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Symplectic Connections and Contact Geometry

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Abstract. Symplectic connection we mean a torsion free connection which is either the Levi-Civita connection of a Bochner-Kahler metric of arbitrary signature. On a given symplectic manifold, there are many symplectic connections, i.e. torsion free connections with respect to which the symplectic form is parallel. We present what is known about preferred connections (critical points of a variational principle). This note also includes a symplectomorphisms on the space of symplectic connections. We also discuss the Curvature tensor of a symplectic connection**.**

***Keywords*:** Bochner-Kahler metric; Curvature tensor

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1. Introduction

The goal of this paper is to provide a fast introduction to symplectic geometry. A symplectic form is a closed nondegenerate 2-form. A symplectic manifold is a manifold equipped with a symplectic form. Symplectic geometry is the geometry of symplectic manifolds. Symplectic manifolds are necessarily even-dimensional and orientable, since non-degeneracy says that the top exterior power of a symplectic form is a volume form. The closedness condition is a natural differential equation, which forces all symplectic manifolds to being locally indistinguishable. The list of questions on symplectic forms begins with those of existence and uniqueness on a given manifold. For specific symplectic manifolds, one would like to understand the geometry and the topology of special submanifolds, the dynamics of certain vector fields or systems of differential equations, the symmetries and extra structure, etc.

Two centuries ago, symplectic geometry provided a language for classical mechanics. Through its recent huge development, it conquered an independent and rich territory, as a central branch of differential geometry and topology. To mention just a few key landmarks, one may say that symplectic geometry began to take its modern shape with the formulation of the Arnold conjectures in the 60’s and with the foundational work of Weinstein in the 70’s. Gromov in the 80’s gave the subject a whole new set of tools: pseudo-holomorphic curves. Gromov also first showed that important results from complex Kahler geometry remain true in the more general symplectic category and this direction was continued rather dramatically in the 90’s in the work of Donaldson on the topology of symplectic manifolds and their symplectic submanifolds, and in the work of Taubes in the context of the Seiberg-Witten invariants.

Symplectic geometry is significantly stimulated by important interactions with global analysis, mathematical physics, low-dimensional topology, dynamical systems, algebraic geometry, integrable systems, microlocal analysis, partial differential equations, representation theory, quantization, equivariant cohomology, geometric combinatorics, etc.

As a curiosity note that two centuries ago the name symplectic geometry did not exist. If you consult a major English dictionary, you are likely to find that symplectic is the name for a bone in a fish's head. However, the word symplectic in mathematics was coined by Weyl who substituted the Latin root in complex by the corresponding Greek root, in order to label the symplectic group. Weyl thus avoided that this group connote the complex numbers, and also spared us from much confusion that would have arisen, had the name remained the former one in honor of Abel: abelian linear group.

2. Differential forms on manifolds

Given a smooth manifold M, a smooth 1-form φ on M is a real-valued function on the set of all tangent vectors to M such that

i. Φ is linear on the tangent space T_xM for each $x \in M$.

ii. For any smooth vector field v on M, the function $\varphi(v) : M \to \mathbb{R}$ is smooth.

So for each $x \in M$, the map

$$\varphi_x: T_xM \to \mathbb{R}$$

is an element of the dual space $(T_xM)^*$.

Wedge products and exterior derivatives are defined similarly as for $\mathbb{R}^n$. If $f: M \to \mathbb{R}$
is a differentiable function, then we define the exterior derivative of f to be the 1-form df with the property that for any $x \in M, v \in T_xM, df_x(v) = v(f)$.

A local basis for the space of $1-$forms on M can be described as before in terms of any local coordinate chart $(x_1,\ldots,x_n)$ on M, and it is possible to show that the coordinate-based notions of wedge product and exterior derivative are in fact independent of the choice of local coordinates and so are well-defined.

More generally, suppose that M_1, M_2 are smooth manifolds and that $F: M_1 \to M_2$ is a differentiable map. For any $x \in M_1$, the differential dF(also denoted F^*): $T_xM_1 \to T_{F(x)}M_2$ may be thought of as a vector-valued 1-form, because it is a linear map from T_xM_1 to the vector space $T_{F(x)}M_2$. There is an analogous map in the opposite direction for differential forms, called the pullback and denoted F^*. It is defined as follows.

