Annals of
Pure and Applied
Mathematics

Connections on Riemannian Geometry and it's Applications

Md. Rifat Hasan[1] and *A. K. M. Nazimuddin*[2]

[1]*Department of Applied Mathematics, Noakhali Science and Technology University*
Sonapur, Noakhali-3814, Bangladesh
Email: rifatmathdu@gmail.com
[2]*Department of Electronics and Communication Engineering, East West University*
Dhaka-1212, Bangladesh
Email: nazimsh027@yahoo.com

Abstract. The main purpose of this paper is to study the connections on vector bundle and apply connections to prove the Bianchi identity and Christoffel symbols.

Keywords: Riemannian geometry, connections, vector bundle

AMS Mathematical Subject Classification (2010): 53Bxx

1. Introduction
In order to differentiate sections of a vector bundle [1] or vector fields on a manifold we need to introduce a structure called the connection on a vector bundle. For example, an affine connection is a structure attached to a differentiable manifold so that we can differentiate its tensor fields. We first introduce the general theorem of connections on vector bundles. Then we study the different kind of connections an affine connection and the Levi-Civita connection with some theorem. We apply the connection to prove the theorem of the Bianchi identity and Christoffel symbols which are very important for tensor analysis [2].

2. Connection on vector bundles
A connection on a fiber bundle [3] is a device that defines a notion of parallel transport on the bundle; that is, a way to "connect" or identify fibers over nearby points. If the fiber bundle is a vector bundle, then the notion of parallel transport must be linear. Such a connection is equivalently specified by a covariant derivative, which is an operator that can differentiate sections of that bundle along tangent directions in the base manifold [4].

Definition 1. A connection on a vector bundle (E, M, π) as the map $\nabla: \mathfrak{X}(M) \times \Gamma(M) \to \Gamma(M)$ written $(X, Y) \mapsto \nabla_X Y$ satisfying the following properties

C1. $\nabla_X Y$ is linear over $C^\infty(M)$ in X :

$$\nabla_{fX_1+gX_2}Y = f\nabla_{X_1}Y + g\nabla_{X_2}Y \qquad\qquad \text{for } f, g \in C^\infty(M)$$

C2. $\nabla_X Y$ is linear over $\mathbb{R}$ in Y:

$$\nabla_X(aY_1 + bY_2) = a\nabla_X Y_1 + b\nabla_X Y_2 \qquad\qquad \text{for } a, b \in \mathbb{R}$$

C3. ∇ satisfy the following product rule:

$$\nabla_X(fY) = f\nabla_X Y + (Xf)Y \qquad\qquad \text{for } f \in C^\infty(M)$$

Theorem 1 [5]. A connection always exists on vector bundles.

Lemma 1. If ∇ is a connection in bundle $X \in \mathfrak{X}(M)$, $Y \in \Gamma(M)$ and $p \in M$, then $\nabla_X Y|_p$ depends only on the values of X and Y in an arbitrarily small neighborhood of p. More precisely if $X = \tilde{X}$ and $Y = \tilde{Y}$ on a neighborhood of p, then $\nabla_X Y|_p = \nabla_{\tilde{X}}\tilde{Y}|_p$.

Proof: first consider Y. Replacing Y by Y it is clearly suffices to show that $\nabla_X Y|_p = 0$ if Y vanishes on a neighborhood U of p.

