Annals of Pure and Applied Mathematics
Vol. 6, No. 1, 2014, 36-40
ISSN: 2279-087X (P), 2279-0888(online)
Published on 5 May 2014
www.researchmathsci.org

Annals of
Pure and Applied Mathematics

Planarity of Some Variants of *p*-Petal Graphs

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Received 19 April 2014; accepted 29 April 2014

Abstract. The main result of the paper [4] was that only a 3-petal graph with even number of petals is planar. In this paper some variants of p-petal graphs are defined and the conditions of planarity of these graphs are studied.

Keywords**:** p-petal graphs, partial p-petal graphs, $(p_1, p_2, \dots, p_r)$-petal graphs, Petersen petal graphs

AMS Mathematics Subject Classification (2010): 05C10

1. Introduction

The A *petal graph* G is a simple connected (possibly infinite) graph with maximum degree three, minimum degree two, and such that the set of vertices of degree three induces a 2-regular graph G_Δ (possibly disconnected)and the set of vertices of degree two induces a totally disconnected graph G_δ [2]. If G_Δ is disconnected, then each of its components is a cycle. The vertex set of G is given by $V = V_1 \cup V_2$, where $V_1 = \{u_i\}, i = 0,1,\dots,2a-1$ is the set of vertices of degree three, and $V_2 = \{v_j\}, j = 0,1,\dots,a-1$ is the set of vertices of degree two. For basic definitions and results on petal graphs, please refer [4].

In section 2 we define *partial* p*-petal graph* and present the necessary and sufficient condition for its planarity. In section 3 we define Petersen petal graph and present some results on this graph.

A petal graph G of size n with petal sequence $\{P_j\}$is said to be a p*-petal graph* denoted $G = P_{n,p}$ if every petal in G is of size p and $l(P_i, P_{i+1}) = 2, i = 0,1,2,\dots,a-1$with $P_{a+1} = P_0$.In a p-petal graph the petal size$p$ is always odd.

It can be easily verified that a p-petal graph $G = P_{n,p}$ is planar when $p(G) = 1$ for any value of $n$as well as a. The graph P^* obtained from the Petersen graph by removing one of the vertices is a 3-petal graph $P_{9,3}$. The petal graph P^* is a subdivision of $K_{3,3}$ and hence not a planar graph. In fact, when $p \geq 3$, a petal graph is not necessarily a planar graph. The number of petals a in a 3-petal graph decides the planarity of the graph.

Theorem 1. *A p-petal graph* $G = P_{n,p}$ $(p \neq 1)$ *is planar if and only if (i)* $p = 3$*; (ii).* a *is an even integer.*

Proof: Let $G = P_{n,3}$ be a 3-petal graph with petal sequence $\{P_i\}, i = 0,1,2, \ldots, a-1$, where a is even. The cycle G_Δ divides the plane into two regions, the inner and the outer region. It is possible to draw the $\frac{a}{2}$ alternate petals $P_0, P_2, \ldots, P_{a-2}$ of G in the inner region so that they do not cross the remaining $\frac{a}{2}$ petals $P_1, P_3, \ldots, P_{a-1}$ that are in the outer region.

If either $p \neq 3$ or a is odd, then it is possible to represent G as graph homeomorphic to $K_{3,3}$ by partitioning the vertex set $V_1(G)$ into three sets $V_1^1(G)$, $V_1^2(G)$and $V_1^3(G)$ such that each of $V_1^1(G)$and $V_1^2(G)$ have three non-adjacent vertices and $V_1^3(G)$has the remaining vertices. For complete proof, refer [4]. □

2. Partial p-petal graphs

A petal graph G is said to be a *partial petal graph* if G_Δ is disconnected. The partial petal graph G is called a *partial* p*-petal graph* if every finite petal in G is of size p and $l(P_i, P_{i+1}) = 2$ for any petal P_i in any component G_{Δ_l}.Two infinite petals P_i and P_j of $P\left(G_{\Delta_k} \cup G_{\Delta_l}\right)$ form an *infinite petal pair* if their base points lie on the bases of two successive finite petals in both $G_{\Delta_k}$and G_{Δ_l}.