Definition 1. If $F: M_1 \to M_2$ is a differentiable map, then

i. If $f: M2 \to \mathbb{R}$ is a differentiable function, then $F^*f: M_1 \to \mathbb{R}$ is the function $(F * f)(x) = (foF)(x)$.

ii. If φ is a $p-$form on M_2, then F∗φ is the $p-$form on M_1 defined as follows:

If $v_1,\ldots,v_p \in T_xM_1$ then

$$(F^*\varphi)(v_1,\ldots,v_p) = \varphi(F*(v_1),\ldots,F^*(v_p)).$$

In terms of local coordinates $(x^1, \dots, x^n)$ on $M_1$and $(y^1, \dots, y^m)$ on M_2, suppose that the map F is described by

$$y^i = y^i(x^1, \dots, x^n), 1 \le i \le m.$$

Then the differential dF at each point $x \in M_1$ may be represented in this coordinate system by the matrix

$$\left[\frac{\partial y^i}{\partial x^j}\right]$$

The dx^j's are forms on M_1, the dy^i's are forms on M_2, and the pullback map F^* acts on the dy^i's by

$$F * (dy^i) = \sum_{j=1}^{n} \frac{\partial y^i}{\partial x^j} d\, x^j.$$

The pullback map behaves as nicely as one could hope with respect to the various operations on differential forms, as described in the following theorem.

Theorem 1. Let $F: M_1 \to M_2$ be a differentiable map, and let φ, η be differential forms on M_2. Then

i. $F^*(\varphi + \eta) = F^*\varphi + F^*\eta$.
ii. $F^*(\varphi \wedge \eta) = F^*\varphi \wedge F^*\eta$.
iii. $F^*(d\varphi) = d(F^*\varphi)$.

3. Symplectic geometry

3.1 Symplectic vector fields

Throughout our discussion, $(\mathbb{R}^{2n}, \omega_o)$ will denote a symplectic vector space with standard symplectic form

$$\omega_o = dx_1 \Lambda dy_1 + \cdots + dx_n \Lambda dy_n.$$

Recall that a symplectic form is a closed non-degenerate 2-form.

Given a symplectic vector space (V^{2n}, ω), and a subspace$W \subset V$, one can define the orthogonal complement W^ω by setting $W^\omega = \{v \in V: \omega(v, w) = 0, \text{for all } w \in W\}$.

Since ω is non-degenerate, the kernel of $I: V \to V^*$ by $v \mapsto \omega(v, .)|W$ will have kernel W^ω. Hence we see that the $dimW + dimW^\omega = dimV$. Clearly W is a symplectic subspace of V when $\omega|W$ is non-degenerate. This happens only when $W \cap W^\omega = \{0\}$ or by the dimension formula above, when$V = W \oplus W^\omega$. When $W = W^\omega$we say that W is Lagrangian and this happens when dim$W$=n.

Proposition 1. Every symplectic vector space (V, ω) is isomorphic to$(\mathbb{R}^{2n}, \omega_o)$.
Proof: Proof is by induction. We wish to construct a standard basis of V. By doing so we can then define a linear map which throws the constructed standard basis onto the standard basis of $\mathbb{R}^{2n}$. Since linear maps preserve symplectic forms, we will be done.
To construct a standard basis, take an element $u_1 \neq 0 \in V$ and choose v_1 such that $\omega(u_1, v_1) = 1$. One can easily do this as ω is non-degenerate and linear.

If $\omega(u_1, v) = \lambda$, then use $v_1 = \frac{v_1}{\lambda}$. Let $W = \text{Span}\{u_1, v_1\}$. Clearly ωW is non-degenerate and so $V = W \oplus W^{\omega}$ by above.

Now by induction, W^{ω} has a standard basis $(u_2, v_2, \cdots, u_n, v_n)$. The combination of the two basis is the basis of V, i.e., $\omega(u_i, u_j) = \omega(v_i, v_j) = 0$ and $\omega(u_i, v_j) = \delta_{ij}$. ∎

3.2. Symplectic Darboux's Theorem

Theorem 2. Every symplectic form ω on $M$is locally diffeomorphic to the standard form ω_o on $\mathbb{R}^{2n}$.

Proof: Let $p\epsilon M$ be a point on the manifold and consider a local chart φ such that $\varphi(p) = 0$. Use this local chart to push ω forward to a form ω'on $\mathbb{R}^{2n}$. We now need to show that ω' is diffeomorphic to standard form ω_o near 0. With the aid of the proposition mentioned above we can choose φ in such a way that $\omega' = \omega_o$ at the origin.

Now define the ω_t using the stupid homotopy trick $\omega_t = (1 - t)\omega_o + t\omega'$.