Choose a bump function $\varphi \in C^\infty(M)$ with support in U such that $\varphi(p) = 1$. The hypothesis that Y vanishes on U implies that $\varphi Y \equiv 0$ on all of M, so $\nabla_X(\varphi Y) = \nabla_X(0 . \varphi Y) = 0\nabla_X(\varphi Y) = 0$. Thus for any $X \in \mathfrak{X}(M)$, the product rule gives

$$0 = \nabla_X(\varphi Y) = \varphi\nabla_X Y + (X\varphi)Y \qquad\qquad (1)$$

Now $Y \equiv 0$ on the support of φ, so the first term on the right is identically zero. Evaluating (1) at p shows that $\nabla_X Y|_p$. The argument for X is similar but easier. ∎

3. Affine connection
In the branch of mathematics called differential geometry[7], an affine connection[8] is a geometric object on a smooth manifold which connects nearby tangent spaces, and so permits tangent vector fields to be differentiated as if they were functions on the manifold with values in a fixed vector space. The notion of an affine connection has its roots in 19th-century geometry and tensor calculus, but was not fully developed until the early 1920s, by Élie Cartan (as part of his general theory of connections) and Hermann Weyl (who used the notion as a part of his foundations for general relativity). The terminology is due to Cartan and has its origins in the identification of tangent spaces in Euclidean space $\mathbb{R}^n$ by translation: the idea is that a choice of affine connection makes a manifold look infinitesimally like Euclidean space not just smoothly, but as an affine space.

Definition 2. An affine connection on a manifold M is a map $\nabla: \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ written $(X, Y) \mapsto \nabla_X Y$ satisfying the following properties:

C1. $\nabla_X Y$ is linear over $C^\infty(M)$ in X :

Connections on Riemannian Geometry and it's Applications

$$\nabla_{fX_1 + gX_2} Y = f\nabla_{X_1} Y + g\nabla_{X_2} Y \qquad\qquad \text{for } f, g \in C^\infty(M)$$

C2. $\nabla_X Y$ is linear over $\mathbb{R}$ in Y:

$$\nabla_X(aY_1 + bY_2) = a\nabla_X Y_1 + b\nabla_X Y_2 \qquad\qquad \text{for } a, b \in \mathbb{R}$$

C3. ∇ satisfy the following product rule:

$$\nabla_X(fY) = f\nabla_X Y + (Xf)Y \qquad\qquad \text{for } f \in C^\infty(M)$$

The definition of an affine connection resembles the characterization of $\binom{2}{1}$-tensor field given by the characterization lemma[6], an affine connection is not a tensor field because it is not linear over $C^\infty(M)$ in Y, but satisfies the product rule.

Next we examine how affine connection appears in components. Let $\{E_i\}$ be a local frame $E_i = \partial_i$, but it is useful to start by doing the components for more general frames. For any choices of the indices i and j, we expand

$$\nabla_{E_i} E_j = \Gamma_{ij}^k E_k \qquad\qquad (1)$$

This defines n^3 functions Γ_{ij}^k on U, called the Christoffel symbols of ∇ with respect to this frame. The following lemma shows that the action of the connection ∇ on U is completely determine by its Christoffel symbols.

Lemma 2. Let ∇ be a affine connection, and let $X, Y \in \mathfrak{X}(M)$ be expressed in terms of a local frame by $X = X^i E_i$, $Y = Y^j E_j$. Then
$$\nabla_X Y = (XY^k + X^i Y^j \Gamma_{ij}^k) E_k \qquad\qquad (2)$$

Proof: Just use the defining rules for affine connection and compute

$$\nabla_X Y = \nabla_X\left(Y^j E_j\right)$$

$$= \left(XY^j\right)E_j + Y^j \nabla_{X^i E_i} E_j$$

$$= \left(XY^j\right)E_j + X^i Y^j \nabla_{E_i} E_j$$

$$= XY^j E_j + X^i Y^j \Gamma_{ij}^k E_k$$

Renaming the dummy index in the first term yields the proof. $\blacksquare$

Lemma 3. Suppose M is a manifold covered by a single coordinate chart. There is a one-to-one correspondence between affine connection on M and choices of n^3 functions Γ_{ij}^k on M, by the rule

$$\nabla_X Y = (X^i \partial_i Y^k + X^i Y^j \Gamma_{ij}^k)\partial_k \qquad\qquad (3)$$

