 = 15$. Further, combining the inequality (3) with the second of formulas (2) one finds that $Y_c \geq \frac{1}{25}$. Therefore the unique minimum for Y_c corresponds to the COG $F_m(15, \frac{1}{25})$.

The ideal case is when $y_1 = y_2 = y_3 = y_4 = 0$ and $y_5 = 1$. Then from formulas (2) we get that $X_c = 33$ and $Y_c = \frac{1}{5}$. Therefore the COG in this case is the point $F_i(33, \frac{1}{5})$. On the

other hand, the worst case is when $y_1 = 1$ and $y_2 = y_3 = y_4 = y_5 = 0$. Then from formulas (2), we find that the COG is the point $F_w(5, \frac{1}{5})$. Therefore the "area" where the COG F_c lies

is the triangle $F_w F_m F_i$. Then, applying an argument analogous to that applied with respect to Figure 4 in section 3 of [12] we obtain the following criterion for comparing the student groups' performances:

- Among two or more groups the group with the greater X_c performs better.
- If two or more groups have the same $X_c \geq 15$, then the group with the greater Y_c performs better.
- If two or more groups have the same $X_c < 15$, then the group with the lower Y_c performs better.

3. A classroom experiment

A Triangular Fuzzy Model for Assessing Problem Solving Skills

The three problems presented in the Appendix of [12] were given for solution to the students of two different Departments of the School of Technological Applications (prospective engineers) of the Graduate T. E. I. of Western Greece as a test for assessing their progress in the mathematics course of their first term of studies. The results of their performance are shown in Table 1 below:

Table 1: The test's data

Department 1

% Scale	Grade	No. of students	% of students
89-100	A	0	0
77-88	B	5	13
65-76	C	6	16
53-64	D	9	24
Less than 53	F	18	47
Total		38	

Department 2

% Scale	Grade	No. of students	% of students
89-10	A	0	0
77-88	B	1	3
65-76	C	5	17
53-64	D	3	10
Less than 53	F	20	70
Total		29	

Then the first of formulas (2) gives that $X_c = 7(0.47+2*0.24+3*0.16+4*0.13)-2=11.65$ for the first and $X_c = 7(0.7+2*0.1+3*0.17+ 4*0.03)-2=8.71$ for the second Department. Thus, according to the TFM the students of the first Department demonstrated a better total performance than the students of the second Department .

4. Discussion and conclusions

The methods of assessing a group's performance usually applied in practice are based on principles of the bivalent logic (yes-no). However such methods probably are not the most suitable ones. On the contrary, fuzzy logic, due to its nature of including multiple values, offers a wider and richer field of resources for this purpose. This gave us the impulsion to introduce here an improved version of the TFM approach, which is a variation of the COG defuzzification technique fitting more properly to the ambiguous cases of students' scores lying in the limits between two different grades. However, there is a need for more classroom experiments to be performed in future for obtaining safer statistical data. On the other hand, since the TFM approach is a general assessment method, our future plans for further research on the subject include also the effort to apply this approach in assessing the individuals' performance in several other human activities.

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Michael Gr. Voskoglou

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