Note that $\omega_t = \omega_o$ at 0 for all t. Since ω_o is non-degenerate and non-degenerecy is an open condition, there exists a neighborhood U of our original neighborhood such that ω_t is non-degenerate for points in U. Since $\frac{d}{dt}\omega_t = \omega' - \omega$ it is closed on U. Hence by Poincare's lemma, there exists a one form σ such that $d\sigma = \omega' - \omega_o = \frac{d}{dt}\omega_t$. By subtracting the constant form $\sigma(0)$ we may assume that $\rho = 0$ at 0. From the equation $\sigma + \iota(X_t)\omega_t = 0$ we know that the corresponding family of vector fields vanish at 0. Using Moser's argument, let φ_t be the partially defined flow of X_t. However, in this case φ_t has 0 as a fixed point for all t. Hence by the general fixed point theorem, there exists a neighborhood $V \subset U$ of 0 such that $\varphi_t(x)$ for $0 \leq t \leq 1$ of points $x \in V$ remain inside U. Hence φ_t is defined on V and we have $\varphi_t^*(\omega') = \omega_o$. ∎

4. Contact geometry

Recall that a contact manifold is a $2n + 1$ dimensional manifold with a contact one form α. A contact form is a 1-form such that

$$\alpha \wedge \underbrace{d\alpha \wedge \cdots d\alpha}_{n \perp times}$$

is the volume form. The standard contact form on $\mathbb{R}^{2n+1}$ is

$$\alpha_0 = x_1 dy_1 + x_2 dy_2 + \cdots + x_n dy_n + dz.$$

For each $p \in M$, let $\mathcal{D}_p$ denote the kernel of the linear mapping $\alpha_p: T_pM \to \mathbb{R}$,

$$\text{i.e., } \mathcal{D}_p = \{X \in T_pM \mid \alpha_p(X) = 0\}.$$

Since α is no where vanishing, the dimension of $\mathcal{D}_p$ is $2n$ for all $p \in M$. Let $\mathcal{D} = \bigcup_{p \in M} \mathcal{D}_p$ be the contact distribution. For each $p \in M$, complete a basis $\{X_1, X_2, \cdots, X_{2n}\}$ of $\mathcal{D}$ into a basis $\{X_1, X_2, \cdots, X_{2n}, \xi\}$ of T_pM. Therefore we get that the volume form applied to the basis gives

$$0 \neq \alpha \Lambda (d\alpha)^n (X_1, X_2, \cdots, X_{2n}, \xi)$$
$$= \alpha(\xi).(d\alpha)^n (X_1, X_2, \cdots, X_{2n})$$

Since $\alpha(X_i) = 0$. Hence $(d\alpha)^n (X_1, X_2, \cdots, X_{2n}) \neq 0$ this means that the restriction of $d\alpha$ to $\mathcal{D}$ is a non-degenerate 2-form, i.e., $d\alpha$ is locally a symplectic form on $\mathcal{D}$.

Note that this means that $\mathcal{D}$ is oriented by restriction of $d\alpha$ to it. It also means that the normal to $\mathcal{D}$ is oriented as well. Since $\alpha(\xi) \neq 0$ and $\iota(\xi) d\alpha = 0$, we may normalize ξ. Hence we may assume $\iota(\xi)\alpha = 1$

$$\iota(\xi) d\alpha = 0.$$

Furthermore, we have that this vector field satisfies

$$\iota(\xi)[\alpha \Lambda (d\alpha)^n] = (\iota(\xi)\alpha)(d\alpha)^n - \alpha \Lambda (\iota(\xi)\alpha)(d\alpha)^n = (d\alpha)^n$$

Since $\alpha \Lambda (d\alpha)^n : \Gamma(M) \to \Omega^{2n}(\mathrm{M})$ is an isomorphism, there is exactly one vector field which satisfies the above conditions. Hence we have shown

Theorem 3. There exists a unique vector field ξ such that $\iota(\xi)\alpha = 1$ and $\iota(\xi) d\alpha = 0$. This vector field is called the Reeb vector field or sometimes the characteristic vector field. ■

Odd symplectic geometry (more generally, odd Poisson geometry) or the geometry of odd brackets is the mathematical basis of the Batalin–Vilkovisky method [3, 4, 5] in quantum field theory. Odd symplectic geometry possesses features connecting it with both classical ("even") symplectic geometry and Riemannian geometry. In particular, *odd Laplace operators* arise naturally on an odd symplectic manifold, i.e., the second order differential operators whose principal symbol is the odd quadratic forms corresponding to the odd bracket [6]. The key difference from the Riemannian case is that the definition of an odd Laplace operator, in general, requires an extra piece of data besides the "metric", namely, a choice of a volume form (even for a Laplacian acting on functions). This is due to the fundamental fact that on an odd symplectic manifold there is no invariant volume element [6]. However, as discovered by one of the authors, there is one isolated case where an odd Laplacian is defined canonically by the symplectic