Proof: Observe that (3) is equivalent to (2) when $E_i = \partial_i$ is a coordinate frame, so for every connection of the $\{\Gamma_{ij}^k\}$ defined by (1) satisfies (3). On the other hand, given $\{\Gamma_{ij}^k\}$, it is easy to see by inspection that (3) is smooth if X and Y are linear over $\mathbb{R}$ in Y and linear over $C^\infty(M)$ in X, so only the product rule requires checking; this is a straight forward computation of the reader. ∎

Proposition 1. Every manifold admits an affine connection.
Proof. Cover M with coordinate charts $\{U_\alpha\}$ the preceding lemma guarantees the existence of a connection ∇^α on each U_α. Choosing a partition of unity $\{\varphi_\alpha\}$ subordinate to $\{U_\alpha\}$ we'd like to patch the ∇^αs together by the formula

$$\nabla_X Y = \sum_\alpha \varphi_\alpha \nabla_X^\alpha Y \qquad\qquad (4)$$

Again, it is obvious by inspection that this expression is smooth, linear over $\mathbb{R}$ in Y, and linear over $C^\infty(M)$ in X. We have to be a bit careful with the product rule, though, since a linear combination of connections is not necessarily a connection. By direct computation,

$$\nabla_X(fY) = \sum_\alpha \varphi_\alpha \nabla_X^\alpha(fY)$$

$$= \sum_\alpha \varphi_\alpha \left((Xf)Y + f\nabla_X^\alpha Y\right)$$

$$= (Xf)Y + f\sum_\alpha \varphi_\alpha \nabla_X^\alpha Y$$

$$= XfY + + f\nabla_X Y \quad \blacksquare$$

4. The Levi-Civita Connection
The Levi-Civita connection is named after Tullio Levi-Civita, although originally "discovered" by Elwin Bruno Christoffel. Levi-Civita, along with Gregorio Ricci-Curbastro, used Christoffel's symbols to define the notion of parallel transport and explore the relationship of parallel transport with the curvature[9], thus developing the modern notion of holonomy.

Definition 3. Let (M, g) be a Riemannian manifold [6]. Then ∇ be an affine connection[8] on a vector bundle (E, M, π) is said to be compatible with the Riemannian metric g if

$$X\big(g(Y, Z)\big) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \qquad \forall\, X, Y, Z \in \mathfrak{X}(M)$$

which can be also written as

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

Connections on Riemannian Geometry and it's Applications

Definition 4. An affine connection ∇ on a M manifold is called torsion free if
$$\nabla_X Y - \nabla_Y X = [X, Y] \text{ for all } X, Y \in \mathfrak{X}(M).$$

Theorem 2. Let (M, g) be a Riemannian manifold. Then there exists a unique torsion free affine connection on compatible with Riemannian metric g. This connection is characterized by the identity
$$\langle \nabla_X Y, Z \rangle = \frac{1}{2} \{ X\langle Y, Z \rangle + Y\langle Z, X \rangle - Z\langle X, Y \rangle - \langle X, [Y, Z] \rangle$$
$$+ \langle Y, [Z, X] \rangle + \langle Z, [X, Y] \rangle \} \quad \forall\, X, Y, Z \in \mathfrak{X}(M) \tag{5}$$