Theorem 2. *Let* G *be a partial* p*-petal graph with* a *petals. Let* $G_{\Delta_1}, G_{\Delta_2}, \ldots, G_{\Delta_r}$ *be the components of* G *with* $a_1, a_2, \ldots, a_r$ *finite petals respectively. Let* a_{kl} *denote the number of infinite petals in* $P(G_{\Delta_k} \cup G_{\Delta_l})$*. The graph* G *is planar if and only if the following conditions are satisfied:*

- *i.* $p = 3$*;*
- *ii.* a *is even;*
- *iii. The number of finite petals in* G_{Δ_k}*on a path joining two consecutive infinite petals* $P_i \in P(G_{\Delta_k} \cup G_{\Delta_l})$*and* $P_j \in P(G_{\Delta_k} \cup G_{\Delta_q})$*, (possibly* $G_{\Delta_l} = G_{\Delta_q}$*) is either zero or odd, when there exists at least one component, except* G_{Δ_k}*, connecting* G_{Δ_l} *and* G_{Δ_q}*.*

Proof: Let G be a partial p-petal graph as given. From Theorem 1, any p-petal graph is planar if and only if $p = 3$ and a is an even integer. Hence it is sufficient to prove that G is planar if and only if condition (iii) is satisfied. Let us assume that condition (iii) holds true. Consider the infinite petals $P_i \in P(G_{\Delta_k} \cup G_{\Delta_l})$and $P_j \in P(G_{\Delta_k} \cup G_{\Delta_q})$. Let a_k^1 be the number of finite petals in a path on G_{Δ_k} joining $P_i$and P_j. From condition (iii), if $a_k^1 > 0$, then a_k^1 is odd. Now, draw in the inner region of G_{Δ_k}, the finite petal to whose base edge the base point of P_i is incident, together with the $\left\lceil \frac{a_k^1}{2} \right\rceil$ alternate petals. Draw the remaining $\left\lfloor \frac{a_k^1}{2} \right\rfloor$petals in the outer region of G_{Δ_k}. This representation of G_{Δ_k} is obviously planar. Since G_{Δ_k} is an arbitrary component of G, we conclude that G is planar.

Conversely, let the partial p-petal graph G be planar. Therefore, each component $G_{\Delta_k}, k = 1,2,\dots,r$ is also planar. From given conditions, $p = 3$ and each a_i is even. The following cases arise:

Case 1: There is only one pair of infinite petals in G_{Δ_k}: Let $P_i = u_s v_i u_s'$ and $P_j = u_t v_j u_t'$ be the infinite petal pair between G_{Δ_k} and G_{Δ_l} where u_s and u_t are in G_{Δ_k}. Let G'_{Δ_k} and G'_{Δ_l} be the components obtained by identifying v_i and v_j to get a new vertex v_{ij}. The paths $u_s v_{ij} u_t$ and $u_s' v_{ij} u_t'$ are finite petals in G'_{Δ_k} and G'_{Δ_l} respectively. Clearly, each of these components is planar. Hence, the number of finite petals other than $u_s v_{ij} u_t$ in G'_{Δ_k} is odd. Similarly, the number of finite petals other than $u_s' v_{ij} u_t'$ in G'_{Δ_l} is also odd.

Case 2: There are more than one pair of infinite petals in G_{Δ_k}: Let $P_i, P_j \in P(G_{\Delta_l} \cup G_{\Delta_k})$ with centers v_i and v_j. Let $P_s, P_t \in P(G_{\Delta_k} \cup G_{\Delta_q})$ with centers v_s and v_t. Identify the pairs of vertices $v_i \& v_j$, and $v_s \& v_t$ to get v_{ij} and v_{st} respectively.

Case 2a: There exists no component of G_Δ except G_{Δ_k}, connecting $\mathrm{G}_{\Delta_\mathrm{l}}$ and $\mathrm{G}_{\Delta_\mathrm{q}}$: Let the number of finite petals on the path between the consecutive infinite petals P_i and P_j be a_k^1. If a_k^1 is odd, then it is possible to draw the finite petal that has the base point of P_i in the inner region of G_{Δ_k} together with the $\left\lceil \frac{a_k^1}{2} \right\rceil$ alternate petals. Draw the remaining $\left\lfloor \frac{a_k^1}{2} \right\rfloor$ petals in the outer region of G_{Δ_k}; If a_k^1 is even, then G_{Δ_l} and that part of G connected to G_{Δ_l} can be drawn in the inner region of G_{Δ_i} to preserve the planarity of G.