Structure without any extra data [8, 9, 10], it is an operator acting on densities of weight 1/2 (half-densities or semi-densities). This fact is not obvious, and there is no simple explanation. A known proof is based on an analysis of the canonical transformations of the odd bracket. In works [11, 13, 15] further phenomena related with odd Laplacians on odd Poisson manifolds were discovered, such as the existence of a natural 'master' groupoid acting on volume forms, its orbits corresponding to Laplacian on half-densities. The symplectic case is distinguished by the existence of a distinguished orbit, which gives the "canonical" Operator.

In a very interesting recent paper [18] suggested a homological interpretation of the canonical odd Laplace operator on half-densities as one of the higher differentials in a certain natural spectral sequence associated with the odd symplectic structure.

In our paper we in particular discuss this interpretation and show that there is a simple but fundamental underlying fact from linear algebra, concerning the Berezinian of a canonical transformation for an odd symplectic bracket. It is the formula

$$\mathrm{Ber}\, J = (det\ J_{00})^2 \quad (1)$$

for J in the odd symplectic supergroup, where J_{00} is the even-even block. Hence the Berezinian is an entire rational function and, moreover, a complete square. There are many geometric facts related with formula (1), which can be found in the literature on odd brackets and the BV formalism. As for example, [17, 18, 6, 7, 8]. We want to draw attention to it as a simple identity for matrices. In view of it, half-densities on an odd symplectic manifold are 'tensor' objects, i.e., transforming according to a polynomial representation. They can be seen as virtual differential forms on a Lagrangian surface. When such a surface is fixed, they become (isomorphic to) actual forms. We see that in the space of differential forms on an ordinary manifold, there is a natural representation of the super group of canonical transformations of the odd bracket. We give a clear description of this action in classical terms. The invariance of the de Rham differential under such a super group, which is absolutely transparent, is equivalent to the existence of the canonical odd Laplacian, but expressed in a different language.

Theorem 4. (Darboux's theorem for Contact Manifolds): Every contact form α on M is locally diffeomorphic to the standard form α_0 on $\mathbb{R}^{2n}$.
Proof [3]: Let α be a contact form on M. Consider $p \in M$ and V a neighborhood of $0 \in T_pM$ of the form $V = V_0 \times (-\varepsilon, \varepsilon)$, where V_0 is a neighborhood of $\mathcal{D}_p$. The geodesic coordinates give a diffeomorphism of ρ of V onto a neighborhood U of p in M. For each $t \in (-\varepsilon, \varepsilon)$, the restriction of $d\alpha$ to $\rho(V_0 \times \{t\})$ is a closed 2-form of maximum rank. In particular, by Darboux's symplectic theorem, we have a change of coordinates at p such that $d\alpha$ is a standard symplectic form, i.e.,

$$d\alpha \Big| V_0 \times \{0\} = \sum dx_i \times dy_i$$

Without loss of generality assume that this is $d\alpha's$ form at $p \in M$.

Next we wish to show that in the neighborhood of p, $d\alpha|\rho(V_0 \times \{t\})$ has the same form. Let $\xi = \frac{d}{dt}$ be the Reeb flow in the neighborhood of p. Then ξ provides us with a partially defined flow of diffeomorphisms φ_t with $\varphi_0 = id$. Again we have that

$$\frac{d}{dt}\big(\varphi_t^*(d\alpha)\big) = \varphi_t^*\big(\mathcal{L}_\xi d\alpha\big)$$

$$= \varphi_t^*\big(\iota(\xi) dd\alpha + d(\iota(\xi) d\alpha)\big) = \varphi_t^*(0) = 0$$

and therefore $d\alpha$ does not depend on time. Hence $\varphi_t^*(d\alpha|\rho(V_0 \times \{t\})) = d\alpha = \sum dx_i \times dy_i$ and therefore exists a neighborhood $U' \subset U$ which has a local coordinate system such that

$$d\alpha = \sum dx_i \times dy_i$$

Notice that this gives us that

$$d\left(\alpha - \sum x_i \, dy_i\right) = 0$$

and by the Poincare lemma, there exists a function z on U' such that when $d\alpha$ is restricted to the field of hyper planes we have

$$\alpha - \sum x_i \, dy_i = dz$$

or

$$\alpha = \sum x_i \, dy_i + dz$$

Since $\alpha \times (d\alpha)^n \neq 0$, the functions are independent and hence make up the desired coordinate system on U'. ■