Proof: Assume ∇ exists with the desired properties. Using compatibility of ∇ with g we get
$$\langle \nabla_X Y, Z \rangle = X\langle Y, Z \rangle - \langle Y, \nabla_X Z \rangle$$
Since ∇ is symmetric we have $\nabla_X Z + \nabla_Z X = [Z, X]$ So replacing this in the expression above gives
$$\langle \nabla_X Y, Z \rangle = X\langle Y, Z \rangle - \langle Y, [Z, X] - \nabla_Z X \rangle = X\langle Y, Z \rangle - \langle Y, [Z, X] \rangle + \langle Y, \nabla_X Z \rangle \tag{6}$$
Cycling X, Y, Z gives two similar formulae:
$$\langle \nabla_Y Z, X \rangle = Y\langle Z, X \rangle - \langle Z, [X, Y] \rangle + \langle Z, \nabla_X Y \rangle \tag{7}$$
$$\langle \nabla_Z X, Y \rangle = Z\langle X, Y \rangle - \langle X, [Y, Z] \rangle + \langle X, \nabla_Y Z \rangle \tag{8}$$
By computing $(6) + (7) - (8)$ we obtain
$$\langle \nabla_X Y, Z \rangle + \langle \nabla_Y Z, X \rangle - \langle \nabla_Z X, Y \rangle$$
$$= X\langle Y, Z \rangle - \langle Y, [Z, X] \rangle + \langle Y, \nabla_X Z \rangle + Y\langle Z, X \rangle - \langle Z, [X, Y] \rangle +$$
$$\langle Z, \nabla_X Y \rangle - Z\langle X, Y \rangle + \langle X, [Y, Z] \rangle - \langle X, \nabla_Y Z \rangle$$
Now using symmetry of metric, we complete the proof. $\blacksquare$

Definition 5. Let (M, g) be a Riemannian manifold. The unique torsion-free affine connection on M which preserves the Riemannian metric is known as the Levi-Civita connection on M.

Proposition 2. Let (M, g) be a Riemannian manifold. Then the Levi-Civita connection is a connection on the tangent bundle.
Proof: By definition
$$g(\nabla_X(\lambda. Y_1 + \mu. Y_2), Z) = \lambda. g(\nabla_X Y_1, Z) + \mu. g(\nabla_X Y_2, Z)$$
and $\quad g(\nabla_{Y_1 + Y_2} X, Z) = g(\nabla_{Y_1} X, Z) + g(\nabla_{Y_2} X, Z) \quad \forall\, X, Y, Z \in \mathfrak{X}(M)$ and $\lambda, \mu \in \mathbb{R}$
Furthermore we have for all $f \in C^\infty(M)$
$$2 \cdot g(\nabla_X fY, Z) = \{ X(f \cdot g(Y, Z)) + f \cdot Y(g(X, Z)) - Z(f \cdot g(X, Y)) +$$
$$f \cdot g([Z, X], Y) \quad + g([Z, f \cdot Y], X) + g(Z, [X, f \cdot Y]) \}$$
$$= \{ X(f) \cdot g(Y, Z) + f \cdot X(g(Y, Z)) + f \cdot Y(g(X, Z))$$
$$-Z(f) \cdot g(X, Y) - f \cdot Z(g(X, Y)) + f \cdot g([Z, X], Y)$$
$$+ g((Z(f) \cdot Y + f \cdot [Z, Y]), X) + g(Z, X(f) \cdot Y + f \cdot [X, Y]) \}$$

$$= 2 \cdot \{X(f) \cdot g(Y,Z) + f \cdot g(\nabla_X Y, Z)\}$$
$$= 2 \cdot g(X(f) \cdot Y + f \cdot \nabla_X Y, Z)$$

and

$$2 \cdot g(\nabla_{f.X} Y, Z) = \{f \cdot X(g(Y,Z)) + Y(f \cdot g(X,Z)) - Z(f \cdot g(X,Y)) +$$
$$g([Z, f \cdot X], Y) + f \cdot g([Z,Y], X) + g(Z, [f \cdot X, Y])\}$$
$$= \{f \cdot X(g(Y,Z)) + Y(f) \cdot g(X,Z) + f \cdot Y(g(X,Z)) -$$
$$Z(f) \cdot g(X,Y) - f \cdot Z(g(X,Y)) + g(Z(f) \cdot X, Y) +$$
$$f \cdot g([Z,Y], X) + f \cdot g(Z, [X,Y]) - g(Z, Y(f) \cdot X)\}$$
$$= 2 \cdot f \cdot g(\nabla_X Y, Z)$$

This proves that ∇ is a connection on tangent bundle. ∎

Definition 6. Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. Then the curvature R is called Riemann curvature tensor if
$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \qquad \forall \, X, Y, Z \in \mathfrak{X}(M)$$

5. Main results

Theorem 3. Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. Then the curvature R satisfies
$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$$
this is known as Bianchi identity.