Case 2b: There exists at least one component of G_Δ except G_{Δ_k}, connecting $\mathrm{G}_{\Delta_\mathrm{l}}$ and $\mathrm{G}_{\Delta_\mathrm{q}}$: In this case it is not possible to draw G_{Δ_l} (or G_{Δ_q}) and that part of G connected to G_{Δ_l} (or G_{Δ_q}) in the inner region of G_{Δ_k} as described in case 2a to preserve the planarity of G, thus ruling out the possibility of a_k^1 being even.□

3. Petersen petal graphs

Coxeter [1] introduced a family of graphs generalizing the Petersen graph in 1950. Watkins [5] denoted these graphs as $G(n,k)$ and named them the generalized Petersen graphs. A *generalized Petersen graph* $P(n,k)$ with parameters n and k, $1 \le k \le n-1, k \le \frac{n}{2}$, is a graph on $2n$ vertices a_i, $0 \le i \le 1$ and b_j, $0 \le j \le n-1$, with $3n$ edges $a_i a_{i\pm 1}$, $b_j b_{j\pm k}$ and $a_i b_i$, where all calculations have to be performed modulo n. These edges are called *ring edges*, *chordal edges* and *spokes* respectively. The graph $P(5,2)$ is the Petersen graph.

A petal graph G is called a $(p_1, p_2, \dots, p_r)$*-petal graph* if a_i petals of G are of size p_i, $i = 1,2,\dots r$, such that $\sum_1^r a_i = a$, and is denoted by $G = P_{n,(p_1,p_2,\dots,p_r)}$. A $(p_1, p_2, \dots, p_r)$-petal graph G is said to be a *Petersen petal graph*, denoted $G = {P^*}_{n,(p_1,p_2,\dots,p_r)}$ if G is isomorphic to a graph that can be obtained from the generalized Petersen graph $P(n,k)$ either by subdivision of some of its edges or deletion of some of its vertices. For basic definitions and the following results, refer [3].

The following graphs are Petersen petal graphs:

a) $P_{9,3}$, the 3-petal graph with 3 petals;

b) $P_{9l,3}$, the 3-petal graph with $3l$ petals, $l \geq 2$;

c) $P_{9l,(3,9)}$, the $(3,9)$-petal graph with $3l$ petals, $l \geq 3$

d) Any planar 3-petal graph.

The graphs specified in result a) is not planar; b) is not planar when l is odd; d) is obviously planar. Theorem 3 will help to identify which of the graphs specified in the result c) are planar.We define the following:

Two petals of a petal graph are said to be intersecting petals if their bases have some common edges in G_Δ. A $(p_1, p_2, \dots, p_r)$-petal graph, where $p_1 < p_2 < \cdots < p_r$, is said to be overlapping if the base of a petal of size p_i lies on the base of a petal of size p_{i+1}. Hence, we have $p_{i-1} = p_i + 2$. A overlapping $(p_1, p_2, \dots, p_r)$-petal graph is said to be a neighborhood $(p_1, p_2, \dots, p_r)$-petal graph if none of the petals in the graph are intersecting. A overlapping $(p_1, p_2, \dots, p_r)$-petal graph is denoted as a p_r-petal graph if $a_i = \frac{a}{r}$ for all i.

Any neighborhood $(p_1, p_2, \dots, p_r)$-petal graph is obviously planar.

Theorem 3. *A p_r-petal graph G is planar if and only if a is even.*

Proof: When a is even, each a_i is also even and it is possible to draw one set of alternate petals in the inner region and the other set in the outer region of G_Δ.

When a is odd, we can prove that the p_r-petal graph is homeomorphic to $K_{3,3}$. Partition the vertex set $V_1(G)$ into three sets $V_1^1(G)$, $V_1^2(G)$ and $V_1^3(G)$such that $V_1^1(G) = \{u_0, u_{2r}, u_{2(a-r)}\}$ and $V_1^2(G) = \{u_{p_r}, u_{2a-1}, u_{2r-1}\}$ and $V_1^3(G)$ has the remaining vertices. We can represent G as a graph homeomorphic to $K_{3,3}$using the following steps:

- Take the cycle $u_0 u_{p_r} u_{2r} u_{2a-1} u_{2(a-r)} u_{2r-1}$ containing the vertices of $V_1^1(G) \cup V_1^2(G)$;
- Connect the pairs of vertices (u_0, u_{2a-1}), $\left(u_{p_r}, u_{2(a-r)}\right)$, (u_{2r}, u_{2r-1});
- Subdivide the edges (u_0, u_{2r-1}) with the vertices $u_1, u_2, \dots, u_{2r-2}$; $\left(u_{2r}, u_{p_r}\right)$ with the vertices $u_{2r+1}, u_{2r+2}, \dots, u_{p_r-1}$; $\left(u_{p_r}, u_{2(a-r)}\right)$ with the vertices $u_{p_r+1}, u_{p_r+2}, \dots, u_{2(a-r)-1}$ and $\left(u_{2(a-r)}, u_{2a-1}\right)$ with the vertices $u_{2(a-r)}, u_{2(a-r)+1}, \dots, u_{2a-1}$.
- Connect all adjacent vertices in this representation so that adjacency is preserved.