5. Symplectic connections and deformation quantization

Symplectic connections are closely related to natural formal deformation quantizations at order 2. Flato, Lichnerowicz and Sternheimer introduced deformation quantization in [9]; quantization of a classical system is a way to pass from classical to quantum results and they "suggest that quantization be understood as a deformation of the structure of the algebra of classical observables rather than a radical change in the nature of the observables." In that respect, they introduce a star product which is a formal deformation of the algebraic structure of the space of smooth functions on a symplectic (or more generally a Poisson) manifold; the associative structure given by the usual product of functions and the Lie structure given by the Poisson bracket are simultaneously deformed.

Definition 2. A star product on a symplectic manifold (M, ω) is a bilinear map

$$C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M)[[v]] \;\; (u, v) \rightarrow u *_v v := \sum_{r \geq 0} v^r C_r(u, v)$$

such that

$(u * v) * w = u * (v * w)$ (when extended $R[[v]]$ linearly);

$$C_0(u, v) = uv, \;\; C_1(u, v) - C_1(v, u) = \{u, v\};$$

$$1 * u = u * 1 = u.$$

If all the C_r are bidifferential operators; one speaks of a differential star product; if, furthermore, each C_r is of order $\leq r$ in each argument, one speaks of a natural star product.

The link between symplectic connections and star products appear already in the seminal paper [3] where the authors observe that if there is a flat symplectic connection ∇ on (M, ω), one can generalize the classical formula for Moyal star product $*_M$ defined on R^{2n} with a constant symplectic 2-form. Fedosov, proved more generally that given any symplectic connection ∇, one can construct a star product (in [8] it was proposed that a triple (M, ω, ∇) be known as a Fedosov manifold):

Theorem 5. [1] Given a symplectic connection ∇ and a sequence $\Omega = \sum_{k=1}^{\infty} v^k \omega_k$ of closed 2 forms on a symplectic manifold (M, ω), one can build a star product $*_\nabla, \Omega$ on it. This is obtained by identifying the space $C^\infty(M)[[v]]$ with a sub algebra of the algebra of sections of a bundle of associative algebras (called the Weyl bundle) on M. The sub algebra is the one of flat sections of the Weyl bundle, when this bundle is endowed with a flat connection whose construction is determined by the choices made of the connection on M and of the sequence of closed 2-forms on M. Reciprocally a natural star product determines a symplectic connection. This was first observed by Lichnerowicz for a restricted class of star products.

Theorem 6. [2] A natural star product at order 2 determines a unique symplectic connection.

6. Recollection of the canonical odd Laplacian

In this section we review the construction of the odd Laplacian on half-densities due to [8]. See also [9, 10, 11].

Let *M* be a super manifold endowed with an odd symplectic structure, given by an odd 2-form *ω*. We shall refer to such super manifolds as odd symplectic manifolds. (We always skip the prefix 'super-' unless required to avoid confusion.) Later we shall discuss the more general case of an odd Poisson manifold. A brief definition of the odd Laplacian acting on half-densities on *M* follows.

Consider a cover of *M* by Darboux charts, in which the symplectic form takes the canonical expression $\omega = dx^i d\xi_i$. Here x^i, ξ_i are canonically conjugate variables of opposite parity. We assume that the x^i are even; hence the ξ_i, odd. Let $\mathrm{D}y$, for any kind of variables y, stand for the Berezin, volume element then half-densities on *M* locally look like $\sigma = s(x, \xi)\big(D(x, \xi)\big)^{1/2}$. (Notice that we skip questions related with orientation.) We set

$$\Delta\sigma := \frac{\partial^2 s}{\partial x^i \partial \xi_i}\big(D(x, \xi)\big)^{1/2} , \tag{2}$$

in Darboux coordinates, and call Δ, the canonical odd Laplacian on half-densities

The simplicity of formula (2) is very deceptive. The expression $\frac{\partial}{\partial x^i}\frac{\partial f}{\partial \xi_i}$ was originally suggested by Batalin and Vilkovisky, and is the famous 'BV operator'. However, the trouble is that it is not well-defined on functions (actually, on any objects) unless we fix a volume form, which should therefore enter the definition. The geometrically invariant construction for functions, using a volume form, was first given in [6].There is no canonical volume form on an odd symplectic manifold (unlike even symplectic manifolds, enjoying the Liouville form). In particular,the coordinate volume form $D(x, \xi)$ for Darboux coordinates is not preserved by the (canonical) coordinate transformations (see later). Hence the invariance of the operator Δ given by (2) is a deep geometric fact.