Proof. This property is a direct consequence of the Jacobi identity of vector fields. Indeed,

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z +$$
$$\nabla_Y \nabla_Z X - \nabla_Z \nabla_Y X - \nabla_{[Y,Z]} X$$
$$+ \nabla_Z \nabla_X Y - \nabla_X \nabla_Z Y - \nabla_{[Z,X]} Y$$
$$= \nabla_X (\nabla_Y Z - \nabla_Z Y) + \nabla_Y (\nabla_Z X - \nabla_X Z)$$
$$+ \nabla_Z (\nabla_X Y - \nabla_Y X) - \nabla_{[Y,Z]} X$$
$$- \nabla_{[Z,X]} Y - \nabla_{[X,Y]} Z$$

and so, since the connection is symmetric, we have

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = \nabla_X [Y,Z] + \nabla_Y [Z,X] + \nabla_Z [X,Y]$$
$$- \nabla_{[Y,Z]} X - \nabla_{[Z,X]} Y - \nabla_{[X,Y]} Z$$
$$= [X, [Y,Z]] + [Y, [Z,X]] + [Z, [X,Y]] = 0$$

This proves the Bianchi identity. ∎

Theorem 4. Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. Then prove the christoffel's symbols
$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$$

Proof. Let $(U, (x^i))$ be any local coordinate chart. Applying (5) to the coordinate vector fields, whose Lie –brackets are zero, we obtain

$$< \nabla_{\partial_i} \partial_j, \partial_l > = \frac{1}{2} (\partial_i < \partial_j, \partial_l > + \partial_j < \partial_l, \partial_i > - \partial_l < \partial_i, \partial_j >) \quad (9)$$

From the definition of metric coefficients and the Christoffel symbols

$$g_{ij} = < \partial_i, \partial_j > , \qquad \nabla_{\partial_i} \partial_j = \Gamma_{ij}^m \partial_m$$

Inserting these into (9) yields

$$\Gamma_{ij}^m g_{ml} = \frac{1}{2} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$$

Finally multiplying both sides by the inverse metric g^{kl} and noting that $g_{ml} g^{kl} = \delta_m^k$, we get

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$$

which complete the theorem. ∎

Acknowledgements

REFERENCES

1. P.P.Gupta and Malik, *Tensors and Differential Geometry,* Pragati Prakashan, Meerut, U.P., India, 2000.
2. M.S.Ali, *Tensor Analysis with Applications*, New Age publications, 2006.
3. S.S.Chern, W.H.Chern and K.S.Lam, *Lectures on Differential Geometry*, World Scientific, 1999.
4. D.J.Struik, *Lectures on Classical Differential Geometry*, Addition-Weslely Publishing Co., Inc. 1950.
5. M. S. Ali, K.M. Ahmed and M.M. Islam, Exterior algebra with differential forms on manifolds, *Dhaka Univ. J. Sci.,* 60(2) (2012) 247-252.
6. J.M.Lee, *Riemannian manifold: An Introduction to Curvature*, Spinger, 1997.
7. M.P. do Carno, *Riemannian geometry*, Birkhauser, Boston,1992.
8. Md.S.Ali, Md.M.Islam, F.Nasrin, Md. A.H. Sarkar and T.Zerin Khan, Connections on Bundles, *Dhaka Univ. J. Sci.*, 60(2) (2012) 191-195.
9. J.C.Islam, *Modern Differential Geometry for Physicists*, World Scientific Publishing Co. Pte. Ltd, 1989.