$G$is homeomorphic to $K_{3,3}$ and hence the result is proved. □

Theorem 4. *Let G be a non-neighborhood, non-overlapping $(p_1, p_2, \dots, p_r)$-petal graph. Let a_i' be the number of petals of size p_i on the base of a petal of size $p_r$in G_Δ. Then G is planar if and only if*

i. $p_1 = 3$;

ii. a_1' *is odd;*

iii. $r = 2$ *and*

iv. a_2*is even.*

Proof: Let G be a non-neighborhood, non-overlapping $(p_1, p_2, \dots, p_r)$-petal graph that satisfies the above conditions. Then it is possible to draw the petal graph such that one set of alternating petals of size p_r (that is p_2) are in the inner region of G_Δ and the remaining set of petals in the outer region. It is also possible to draw the petals of size p_1 in the regions bounded by G_Δ and petals of size p_r alternately such that no petals are intersecting.

Theorem 1 demands conditions (i) and (iv). If conditions (ii) and (iv) do not hold, then we can prove that the petal graph is homeomorphic to $K_{3,3}$.We assume that r is at least three. Then, there is at least one more set of petals of size p_2 such that $p_1 < p_2 < p_r$. Let $P_1', P_2', \ldots$ be the sequence of p_2-petals. Let u_2 be the base point of the p_2-petal P_1'. Partition the vertex set $V_1(G)$ of the base points of G on G_Δ in to three subsets $V_1^1(G)$, $V_1^2(G)$ and $V_1^3(G)$such that $V_1^1(G) = \{u_2, u_{p_2+1}, u_{2p_2}\}$, $V_1^2(G) = \{u_3, u_{p_2+2}, u_{2p_2+1}\}$ and $V_1^3(G)$has the remaining vertices. We can represent G as a graph homeomorphic to $K_{3,3}$using the following steps:

- Take the cycle $u_2u_3u_{p_2+1}u_{p_2+2}u_{2p_2}u_{2p_2+1}$ containing the vertices of $V_1^1(G) \cup V_1^2(G)$;
- Connect the pairs of vertices (u_2, u_{p_2+2}), (u_{p_2+1}, u_{2p_2+1}), (u_{2p_2}, u_3);
- Subdivide the edge (u_{2p_2}, u_3) with the vertices on a path from $u_{3p_2}$to u_{2a-p_2+3}.
- Plot all the other vertices and connect all the adjacent vertices in this representation so that adjacency is preserved.

$G$is homeomorphic to $K_{3,3}$ and hence r must be two.

Now we prove that if a_1' is not odd, then$G$ cannot be planar.

Let $P_1'', P_2'', \ldots$ be the sequence of 3-petals. Let u_2 be the base point of the 3-petal P_1''. When a_1' is even, then it is possible to partition the vertex set $V_1(G)$ of the base points of G on G_Δ in to three subsets $V_1^1(G)$, $V_1^2(G)$ and $V_1^3(G)$such that $V_1^1(G) = \{u_2, u_4, u_6\}$, $V_1^2(G) = \{u_3, u_5, u_7\}$ and $V_1^3(G)$has the remaining vertices. We can represent G as a graph homeomorphic to $K_{3,3}$using the following steps:

- Take the cycle $u_2u_3u_4u_5u_6u_7$ containing the vertices of $V_1^1(G) \cup V_1^2(G)$;
- Connect the pairs of vertices (u_2, u_5), (u_4, u_7), (u_6, u_3);
- Subdivide the edge (u_6, u_3) with the vertices on a path from $u_{6+p_r}$to u_{2a-p_r+3}.
- Plot all the other vertices and connect all the adjacent vertices in this representation so that adjacency is preserved.

$G$is homeomorphic to $K_{3,3}$ and hence the result is proved. □

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