As we showed in [11], on any odd Poisson, in particular, odd symplectic, manifold there is a natural *master groupoid* of 'changes of volume forms' $\rho \mapsto e^s \rho$ satisfying the master equation $\Delta_p e^{s/2} = 0$ (note 1/2 in the exponent; without it there would be no groupoid). Here Δ_p is the odd Laplacian on functions with respect to the given volume form ρ. It is defined by $\Delta_p f \coloneqq div_\rho x_{f}$, where x_f is the Hamiltonian vector field corresponding to *f*. (See [6]; note also [14] for another approach.) In a similar way one can define the odd Laplacian on any densities — again, depending on a chosen volume form. Now, half-densities are distinguished from densities of other weights precisely by the fact that for them the corresponding odd Laplacian would depend only on the orbit of a volume form with respect to the action of the above groupoid [11]. It turns out that on an odd symplectic manifold, all Darboux coordinate volume forms belong to the same orbit of the master groupoid. We can regard it as a 'preferred orbit'; hence, in the absence of an invariant volume form, the odd Laplacian on half-densities defined by an arbitrary Darboux coordinate volume form is invariant. It is just (2).

7. Homological interpretation of the odd Laplacian

Now we are going to approach Δ on half-densities from a very different angle.

Let $\Omega(M)$ be the space of all pseudo differential forms on M, i.e., functions on ΠTM. (As usual, Π stands for the parity reversion function on vector spaces, vector bundles, etc.) In coordinates such functions have the form $s = s(x, \xi, dx, d\xi)$, where the differentials of coordinates are commuting variables of parity opposite to that of the respective coordinate. In our case dx^i are odd and $d\xi_i$ are even. We do not assume that functions $s(x, \xi, dx, d\xi)$ are polynomial in $d\xi_i$. Of course they are (Grassmann) polynomial indx_i, because these variables are odd.

Consider the odd symplectic form ω. Since ω^2 = 0, multiplication by ω can be considered as a differential. Define the operator D = d + ω, where d is the de Rham differential. Since $d\omega$ = 0, it follows that D^2 = 0 and we have a 'double complex'.$(\Omega(M), D = \omega + d)$.

The reader should bear in mind that since $\omega = d\theta$ for some even1-form θ, which is true globally, we have $D = e^{-\theta} d e^{\theta}$ and the multiplication by the inhomogeneous differential form e^{θ} sets an isomorphism between the complexes $(\Omega(M), D)$ and $(\Omega(M), d)$. It follows that $H(\Omega(M), D)$ is isomorphic to$H(\Omega(M), d)$, which is just the de Rham cohomology of the underlying manifold M_0. (Note that the isomorphism e^{θ} preserves only parity, but not $\mathbb{Z}$-grading, even if we restrict it to differential forms on M, i.e., polynomials in $dx, d\xi$.)

The operator D = d + ω was introduced in [18]. The idea was to consider the spectral sequence for $(\Omega(M), D)$ regarded as a double complex. We shall follow it in a form best suiting our purposes and which is slightly different from [18]. (In particular, we do not assume grading in the space of forms.)

Although there is no Z-grading present, single or double, one can still develop the machinery of spectral sequences as follows.

We define linear relations (see [15]) on Ω (*M*):

$$\partial_\circ \coloneqq \{(\alpha,\beta) \in \Omega(M) \times \Omega(M) : \omega\alpha = \beta\},$$

$$\partial_r \coloneqq \{(\alpha,\beta) \in \Omega(M) \times \Omega(M) : \exists \alpha_1, \dots, \alpha_r \in \Omega(M) : \omega\alpha = 0,$$
$$d\alpha + \omega\alpha_1 = 0, \dots, d\alpha_{r-2} + \omega\alpha_{r-1} = 0, d\alpha_{r-1} + \omega\alpha_r = \beta\}$$

for all $r = 1, 2, 3, \dots$. We also set $\partial_{-1} \coloneqq \{(\alpha, 0)\}$. We have subspaces Ker ∂_r, Def ∂_r (the domain of definition), Ind ∂_r (the indeterminacy), and Im ∂_r in $\Omega(M)$, and by a direct check

$$Im\, \partial_r \subset Ker \partial_r,$$
$$Def\ \partial_r = Ker\ \partial_{r-1},$$
$$Ind\ \partial_r = Im\ \partial_{r-1}.$$

That is, we have a sequence of differential relations on (M), defining a spectral sequence (*Er, dr*) where

$$E_r \coloneqq \frac{\mathrm{Ker}\partial_{r-1}}{\mathrm{Im}\partial_{r-1}} = \frac{\mathrm{Def}\,\partial_r}{\mathrm{Ind}\,\partial_r}$$

and the homomorphism $dr : Er \to Er$ is induced by ∂_r in the obvious way. (In fact, differential relations like this is the shortest way of defining spectral sequences, see [13]) Clearly $E_\circ = \Omega(M)$. The relation $\partial_\circ$ is simply the graph of the linear map

$d_\circ : \Omega(M) \to \Omega(M), d_\circ\alpha = \omega\alpha$. What is E_1?

Theorem 7. The space E_1 can be naturally identified with the space of half-densities on *M.*

A proof consists of two independent steps. First, we find the cohomology of d0 using algebra. Second, we identify the result with a geometrical object. The first part goes as follows.

The operator $d_o = \omega$ is a Koszul type differential, since in an arbitrary Darboux chart ω$= dx^i d\xi_i$. Introduce a ℤ-grading by the degreein the odd variables dx^i. The operator d_o increases the degree by one. (This grading is not preserved by changes of coordinates). From general theory it follows that the cohomology should be concentrated in the“maximal degree”. Indeed, suppose that dim $M = n|n$ and consider the linear operator *H* on pseudodifferential forms defined as follows.

For $\sigma = \sigma(x, \xi, dx, d\xi)$,

$$H\sigma(x,\xi,dx,d\xi) \coloneqq \int_0^1 dt\ t^{n-1} \frac{\partial^2 \sigma}{\partial dx^i \partial d\xi_i}(x,\xi,t^{-1}dx, t\, d\xi),$$

— notice the similarity with the Δ-operator. The operator *H* is well defined on all forms of degree less than n in dx^i and on forms of ‘top’ degree if they vanish at $d\xi_i$ = 0. (In

both cases there will be no problem with division by *t*.) For forms on which H makes sense one can check
that

$$(Hd_o + d_oH)\sigma = \sigma$$

In particular, if a form σ is d_o-closed and of degree less than *n* in dx^i, then σ = $d_oH\alpha$. The same applies for a top degree form taking anon-zero value at $d\xi_i = 0$. Hence the d_o-cohomology "sits on" pseudo differential forms of degree *n* in dx^i that do not depend on

$$d\xi_i: \sigma = s(x,\xi)\ [dx^1 \dots dx^n].$$

No non-zero form of this appearance can be cohomologous to zero: indeed, any d_o-exact form, $d_o\tau = \omega\tau$, vanishes at $d\xi_i$ = 0.
Hence, each d_o-cohomology class has a unique representative in a given Darboux coordinate system x^i, ξ_i. It is obtained by taking an arbitrary form from the class, extracting its component of degree *n* in dx^i and evaluating at $d\xi_i$ = 0. By applying this to the class of $dx^1, \dots, dx^n$, we immediately arrive at the following Lemma.

Lemma 1. Elements of the cohomology space $E_1 = H(\Omega(M), \omega)$ are represented in Darboux coordinates as classes

$$\sigma = s(x,\xi)[dx^1 \dots dx^n]$$

where under a change of Darboux coordinates

$$x^i = x^i\left(x^i, \xi_i\right),$$
$$\xi_i = \xi_i\left(x^i, \xi_i\right)$$

the class $[dx^1 \dots dx^n]$ transforms as follows:

$$[dx^1 \dots dx^n] = detJ_{00}.\left[dx^{1'} \dots dx^{n'}\right].$$

Here $J_{00} = \frac{\partial x}{\partial x'}'$ is the even-even block of the Jacobi matrix = $\frac{\partial(x,\xi)}{\partial(x',\xi')}$.
.
To better appreciate the statement, notice that

$$dx^i = dx^{i'}\frac{\partial x^i}{\partial x^{i'}} + \ d\xi_i \frac{\partial x^i}{\partial \xi_{i'}}.$$

Hence

$$[dx^1 \dots dx^n] = \ [dx^{1'} \dots dx^{n'}].det\left(\frac{\partial x^i}{\partial x^{i'}}\right) + \text{terms containing }\ d\xi_i\ .$$

Passing to cohomology is equivalent to discarding these lower orders terms.

What kind of geometrical object is this?

Lemma 2. Objects of the form $\sigma = s(x,\xi)[dx^1 \dots dx^n]$, in Darboux coordinates, with the transformation law given in Lemma *7.1* can be identified with half-densities on *M*.
This is the crucial claim. There is a simple but fundamental fact from linear algebra behind Lemma 7.2, which will be proved in the next section.

The transformation law for $[dx^1 \dots dx^n]$ can be obtained from the formal "law" $[dx_i] = [dx_i]\frac{\partial x^i}{\partial x^{i}}'$. Unfortunately, it does not define a geometric object, because it does not obey the co cycle condition. In a way, it is only a 'virtual' transformation law, which will make sense only if an extra structure is imposed on *M*.

Now as we have the space E_1, let us check the differential d_1 on it. It is induced by the differential relation ∂_1 on $\Omega(M)$. Take an element $\sigma = s(x,\xi)[dx^1 \dots dx^n] \in E_1$, take its representative $\sigma = s(x,\xi)[dx^1 \dots dx^n]$ and consider $\beta \in \Omega(M)$ such that $d\alpha + \omega\alpha_1 = \beta$, for $\alpha_1 \in \Omega(M)$. We will have $[\beta] = d_1\alpha$ for the class $[\beta]$ in E_1. Notice that $d\alpha = d\xi_i \frac{\partial s}{\partial \xi_i}[dx^1 \dots dx^n]$ and it will vanish at $d\xi_i = 0$, therefore it is an ω-exact form, according to our previous analysis. Thus $d_1 = 0$ identically and $E_2 = E_1$

Consider d_2 on $E_2 = E_1 = H(\Omega(M),\omega)$. By definition, d_2 maps the class $\sigma = s(x,\xi)[dx^1 \dots dx^n]$, with a local representative $\sigma = s(x,\xi)[dx^1 \dots dx^n]$, to the class of $\beta \in \Omega(M)$ such that $d\alpha + \omega\alpha_1 = 0$, $d\alpha_1 + \omega\alpha_2 = \beta$, for some α_1 and α_2. We may set $\alpha_1 = -Hd\alpha$, where H is the homotopy operator defined above, and $\beta := d\,\alpha_1 = -dHd\alpha$. Directly:

$$Hd\alpha = H\left(d\xi_i \frac{\partial s}{\partial \xi_i}[dx^1 \dots dx^n]\right) = \sum (-1)^{i+\bar{s}} \frac{\partial s}{\partial \xi_i}[dx^{\widehat{1 \dots d}}x^n]$$

and

$$\begin{aligned}\beta &= -dHd\alpha \\ &= -d\sum(-1)^{i-1+\bar{s}}\frac{\partial s}{\partial \xi_i}[dx^1 \dots \widehat{dx^i} \dots dx^n] \\ &= -dx^j\frac{\partial}{\partial x^j}\sum(-1)^{i+\bar{s}}\frac{\partial s}{\partial \xi_i}[dx^1 \dots \widehat{dx^i} \dots dx^n]\end{aligned}$$

$+$ lower order terms in $dx = -\frac{\partial^2 s}{\partial x^i \partial \xi_i}[dx^1 \dots dx^n] +$ lower order terms in dx .

Hence in E_1 we get:

$$d_2\sigma = d_2(s(x,\xi)[dx^1 \dots dx^n]) = -\frac{\partial^2 s}{\partial x^i \partial \xi_i}[dx^1 \dots dx^n] = -\Delta\sigma ,$$

which is quite remarkable. What about the space $E3$ and the differential $d3$ and so on?

It is not hard to notice that the cohomology of the Δ operator on half-densities on *M* is isomorphic to the de Rham cohomology of the underlying ordinary manifold M_0 (we shall say more about this later). Locally the cohomology vanishes except for constants: $\sigma = const.[dx^1 \dots dx^n]$. Thus, $d_3 = 0$, and $E_4 = E_3$; the same continues for $d_4 = 0, E_5 = E_4 = E_3$, and so on. We arrive at the following theorem.

Theorem 8. With the identification of the space $E_1 = H(\Omega(M),\omega),\ \omega)$ with half-densities on *M*, the differential d_1 vanishes and the next differential d_2 coincides up to a sign with the canonical odd Laplacian. The spectral sequence $(E_r,\ d_r)$ degenerates at the term E_3, which is the cohomology of the operator Δ.

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